Spherical 2-type hypersurfaces

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Resumen

Este trabajo va esencialmente encaminado a resolver la conjetura de Barros: *Una hipersuperficie inmersa en la esfera $\mathbb{S}^m$ de $\mathbb{R}^{m+1}$ es de tipo dos si, y sólo si, tiene curvatura escalar constante y curvatura media constante no nula.* Esta conjetura surgió a raíz de los primeros intentos de clasificar aquellas hipersuperficies esféricas cuya inmersión se puede construir con únicamente dos valores propios de su laplaciano. Los primeros resultados en esta línea se deben a los profesores Barros, Chen y Garay.

Este artículo se ha desarrollado en el marco de un proyecto de investigación conjunto entre las universidades de Granada y Murcia, y su versión definitiva se concretó en el segundo semestre del 90 con motivo de una estancia corta del profesor Barros en la Universidad de Murcia.

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Abstract

In this technical report, 2-type spherical hypersurfaces in the sphere S^{n+1} are characterized as those having non-zero constant mean curvature and constant scalar curvature. Some applications of this result are presented and discussed.

1. Introduction

In this paper, we deal with connected (but not necessarily compact) submanifolds of a Euclidean m-space $\mathbb{E}^m$. Such a submanifold M is said to be of finite type if each component of its position vector x can be written as a finite sum of eigenfunctions of the Laplacian Δ of M (through this paper we will use $\Delta f = -\mathrm{div}\nabla f$, the Laplacian of M acting on $\mathcal{C}^\infty(M)$), that is,

$$x = x_0 + \sum_{t=1}^{k} x_t,$$

where $\Delta x_t = \lambda_t x_t$ $(t = 1, \ldots, k)$ and x_0 being a constant vector, which is nothing but the center of mass of M in $\mathbb{E}^m$ when M is compact. If all eigenvalues $\{\lambda_1, \ldots, \lambda_k\}$ are mutually different, then M is called a k-type submanifold (see, for instance, [Ch1] for details).

In terms of finite type submanifolds, a well-known result of T. Takahashi [Ta] says that a submanifold M in $\mathbb{S}^{n+1}$ is of 1-type if and only if M is a minimal submanifold of $\mathbb{S}^{n+1}$. Moreover, when M is compact the center of mass of M in $\mathbb{E}^{n+2}$ is the center of $\mathbb{S}^{n+1}$ in $\mathbb{E}^{n+2}$. Thus, M is mass-symmetric in $\mathbb{S}^{n+1} \subset \mathbb{E}^{n+2}$.

If one takes a compact hypersurface M (which is not a small hypersphere) of $\mathbb{S}^{n+1} \subset \mathbb{E}^{n+2}$ with non-zero constant mean curvature and constant scalar curvature (for example, a non-minimal isoparametric hypersurface) then it can be proved that M is of 2-type (see Corollary 4.2 or [Ch2]). The converse will be obtained in this paper, which can be viewed as a partial solution to a first open problem stated by B.Y. Chen [Ch4, I.4]. More precisely, we prove (Theorems 3.2 and 4.1) that

A 2-type hypersurface M of the hypersphere $\mathbb{S}^{n+1}$ in $\mathbb{E}^{n+2}$ has non-zero constant mean curvature and constant scalar curvature.

Our first approach proves that given a 2-type spherical hypersurface M, then it has constant mean curvature if and only if it has constant scalar curvature, (see Theorem 3.2). This gives an affirmative answer to another open problem stated by B.Y. Chen [Ch4, I.6].

The behaviour of compact 2-type submanifolds in $\mathbb{S}^{n+1}$ is quite different to that of compact minimal ones in $\mathbb{S}^{n+1}$. In fact, while compact minimal submanifolds in $\mathbb{S}^{n+1}$ are always mass-symmetric, this does not happen for compact 2-type submanifolds in $\mathbb{S}^{n+1}$. In a joint paper of first author and B.Y. Chen, [BCh], examples of non-mass-symmetric compact 2-type submanifolds in the sphere are given. These examples were obtained with high codimension. However, as a consequence of our main result, it is proved that

Compact 2-type hypersurfaces of $\mathbb{S}^{n+1}$ are always mass-symmetric.

This also gives a partial answer to a third Chen's problem [Ch4, I.1].

Finally, we will apply our results to study spherical Dupin hypersurfaces to get a local rigidity theorem (see Theorem 5.1). In particular, the following is proved:

2-type Dupin hypersurfaces in the sphere with at most three distinct principal curvatures are isoparametric hypersurfaces.

This partially solves a fourth Chen's open problem, [Ch4, II.11].

Our Main Theorem and its corollaries (see Section 4) generalize the main results of [BG], [HV1], Theorems 1 and 3 of [BChG], Theorem 2 of [Ch2], Theorem 4.5 of [Ch1, p. 279] and also Theorems 1 and 2 of [Ch3].

2. Some basic preliminaries and lemmas

Let M be a hypersurface of the unit hypersphere $\mathbb{S}^{n+1}$ in $\mathbb{E}^{n+2}$ which we will assume (without loss of generality) centred at the origin of $\mathbb{E}^{n+2}$. Denote by x the position vector of M in $\mathbb{E}^{n+2}$ and by ∇ and D the Levi-Civita connection of M and the normal connection of M in $\mathbb{E}^{n+2}$, respectively. We also denote by σ, A and H (respectively, H') the second fundamental form of M in $\mathbb{E}^{n+2}$, the Weingarten map of M in $\mathbb{S}^{n+1}$ and the mean curvature vector field of M in $\mathbb{E}^{n+2}$ (respectively, in $\mathbb{S}^{n+1}$). If Δ denotes the Laplacian of M, then the following formula for ΔH was computed in [BChG]:

$$\Delta H = \frac{n}{2}\nabla\alpha^2 + 2\,\mathrm{tr}A_{DH'} + (\Delta\alpha + \alpha|\sigma|^2)N - (n\alpha^2 + n)x,$$

where $H' = \alpha N$, N being the unit normal vector field of M in $\mathbb{S}^{n+1}$. Here $\nabla\alpha^2$ denotes the gradient of α^2 and $\mathrm{tr}A_{DH'} = \sum_{i=1}^{n} A_{D_{E_i}H'}E_i$, where $\{E_1, \ldots, E_n\}$ is a local orthonormal frame tangent to M.

Now, assume that M is of 2-type. Then its position vector in $\mathbb{E}^{n+2}$ can be written as

$$x = x_0 + x_1 + x_2, \quad \text{with} \quad \Delta x_1 = \lambda_1 x_1 \text{ and } \Delta x_2 = \lambda_2 x_2,$$

where x_0 is a constant vector in $\mathbb{E}^{n+2}$ and $x_1,\ x_2$ are $\mathbb{E}^{n+2}$–valued non-constant differentiable functions on M.

From (2) and the well-known fact of $\Delta x = -nH$, we have

$$\Delta H = bH + c(x - x_0),$$

where $b = \lambda_1 + \lambda_2$ and $c = \dfrac{1}{n}\lambda_1\lambda_2$.

Remark 2.1 Through this paper, we can assume that $c \neq 0$, otherwise last two authors have proved in [FL] the non-existence of such hypersurfaces. Of course, if M is compact then $c \neq 0$.

From (1) and (3) one gets the following formulae:

$$n\alpha^2 + n = b - c + c < x, x_0 >$$

and

$$< \Delta H, X >= -c < x_0, X >,$$

where X denote any tangent vector field to M.

By using (4) and (5) a nice expression for the tangential component of ΔH is found:

$$(\Delta H)^T = -n\nabla\alpha^2.$$

On the other hand, from (1) one has

$$(\Delta H)^T = \frac{n}{2}\nabla\alpha^2 + 2\,\mathrm{tr}A_{DH'}.$$

Finally, an easy computation involving (6), (7) and Codazzi equation gives

$$A(\nabla\alpha) = \mathrm{tr}A_{DH'} = -\frac{3n}{4}\nabla\alpha^2.$$

Therefore, the following lemma is proved.

Lemma 2.2 ([BChG]) *Let M be a 2-type hypersurface of $\mathbb{S}^{n+1}$. Then $\nabla\alpha^2$ is a principal direction with principal curvature $-\dfrac{3n}{2}\alpha$ on the open set $\mathcal{U} = \{p \in M : \nabla\alpha^2(p) \neq 0\}$.*

Next lemma, which can also be found in [BChG], allows us to get a good information about the above quoted open set $\mathcal{U}$.

Lemma 2.3 *Let M be a 2-type hypersurface of $\mathbb{S}^{n+1}$. Then either M has constant mean curvature or $\mathcal{U}$ is dense in M.*

For short, we write $h = (b - |\sigma|^2)\alpha - \Delta\alpha$ and $g = n\alpha^2 + n + c - b$, and use (1), (3) and (6) to get

$$cx_0 = n\nabla\alpha^2 + hN + gx.$$

Now, working on $\mathcal{U}$, choose a local orthonormal frame of principal directions $\{E_1, \dots, E_n\}$ with associated principal curvatures $\{\mu_1, \dots, \mu_n\}$, E_1 being in the direction of $\nabla\alpha^2$, so that $\mu_1 = -\frac{3n}{2}\alpha$. By using (9) we find the following formulae:

$$\begin{aligned} 0 \;=\; E_1(cx_0) \;\;&=\;\; \{nE_1E_1(\alpha^2) + \frac{3n}{2}\alpha h + g\}E_1 \\ &\quad + nE_1(\alpha^2)\nabla_{E_1}E_1 \\ &\quad + \{E_1(h) - \frac{3n^2}{2}\alpha E_1(\alpha^2)\}N \end{aligned} \tag{10}$$

and

$$\begin{aligned} 0 \;=\; E_j(cx_0) \;\;&=\;\; nE_jE_1(\alpha^2)E_1 + nE_1(\alpha^2)\sum_{k=2}^{n}\omega_1^k(E_j)E_k \\ &\quad + E_j(h)N - \mu_j hE_j + gE_j, \quad j = 2, \dots, n, \end{aligned} \tag{11}$$

where we have written $\nabla_{E_j}E_i = \sum_{k=1}^n \omega_i^k(E_j)E_k$. In particular, one has

Lemma 2.4 *Let M be a 2-type hypersurface of $\mathbb{S}^{n+1}$. Then the following formulae hold on $\mathcal{U}$:*

$$E_1(h) = \frac{3n^2}{2}\alpha E_1(\alpha^2), \tag{12}$$

$$E_j(h) = 0, \quad j = 2, \dots, n, \tag{13}$$

$$nE_1E_1(\alpha^2) + \frac{3n}{2}\alpha h + g = 0. \tag{14}$$

Finally, a straighforward computation from (12), (13) and Lemma 2.3 gives

$$h = n^2\alpha^3 + k, \tag{15}$$

for a constant k, holding anywhere on M.

3. Spherical 2-type hypersurfaces with constant scalar curvature

We are going to compute $\Delta\alpha^2$ in two different ways. First, by using (4) we find

$$n\Delta\alpha^2 = \Delta < cx_0, x >= -n < cx_0, H' > +n < cx_0, x >$$

and then, from (9), we get

$$\Delta\alpha^2 = -\alpha h + g.$$

On the other hand,

$$\begin{aligned} \Delta\alpha^2 \;\;&=\;\; 2\alpha\Delta\alpha - 2|\nabla\alpha|^2 \\ &=\;\; 2(b - |\sigma|^2)\alpha^2 - 2\alpha h - 2|\nabla\alpha|^2. \end{aligned} \tag{2}$$

Now, we are ready to prove the following

Proposition 3.1 *Let M be a 2-type hypersurface of $\mathbb{S}^{n+1}$. Then the mean curvature α does not vanish anywhere on M.*

Proof. First, we use (9) and (15) to get

$$\begin{aligned} c^2|x_0|^2 &= 4n^2\alpha^2|\nabla\alpha|^2 + n^4\alpha^6 + n^2\alpha^4 + 2kn^2\alpha^3 \\ &\quad + 2n(n+c-b)\alpha^2 + k^2 + (n+c-b)^2. \end{aligned} \tag{3}$$

Now, assume there is a point $p \in M$ such that $\alpha(p) = 0$. Then (3) yields

$$n^4\alpha^4 + n^2\alpha^2 + 2kn^2\alpha + 2n(n+c-b) + 4n^2|\nabla\alpha|^2 = 0, \tag{4}$$

which holds good on $\mathcal{U}$, and then on M, because $\mathcal{U}$ is dense in M. In particular, we have

$$n + c - b = -2n|\nabla\alpha(p)|^2.$$

On the other hand, (1) and (2) give us

$$n + c - b = -2|\nabla\alpha(p)|^2.$$

Hence

$$n + c - b = 0 = |\nabla\alpha(p)|^2.$$

Now, by carrying (7) into (4) we get $k\alpha \leqslant 0$, so that α does not change sign, because $k \neq 0$ (otherwise, M should be minimal in $\mathbb{S}^{n+1}$). So, we can assume $\alpha \geqslant 0$ and $k < 0$. Then $\Delta\alpha(p) = -h(p) = -k > 0$, which cannot be hold because p is a minimun of α (notice we are using $\Delta f = -\mathrm{div}\nabla f$).

Next, we are going to give the main result of this section, which gives an affirmative answer to an open problem stated by B.Y. Chen [Ch4, I.6].

Theorem 3.2 *Let M be a 2-type hypersurface of $\mathbb{S}^{n+1}$. Then M has constant mean curvature if and only if M has constant scalar curvature.*

Proof. If α is a constant, then h so is because (15) and then $|\sigma|^2$ is also a constant. As a consequence, we use the Gauss equation

$$|\sigma|^2 = n^2\alpha^2 - n(n-1)\tau + n$$

to get M has constant scalar curvature.

Conversely, suppose now M has constant scalar curvature. From (9) we find

$$|\nabla\alpha|^2 = \frac{1}{4n^2\alpha^2}\{c^2|x_0|^2 - h^2 - g^2\},$$

that jointly with (1) and (2) leads to

$$4n^2(b - |\sigma|^2)\alpha^4 + (h - 2n^2\alpha^3)h + (g - 2n^2\alpha^2)g - c^2|x_0|^2 = 0.$$

Finally, from here, (15) and Gauss equation α must be a root of a polynomial with constant coefficients and therefore α is a constant.

4. The main result

Let M be a 2-type hypersurface of $\mathbb{S}^{n+1} \subset \mathbb{E}^{n+2}$. Consider $\mathcal{U} = \{p \in M : \nabla\alpha^2(p) \neq 0\}$ which is a dense open subset of M unless it was empty and so M has constant mean curvature (see Lemma 2.3). Let p be any point of $\mathcal{U}$ and denote by $\gamma(t)$ the integral curve of $\nabla\alpha^2$ through the point $p \in \mathcal{U}$. Now, (15) allows us to rewrite (14) along $\gamma(t)$ as follows:

$$\frac{d^2}{dt^2}(\alpha^2) + \frac{3}{2}n^2\alpha^4 + \alpha^2 + \frac{3}{2}k\alpha + \frac{1}{n}(n + c - b) = 0.$$

Let $\beta = \left(\dfrac{d\alpha}{dt}\right)^2$. Then it is easy to see that equation (1) can be reduced to the following first order differential equation:

$$\alpha\frac{d\beta}{d\alpha} + 2\beta = -\frac{3}{2}n^2\alpha^4 - \alpha^2 - \frac{3}{2}k\alpha - \frac{1}{n}(n + c - b).$$

From this equation we obtain the following solution:

$$\begin{aligned} 4n^2\alpha^2\beta &= -\frac{3}{2}n^4\alpha^4 - 2n^2\alpha^2 - 6kn^2\alpha \\ &\quad - 4n(n + c - b)\ln(\alpha) + C_1, \end{aligned} \tag{3}$$

where C_1 is some constant.

On the other hand, from (9) one has

$$4n^2\alpha^2\beta = c^2|x_0|^2 - (n^2\alpha^3 + k)^2 - (n\alpha^2 + n + c - b)^2. \tag{4}$$

Therefore, (3) and (4) prove the following

Theorem 4.1 *Let M be a 2-type hypersurface of $\mathbb{S}^{n+1} \subset \mathbb{E}^{n+2}$. Then M has constant mean curvature.*

The following result gives a nice characterization of compact 2-type hypersurfaces in the hypersphere $\mathbb{S}^{n+1} \subset \mathbb{E}^{n+2}$ and partially solves an open problem stated by B.Y. Chen [Ch4, I.4].

Corollary 4.2 *Let M be a compact hypersurface of $\mathbb{S}^{n+1} \subset \mathbb{E}^{n+2}$ which is not a small hypersphere of $\mathbb{S}^{n+1}$. Then M is of 2-type if and only if M has non-zero constant mean curvature α and constant scalar curvature τ. Moreover, if M is of 2-type, α and τ are completely determined for the eigenvalues $\{\lambda_1, \lambda_2\}$ involved in the 2-type condition.*

Proof. The necessary condition follows automatically from Theorems 4.1 and 3.2. Now, if α and τ are constant, then $|A|^2$ is also constant and so (1) allows us to write

$$\Delta H = (|A|^2 + n)H + (|A|^2 - n\alpha^2)x,$$

where we have used $H = H' - x$. As a consequence there exist two constants, say r and s, such that $\Delta H = rH + sx$, with $s \neq 0$ because M is not a small hypersphere of $\mathbb{S}^{n+1}$. Therefore, we use Theorem 2.2 of [Ch1, p. 257] to get that M is of 2-type. Last claim of the statement follows from Theorem 4.2 of [Ch1, p. 276].

Next result gives a partial answer to another open problem stated by B.Y. Chen [Ch4, I.1].

Corollary 4.3 *Let M be a compact 2-type hypersurface of $\mathbb{S}^{n+1} \subset \mathbb{E}^{n+2}$. Then M is mass-symmetric in $\mathbb{S}^{n+1}$.*

Proof. First, we use Theorem 4.1 to have (5), where both coefficients $|A|^2+n$ and $|A|^2-n\alpha^2$ are constant. Moreover, $|A|^2-n\alpha^2 \neq 0$ because M is assumed to be of 2-type in $\mathbb{E}^{n+2}$ (notice that $|A|^2 = n\alpha^2$ implies M is a small hypersphere and so of 1-type in some hyperplane of $\mathbb{E}^{n+2}$ and then of 1-type in $\mathbb{E}^{n+2}$). Thus we have

$$0 = \int_M \Delta H dv = (|A|^2+n)\int_M H dv + (|A|^2-n\alpha^2)\int_M x dv,$$

and so

$$\int_M x dv = 0,$$

this means, the center of mass of M is nothing but the origin of $\mathbb{E}^{n+2}$.

Remark 4.4 We would like to point out that Theorem 4.1 and Corollaries 4.2 and 4.3 have been also obtained, simultaneously and independently, by Hasanis and Vlachos in [HV2], where they use a different method of proof.

5. Spherical 2-type Dupin hypersurfaces

A hypersurface M of $\mathbb{S}^{n+1} \subset \mathbb{E}^{n+2}$ is called a Dupin hypersurface if the multiplicity of each principal curvature is constant on M and each principal curvature is constant along its associated principal directions. In [CR1] it is proved that compact embedded Dupin hypersurfaces are conformal images of isoparametric hypersurfaces when the number g of principal curvatures is $g \leqslant 2$, but this is not the case when $g \geqslant 3$. In [Th], G. Thorbergsson proves that, in cohomology level, compact embedded Dupin hypersurfaces are isoparametric. That result leads to the Cecil-Ryan's conjecture [CR2]: *A compact embedded Dupin hypersurface is Lie equivalent to an isoparametric hypersurface.* That holds when $g \leqslant 3$ [CR1, Mi]; otherwise, it can be found counterexamples to the conjecture [PT, MO]. These facts suggest a close relation between compact embedded Dupin hypersurfaces and isoparametric ones.

It is a well-known fact that isoparametric hypersurfaces of $\mathbb{S}^{n+1} \subset \mathbb{E}^{n+2}$ with $g \leqslant 2$ are spheres and Riemannian products of spheres. When $g = 3$, they were completely classified by E. Cartan [Ca]. They are all homogeneous spaces and their multiplicities of principal curvatures (m_1, m_2, m_3) and dimensions n are listed in the adjoint table:

M^n	(m_1, m_2, m_3)	n
$SO(3)/Z_2+Z_2$	$(1,1,1)$	3
$SU(3)/T^2$	$(2,2,2)$	6
$SP(3)/SP(1)^3$	$(4,4,4)$	12
$F_4/Spin(8)$	$(8,8,8)$	24

TABLE 1

Now, we are going to state and prove the main result of this section.

Theorem 5.1 *Let M be a Dupin hypersurface of* $\mathbb{S}^{n+1}$ *with at most three distinct principal curvatures which is not a small hypersphere of* $\mathbb{S}^{n+1}$*. Then M is of 2-type if and only if one the following statements holds:*

1) M is an open piece of a Riemannian product $\mathbb{S}^p \times \mathbb{S}^{n-p}$*.*

2) M is an open piece of one of the hypersurfaces exhibited in Table 1.

Proof. The sufficient condition follows easily from results of above sections. Now, let us suppose M is a 2-type hypersurface of $\mathbb{S}^{n+1}$. Then from Theorems 4.1 and 3.2 we know that M has constant mean curvature and constant scalar curvature. Since M is a Dupin hypersurface it is not difficult to see that M is, in fact, an isoparametric hypersurface. Thus, we obtain the desired conclusion, because M cannot have only one principal curvature.

As a consequence, we obtain the following.

Corollary 5.2 *Let M be a Dupin hypersurface of* $\mathbb{S}^4$ *which is not a small hypersphere. Then M is of 2-type if and only if M is an open piece of one of the following hypersurfaces:* $\mathbb{S}^1 \times \mathbb{S}^2$, $SO(3)/Z_2 + Z_2$.

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