Finite Type Ruled Manifolds shaped on Spherical Submanifolds

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1. Introduction.

Euclidean submanifolds of finite type were introduced few years ago by B.-Y.Chen, [2], [3]. Since then there has been an increasingly interesting development of this subject, where one can observe that most papers concern to compact finite type submanifolds. This is because in such case the finite type structure can be characterized by a practical condition on the Laplacian of the mean curvature vector (see $\mathbb{S}^1$), which turns out to be only a necessary condition when the submanifold is non-compact.

Frequently, the submanifolds constructed over a given submanifold are non-compact. For example, given a compact submanifold M of the unit sphere S^{n+1} centered at the origin in E^{n+2}, one constructs the punctured cone CM-$\{0\}$ shaped on M. In [8], J.Simons proved that if M is minimal in S^{n+1} then CM-$\{0\}$ is minimal in E^{n+2}. In the finite type submanifolds terminology, see [9], Simons'result says that if M is of 1-type then CM-$\{0\}$ is also of 1-type.

In that context, the following problem arises in a natural way.

> PROBLEM:*"To what extent the finite type character of an Euclidean submanifold affects the finite type condition of a manifold shaped on it?"*

In order to solve that question the second author, [6], studied finite type punctured cones CM-$\{0\}$ and got a first answer. Actually he proved that a punctured cone shaped on M is of finite type if and only if M is minimal in S^{n+1}.

In this paper we construct non-compact ruled manifolds on a certain class of compact spherical submanifolds with the aim of testing an answer to the proposed problem. Just now, we want to point out the important differences between our case and the cone one. In fact, our ruled manifolds are never minimal. Furthermore, they are of finite type if and only if they are generalized cylinders shaped on a finite type spherical submanifold. Hence, whereas we could find only a special kind of 1-type punctured cones, and then minimal, here we can get a k-type ruled manifold for any $k \in Z^+$, $k \geqslant 2$, all of them being generalized cylinders.

It seems to be that cylinders can be guessed to play a chief role in order to give a classification of non-compact finite type euclidean submanifolds. Indeed it was shown, in [4], that the only finite type tubes in E^3 are the circular cylinders and, in [7], that euclidean hypersurfaces whose coordinate functions are eigenfunctions of its Laplacian are either minimal or spheres or circular cylinders. Also, in a recent paper, [5], B.-Y.Chen has shown that a null 2-type surface in E^3 is an open piece of a circular cylinder.

Finally we mention that if we choose a totally umbilical hypersurface M of S^{n+1}, our ruled manifolds are nothing but the classical circular cylinders and circular cones, depending on M is totally geodesic or not. Therefore circular cones are not of finite type. This fact was also implicitly contained in [6].

2. Preliminaries.

A p-dimensional submanifold M^p of the Euclidean space E^{n+2} is said to be of k-type if the position vector x of M^p in E^{n+2} can be decomposed as

$$x = c + x_{i_1} + \cdots + x_{i_k},$$

such that

$$\Delta x_{i_j} = \lambda_{i_j} x_{i_j},$$

and $\lambda_{i_1} < \cdots < \lambda_{i_k}$, where $c \in E^{n+2}, \lambda_{i_j} \in \mathbb{R}$ and Δ represents the Laplacian of M^p with respect to the induced metric. M^p is called of null k-type if one of the λ 's is zero. If M^p is a k-type Euclidean submanifold there exists a polynomial of degree k, $P(t)$, such that $P(\Delta)\tilde{H} = 0$, $\tilde{H}$ being the mean curvature vector of M^p in E^{n+2}. When M^p is compact that is also a sufficient condition of M^p to be of finite type (see [2], pg. 255).

Let us denote by S^{n+1} the (n+1)-dimensional unit sphere centered at the origin of E^{n+2}, choose a totally umbilical compact hypersurface $\bar{M}^n$ of S^{n+1} and suposse $\bar{M}^n$ is given as the intersection of S^{n+1} with an affine hyperplane P. Finally take M^p as a compact p-dimensional submanifold of $\bar{M}^n$.

Since S^{n+1} is simply connected, we may choose a global unit vector field v, normal to $\bar{M}^n$ in S^{n+1} which satisfies $A_v X = \rho X$ for a constant $\rho \in \mathbb{R}$ and any vector field X of $\bar{M}^n$. Here A is the Weingarten map of $\bar{M}^n$ in S^{n+1}. We construct a (p+1)-dimensional ruled submanifold over M^p, say M^*, in the following way:

$$\begin{array}{ccc} M^p \times (-\varepsilon, \varepsilon) & \longrightarrow & E^{n+2} \\ (m,t) & \longrightarrow & m + tv \end{array}$$

where $\varepsilon > 0$ is the largest real number for which M^* is isometrically imbedded in E^{n+2}.

Our first task will be to compute the mean curvature vector fields $\bar{H}$ and $\tilde{H}$ of M^* and M^p, respectively, in E^{n+2}. To do that let σ be the second fundamental form of M^p in E^{n+2}. Let us write H' and ξ for the mean curvature vector of M^p in $\bar{M}^n$ and the position vector on M^p, respectively. Let m be any point of M^p and choose a local frame $\{E_i\}_{i=1}^p$ tangent to M^p and so that $\nabla_{E_j} E_i(m) = 0$, ∇ being the Levi–Civita connection on M^p. By parallel transport in E^{n+2} along the rays of $\bar{M}^n$, we can extend $\{E_i\}_{i=1}^p$, ξ and v to vector fields in $\bar{M}^n$ which we also denote by the same letters.

First, by a straightforward computation, one gets

$$\tilde{H}\ (m) = (H' + \rho v - \xi)(m).$$

At each "time" t $\in (-\varepsilon, \varepsilon)$ we have on M^* a homotetic copy of M^p, M^p_t, which is located on a certain sphere $S^{n+1}_t(r)$ of radius $r(t)$ depending on t. Then we have

$$\overline{\nabla}_{E_i} E_i(m,t) = \nabla^t_{E_i} E_i(m,t) + \sigma^t(E_i, E_i) - \frac{1}{r}\xi, (i = 1, \ldots, p),$$

where ∇^t and σ^t are the Levi–Civita connection and the second fundamental form of M_t^p in E^{n+2}, respectively. Now since M^p and M_t^p are homotetic and $\nabla_{E_i} E_i(m) = 0$, we obtain

$$\overline{\nabla}_{E_i} E_i(m,t) = \frac{1}{r(t)}[\sigma(E_i, E_i)(m) - \xi(m)].$$

Observe that $\{E_1, \ldots, E_p, v\}$ is a local tangent frame to M^*. Furthermore, as v is the unit tangent field along the rays $\partial/\partial t$ we have $\overline{\nabla}_v v(m,t) = 0$ and then

$$\bar{H}\ (m,t) = \frac{1}{p+1}\{\sum_{i=1}^{p} \overline{\nabla}_{E_i} E_i(m,t)\}^N = \frac{p}{p+1}\frac{1}{r}\{H'(m) - \xi(m)\},$$

where N means normal component. Thus if we write $f(t) = \dfrac{1}{r(t)}$ we get

$$\bar{H}\ (m,t) = \frac{p}{p+1} f(t)\{H'(m) - \xi(m)\}.$$

Before going any further, we would like to derive some easy but quite interesting consequences from the formula (3). First we want to point out that even M^p is minimal in either $\bar{M}^n$ or S^{n+1}, M^* is never minimal. This fact follows directly from (3) and makes the difference with that situation given for the punctured cones shaped on M^p (see [8]).

Let us denote by η a unit vector field in the direction of $\bar{H}$, so that $\bar{H}\ = \overline{\alpha}\eta$, $\overline{\alpha}$ being the mean curvature of M^* in E^{n+2}. Then we have:

Proposition 2.1 *With the above notations, M^p is a minimal submanifold of $\bar{M}^n$ if and only if*

$$\overline{\alpha}^2(m,t) = (\frac{p}{p+1})^2 f^2(t).$$

Proof. An easy computation from (3) yields

$$\overline{\alpha}^2(m,t) = (\frac{p}{p+1})^2 f^2(t)\{\alpha'^2(m) + 1\},$$

$\alpha'^2 =< H', H' >$ being the mean curvature of M^p in $\bar{M}^n$.

In other words, if M^p is minimal in $\bar{M}^n$ then the mean curvature of M^* at a point depends exclusively on the height from such point to M^p.

Similarly one gets:

Proposition 2.2 *M^* is a generalized cylinder shaped on M^p if and only if the mean curvature $\overline{\alpha}$ of M^* is a constant along the rays of M^*. Moreover, in such case, M^p is minimal in S^{n+1} if and only if $\overline{\alpha}^2 = (\frac{p}{p+1})^2$.*

Proof. We only need to prove the sufficient condition. Suppose $\overline{\alpha}^2$ is a constant $\mu(m)$ along the rays. Then from (4)

$$\mu(m) = (\frac{p}{p+1})^2 f^2(t)\{\alpha'^2(m) + 1\}$$

holds along the rays, for each fixed point $m \in M^p$. Therefore f does not depend on t. By recalling the construction of M^* this means that $r(t) = 1$, for all t, and then M^* is isometric to $M^p \times \mathbb{R}$, a generalized cylinder over M^p.

Second part follows from Proposition 2.1.

Remark. It can be compared Proposition 2.2 and Theorem 2.15 in [1], where it has been shown an analogous result for another kind of ruled submanifolds of E^{n+2}.

At this point we go back to our central computations. By using arguments of Elementary Geometry one can find

$$f(t) = \frac{1}{1-\gamma t},$$

where γ is a constant given by $\gamma = \tan\theta$, θ being the angle between v and e, here e means the unit normal vector to the plane P which determines $\bar{M}^n$. Note that $\bar{M}^n$ has to be totally umbilical to make sure that v and e make a constant angle on $\bar{M}^n$. Moreover, $f(t)$ is always positive since $r(t)$ is the radius of a sphere.

In order to know the behaviour of the higher order Laplacians of the mean curvature vector we are going to give a general formulation of the Laplacian of certain vector fields defined on M^*. Let g be a $\mathcal{C}^\infty$ real function on the open interval $(-\varepsilon, \varepsilon)$ and X a $\mathcal{C}^\infty$ vector field on M^p, not necessarily tangent. The parallel transport of X along the rays on M^* will also be denoted by X. Then $gX(m,t) = g(t)X(m)$ is a $\mathcal{C}^\infty$ vector field on M^*. Now let us consider a $\mathcal{C}^\infty$ function $F(m,t)$ on M^* and define for each $t \in (-\varepsilon, \varepsilon)$ the following associated $\mathcal{C}^\infty$ function on M^p :

$$F_t : M^p \longrightarrow \mathbb{R};\ F_t(m) = F(m,t).$$

For a tangent vector field E_i, i=1, ..., p, choose a curve $\alpha_i : J_i \longrightarrow M^p$ starting at m in the direction of $E_i(m)$, i.e., $\alpha_i(0) = m$ and $\alpha_i^{'}(0) = E_i(m)$. Then the curve $\beta_i : J_i \longrightarrow M^*$ given by $\beta_i(s) = \alpha_i(s/k) + tv(s/k)$, where $k = 1 - \rho t$ (recall that $A_v = \rho I$), takes initial conditions $\beta_i(0) = m + tv$ and $\beta_i^{'}(0) = E_i(m,t)$. Since

$$(E_iF)(m,t) = \frac{d}{ds}|_{s=0}(F\circ\beta)(s) = \frac{d}{ds}|_{s=0}F_t(\alpha(s/k)) = \frac{1}{k}(dF_t)_m(E_i(m)),$$

we get

$$(E_iF)(m,t) = \frac{1}{k}(E_iF_t)(m).$$

Thus using (2) and (6) we have

$$k = r \quad \text{and} \quad \gamma = \rho = \tan\theta.$$

On the other hand if Z is a $\mathcal{C}^\infty$ vector field over an Euclidean p-dimensional submanifold N and Δ represents its Laplacian, we can write ΔZ in the following way (see [2], pg. 270):

$$\Delta Z(q) = -\{\sum_{i=1}^{p} \overline{\nabla}_{E_i}\overline{\nabla}_{E_i}Z - \overline{\nabla}_{\nabla_{E_i}E_i}Z\}(q),$$

where $\{E_i\}_{i=1}^p$ is a local tangent frame in a neighbourhood of $q \in N$, $\overline{\nabla}$ is the Euclidean connection and ∇ the induced connection on N.

Finally we have the following:

Lemma 2.3 *Write $\overline{\Delta}$ and Δ for the Laplacians of M^* and M^p, respectively, and let g and X be as before. Then we have*

$$\overline{\Delta}(gX)(m,t) = gf^2(t)\Delta X(m) + \{pf(t)\rho\frac{\partial g}{\partial t} - \frac{\partial^2 g}{\partial t^2}\}X(m).$$

Proof. Choose a local frame $\{E_i\}_{i=1}^p$ of tangent fields to M^p satisfying the equality $\nabla_{E_i} E_j(m) = 0$ at a point $m \in M^p$ and let $v = \partial/\partial t$ be the unit tangent field to M^* along the rays. Then if ∇^* is the Levi-Civita connection on M^* we have $\nabla^*_v v = 0$. From (8) we get

$$\overline{\Delta}(gX)(m,t) = -\{\sum_{i=1}^{p}(\overline{\nabla}_{E_i}\overline{\nabla}_{E_i}(gX) - \overline{\nabla}_{\nabla^*_{E_i}E_i}(gX))(m,t) + \overline{\nabla}_v\overline{\nabla}_v(gX)\}.$$

Now if we write $gX = (gX_j)_j$, j=1, …,n+2, then $gX_j \in \mathcal{C}^\infty(M^*)$, and we can use (6) to obtain

$$\overline{\nabla}_{E_i}(gX)(m,t) = (E_i(gX_j))_j(m,t) = (fg(E_iX_j(m)))_j$$

and

$$\overline{\nabla}_{E_i}\overline{\nabla}_{E_i}(gX)(m,t) = gf^2(t)((E_iE_iX_j)(m))_j = gf^2(t)\overline{\nabla}_{E_i}\overline{\nabla}_{E_i}X(m).$$

On the other hand from (2) one sees that $\nabla^*_{E_i}E_i(m,t) = f\rho v$, and then

$$\overline{\nabla}_{\nabla^*_{E_i}E_i}(gX)(m,t) = \rho f(t)\frac{\partial g}{\partial t}X(m).$$

One also has

$$\overline{\nabla}_v\overline{\nabla}_v(gX)(m,t) = \frac{\partial^2 g}{\partial t^2}X(m),$$

so that putting together (9) through (12) we get the lemma.

3. Higher Order Laplacians of the Mean Curvature Field.

In order to get a formula for the Laplacian of $\bar{H}$, an easy computation from (8) yields

$$\Delta\xi = -\sum_{i=1}^{p}\overline{\nabla}_{E_i}E_i = -p\tilde{H} \; = -p\{H' + \rho v - \xi\},$$

having chosen, as usually, a local frame at m, $\{E_i\}_{i=1}^p$, satisfying $\nabla_{E_i}E_j(m) = 0$. Then, from (3) and Lemma 2.3 , one gets

$$\overline{\Delta}\bar{H}\;(m,t) = \frac{p}{p+1}f^3(t)\{\Delta H' + [p + \rho^2(p-2)](H' - \xi) + p\rho v\}(m).$$

Let us seek for a general expression of the higher order Laplacians of $\bar{H}$. To this end, we define the functions

$$F_{i,j}(t) = \phi_{i,j}f^{2i+1}(t)$$

and

$$G_i(t) = a_i f^{2i+1}(t),$$

where the constans $\phi_{i,j}$ and a_i are defined by

$$\phi_{i,i} = 1,\; i = 0, 1, \ldots$$

$$\phi_{i,j} = \phi_{i-1,j-1} + \rho^2(p-2i)(2i-1)\phi_{i-1,j},\; 1 \leqslant j < i,\; i = 2, 3, \ldots$$

$$\begin{cases} a_0 &= 0 \\ a_i &= p\rho \sum_{j=1}^{i} (p\rho^2)^{j-1} \phi_{i,j} \quad i = 1, 2, \ldots \end{cases}$$

$$\phi_{i,0} = [p + \rho^2 (p - 2i)(2i - 1)]\phi_{i-1,0} + p\rho a_{i-1}, \; i = 1, 2, \ldots$$

On the other hand, from (8) one has

$$\Delta v = p\rho\{H' + \rho v - \xi\}.$$

Then we obtain the following:

Proposition 3.1 *With the above notation, we have for any* $k \in \mathbf{N}$:

$$\overline{\Delta}^k \bar{H}\ (m,t) = \frac{p}{p+1}\{F_{k,k}\Delta^k H' + F_{k,k-1}\Delta^{k-1} H' + \ldots + F_{k,1}\Delta H' + F_{k,0}(H' - \xi) + G_k v\}.$$

Proof. It is a simple induction following Lemma 2.3 and the above set of formulas (13) through (19).

4. Statement and Proof of the Main Result.

At this point it may be worth recalling the situation we are. We have a compact p-dimensional submanifold M^p of a totally umbilical hypersurface $\bar{M}^n$ of the unit sphere S^{n+1} centered at the origin of E^{n+2}. By using the umbilicity of $\bar{M}^n$ we have constructed in $\mathbb{S}^1$ a ruled (p+1)-dimensional submanifold of E^{n+2} which we have denoted by M^*. Now we are ready to give a solution to the problem stated in $\mathbb{S}^0$ for such a class of ruled manifolds.

Theorem 4.1 *The ruled submanifold M^* is of finite type k if and only if it is a generalized cylinder of null k-type, $M^* = M^p \times \mathbb{R}$, constructed on a $(k-1)$ -type spherical submanifold M^p.*

Proof. As one said in $\mathbb{S}^1$, if M^* is of k -type, then its mean curvature vector satisfies

$$\overline{\Delta}^k \bar{H}\ + d_1 \overline{\Delta}^{(k-1)} \bar{H}\ + \cdots + d_{k-1}\overline{\Delta}\bar{H}\ + d_k \bar{H}\ = 0,$$

where d_i , i=1, ..., k , are real numbers (d_k being nonzero because it is the product of the nonzero eigenvalues used to build the immersion of M^* in E^{n+2}) and $\overline{\Delta}$ is the Laplacian of M^*.

Assume $\rho \neq 0$, i.e., $\bar{M}^n$ is not totally geodesic in S^{n+1}. Hence by using Proposition 3.1 and (20) we have

$$\begin{aligned} F_{k,k}\Delta^k H' + \{F_{k,k-1} + d_1 F_{k-1,k-1}\}\Delta^{k-1} H' + \cdots + \\ +\{F_{k,1} + d_1 F_{k-1,1} + \cdots + d_{k-1}F_{1,1}\}\Delta H' + \\ +\{F_{k,0} + d_1 F_{k-1,0} + \cdots + d_{k-1}F_{1,0} + d_k F_{0,0}\}(H' - \xi) + \\ +\{G_k + d_1 G_{k-1} + \cdots + d_{k-1}G_1\}v = 0, \end{aligned} \tag{21}$$

where, as always, Δ is the Laplacian of M^p and H' the mean curvature vector of M^p in $\bar{M}^n$. From (5) and (7) one gets $r(t) = 1 - \rho t$. Now multiplying (21) by r^{2k+1} and using $F_{h,j}(t) = \phi_{h,j} f^{2h+1}(t)$ we obtain

$$\begin{aligned} \phi_{k,k}\Delta^k H' + \{\phi_{k,k-1} + d_1\phi_{k-1,k-1} r^2\}\Delta^{k-1} H' + \cdots + \\ +\{\phi_{k,1} + d_1\phi_{k-1,1} r^2 + \cdots + d_{k-1}\phi_{1,1} r^{2(k-1)}\}\Delta H' + \\ +\{\phi_{k,0} + d_1\phi_{k-1,0} r^2 + \cdots + d_{k-1}\phi_{1,0} r^{2(k-1)} + d_k\phi_{0,0} r^{2k}\}(H' - \xi) + \\ \{a_k + d_1 a_{k-1} r^2 + \cdots + d_k a_1 r^{2(k-1)}\}v = 0. \end{aligned} \tag{22}$$

Therefore the coefficients of $\Delta^i H'$, $(i = 1, \ldots, k)$, $(H' - \xi)$ and v are polynomials of different degrees. Then for any fixed point $m \in M^p$, as $d_k \neq 0$, we will have $H' - \xi = 0$, which is a contradiction.

Hence $\rho = 0$. That means $f \equiv 1$ and then $\bar{M}^n$ is a totally geodesic hypersurface of S^{n+1}. Thus M^* is isometric to the product manifold $M^p \times \mathbb{R}$, i.e., M^* is a generalized cylinder. By the relationship between the second fundamental form of $M^p \times \mathbb{R}$ and that of M^p we can assert that if $\bar{H}$ satisfies (20), then $\tilde{H}$ satisfies also the same equation, where $\tilde{H}$ denotes the mean curvature vector of M^p in E^{n+2}. As M^p is compact, we have that M^p is of l -type, with $l \leqslant k$. Since M^* is a cylinder, $l = k - 1$. The converse is trivial.

Remark. In the above proof we can also deduce M^p is of finite type by means of formula (22) . Indeed if we use $\rho = 0$ in the equations (16) through (19) we obtain

$$F_{h,j} = \phi_{h,j} \text{ and } G_h = 0 \text{ for all h and j.} \tag{23}$$

Thus taking (23) in (22) and recalling (1), we finally get

$$\Delta^k \tilde{H} + d_1 \Delta^{k-1} \tilde{H} + \cdots + d_{k-1} \Delta \tilde{H} + d_k \tilde{H} = 0.$$

Since M^p is compact, then from (24) M^p is of finite type.

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