On a certain class of conformally flat Euclidean hypersurfaces

Angel Ferrández, Oscar J.Garay and Pascual Lucas
Lecture Notes in Math., n. 1481, (1991), 48–54

(Partially supported by DGICYT grant PS87-0115)

1. Introduction

Let $x : M^p \longrightarrow \mathbb{R}^{n+1}$ be an isometric immersion of a manifold M^p into the Euclidean space $\mathbb{R}^{n+1}$ and Δ its Laplacian. The family of such immersions satisfying the condition $\Delta x = \lambda x,\ \lambda \in \mathbb{R}$, is characterized by a well-known result of Takahashi, [10]: they are either minimal in $\mathbb{R}^{n+1}$ or minimal in some Euclidean hypersphere. These submanifolds obviously satisfy the condition $\Delta H = \lambda H,\ H$ being the mean curvature vector field in $\mathbb{R}^{n+1}$. Let us write $\mathcal{C}_\lambda$ as the family of those submanifolds satisfying $\Delta H = \lambda H$. Then $\mathcal{C}_\lambda$ contains the Takahashi's family as a proper subfamily as the cylinder $S^p \times \mathbb{R}^q$ shows. When M^p is compact, both families are the same. Therefore, it is interesting to ask for the following geometric question:

> *"Are there any other submanifolds in $\mathcal{C}_\lambda$ apart from cylinders and Takahashi's family?"*

In this context it is worth exploring the existence of non-minimal Euclidean submanifolds whose mean curvature vector be harmonic, i.e., $\Delta H = 0$. As a first stage, only a special case of Euclidean hypersurfaces will be involved in our study. More concretely, we analize the conformally flat hypersurfaces of $\mathbb{R}^{n+1}$ in $\mathcal{C}_\lambda$. Actually, we show that this class of hypersurfaces is rather small. Essentially they are either minimal, or hyperspheres or right circular cylinders (see Theorem 3.2). Although there have been several attempts of clasifying conformally flat hypersurfaces of $\mathbb{R}^{n+1}$, [7], [8], these results are not complete as Cecil and Ryan show in [3], where they also obtained a classification of the conformally flat hypersurfaces tauthy embedded in $\mathbb{R}^{n+1}$, $n > 3$.
In proving our result, we have used a method developed by B-Y Chen in [6]. As a result, a significant fact is crucial in our computations: A conformally flat hypersurface $M^n,\ n > 3$, is characterized by having two principal curvatures (not necessarily distinct) of multiplicities 1 and $n-1$ respectively. Thus even in the case of hypersurfaces with two principal curvatures of arbitrary multiplicities the method does not work. However the procedure also holds good in the case of Euclidean surfaces. This fact allow us to classify the Euclidean surfaces in $\mathcal{C}_\lambda$ (Proposition 3.1). This result was also implicitly contained in the proof of main theorem of [6].

2. Basic Lemmas

Let M^n be an orientable hypersurface in the Euclidean space $\mathbb{R}^{n+1}$. Let us denote by σ, A, H, ∇ and D the second fundamental form, the Weingarten endomorphism, the mean curvature vector, the Riemannian connection of M^n and the normal connection of M^n in $\mathbb{R}^{n+1}$. Then following [4, p. 271] we have

Lemma 2.1 *If M^n is a hypersurface in $\mathrm{I\!R}^{\mathrm{n+1}}$ then*

$$\Delta H = \Delta^D H + |\sigma|^2 H + Tr(\bar{\nabla} A_H),$$

Δ being the Laplacian of M^n acting on (n+1)-valued functions, $\Delta^D H$ the Laplacian in the normal bundle and $Tr(\bar{\nabla} A_H)$ is the trace of $\bar{\nabla} A_H = \nabla A_H + A_{DH}$.

Now next lemma tell us how to compute $Tr(\bar{\nabla} A_H)$ in a more familiar form.

Lemma 2.2 *Suppose M^n is an orientable hypersurface of $\mathrm{I\!R}^{\mathrm{n+1}}$ and ξ a global unit normal vector field. Let α be the mean curvature with respect to ξ, that is $H = \alpha\xi$. Then*

$$Tr(\bar{\nabla} A_H) = \frac{n}{2}\nabla\alpha^2 + 2A(\nabla\alpha),$$

$\nabla\alpha$ being the gradient of α.

Proof. Choose an orthonormal local frame $\{E_i\}_{i=1}^{n+1}$ in such a way that $\{E_i\}_{i=1}^{n}$ are tangent vector fields to M^n and $E_{n+1} = \xi$. Moreover we assume that $\{E_i\}_{i=1}^{n}$ are eigenvectors of $A_\xi = A$ corresponding to the eigenvalues μ_i, $AE_i = \mu_i E_i,\ i = 1, \ldots, n$. Denote by $\{\omega^1, \ldots, \omega^{n+1}\}$ and $\{\omega_i^j\},\ i, j = 1, \ldots, n+1$, the dual frame and connection forms associated to $\{E_i\}_{i=1}^{n+1}$, respectively. Then, using the connection equations:

$$\nabla_{E_i} E_j = \sum_{k=1}^{n} \omega_j^k(E_i) E_k,$$

we obtain

$$(\nabla_{E_i} A_H) E_j = E_i(\alpha)\mu_j E_j + \alpha E_i(\mu_j) E_j + \alpha \sum_k (\mu_j - \mu_k)\omega_j^k(E_i) E_k.$$

But then by Codazzi's equation

$$0 = \alpha E_i(\mu_j) E_j - \alpha E_j(\mu_i) E_i + \alpha \sum_k \{(\mu_j - \mu_k)\omega_j^k(E_i) - (\mu_i - \mu_k)\omega_i^k(E_j)\} E_k.$$

Therefore

$$\alpha E_j(\mu_i) = \alpha(\mu_j - \mu_i)\omega_i^j(E_i).$$

Consequently

$$Tr(\bar{\nabla} A_H) = \frac{n}{2}\nabla\alpha^2 + 2\, Tr A_{DH},$$

and since $Tr A_{DH} = \sum_i A_{D_{E_i} H} E_i = A(\nabla\alpha)$ the lemma follows.

Remark 2.3 We denote by $\mathcal{U} = \{p \in M^n : \nabla\alpha^2(p) \neq 0\}$. $\mathcal{U}$ is an open set in M^n and as a consequence of Lemma 2.2, $Tr\bar{\nabla} A_H = 0$ if and only if $A(\nabla\alpha^2) = -\dfrac{n}{2}\alpha\nabla\alpha^2$ on $\mathcal{U}$.

Before ending this section we would like to give a first application of Lemma 2.2.

Proposition 2.4 *Let M^n be a compact hypersurface immersed in $\mathrm{I\!R}^{\mathrm{n+1}}$. Then M^n has constant mean curvature α if and only if $\Delta H = |\sigma|^2 H$, $|\sigma|^2$ being the length of the second fundamental form.*

Proof. Suppose M^n has constant mean curvature α. Then using Lemma 2.2 we have $Tr(\bar{\nabla} A_H) = 0$. Now take a local orthonormal frame $\{E_i\}_{i=1}^n$ in a neighborhood of a given point p. We can choose the fields $\{E_i\}_{i=1}^n$ in such a way that $\nabla_{E_i} E_j(p) = 0$. Therefore

$$\Delta^D H(p) = -\left(\sum_{i=1}^n D_{E_i} D_{E_i} H - D_{\nabla_{E_i} E_i} H\right)(p) = -\sum_{i=1}^n D_{E_i} D_{E_i} H(p) = \Delta(\alpha)\xi(p).$$

Thus $\Delta^D H = \Delta(\alpha)\xi$ on the whole M^n. Since α is constant, $\Delta^D H = 0$. Consequently we get from Lemma 2.1 $\Delta H = |\sigma|^2 H$.
Conversely, suppose $\Delta H = |\sigma|^2 H$. Then from Lemma 2.1, $\Delta^D H + Tr(\bar{\nabla} A_H) = 0$. Hence if we write normal and tangent components, we have $\Delta^D H = Tr(\bar{\nabla} A_H) = 0$. As before, $\Delta^D H = \Delta(\alpha)\xi$, then $\Delta(\alpha) = 0$. Since M^n is compact, α is constant.

Remark 2.5 Let us denote by τ the scalar curvature of M^n. From the above proposition and Takahashi's theorem, it is easy to prove that the only compact immersed hypersurfaces of $\mathbb{R}^{n+1}$ having constant two of the three following quantities α, $|\sigma|^2$ and τ, are hyperspheres. See also [5].

3. Main Result

Our goal is to prove the following theorem.

Theorem 3.1 *Let M^n be a conformally flat orientable hypersurface of $\mathbb{R}^{n+1}$, $n > 3$. If M^n is in the family $\mathcal{C}_\lambda$ (i.e. $\Delta H = \lambda H$), for a constant λ, then it is either minimal or isoparametric.*

Proof. Suppose M^n is conformally flat in $\mathbb{R}^{n+1}$, $n > 3$. If M^n is totally umbilical, then M^n is a piece of $\mathbb{R}^n$ or S^n. Otherwise, from Theorem 3 of [8] the Weingarten map of M^n has two distinct eigenvalues of multiplicities 1 and $n-1$, respectively. Our next step is to prove that M^n has constant mean curvature. If α were not constant, then by the Remark 2.1 $\mathcal{U}$ is not empty and the vector $\nabla\alpha^2$ is an eigenvector of A corresponding to the eigenvalue $-\dfrac{n}{2}\alpha$. Choose a local frame $\{E_i\}_{i=1}^{n+1}$, $E_{n+1} = \xi$, in an open set of $\mathcal{U}$ satisfaying that $\{E_i\}_{i=1}^n$ are eigenvectors of A and E_1 is parallel to $\nabla\alpha^2$. Thus one of the eigenvalues is $-\dfrac{n}{2}\alpha$. We have then two possible cases:
a) $-\dfrac{n}{2}\alpha$ has multiplicity 1, and therefore the other eigenvalue is $\dfrac{3}{2}\dfrac{n}{n-1}\alpha$ with multiplicity $n-1$.
b) $-\dfrac{n}{2}\alpha$ has multiplicity $n-1$ and the other eigenvalue is $\dfrac{n(n+1)}{2}\alpha$ with multiplicity 1.
Either choice of the multiplicity of $-\dfrac{n}{2}\alpha$ will lead to the same conclusion, so there is no loss of generality in assuming we are in the first case.
Now by hypothesis $\Delta H = \lambda H$ so that from Lemmas 2.1 and 2.2 we have

$$\Delta^D H = (\lambda - |\sigma|^2)H; \qquad A(\nabla\alpha) + \frac{n}{2}\alpha\nabla\alpha = 0.$$

Let $\{\omega^1, \ldots, \omega^{n+1}\}$ and $\{\omega_i^j\}_{i,j=1,\ldots,n+1}$ the dual frame and the connection forms of the choosen frame. Then we have

$$\omega_{n+1}^1 = \frac{n}{2}\alpha\omega^1; \qquad \omega_{n+1}^j = -\frac{3}{2}\frac{n}{n-1}\alpha\omega^j, j = 2, \ldots, n.$$

$$d\alpha = E_1(\alpha)\omega^1.$$

From the first equation of (2) we have

$$d\omega^1_{n+1} = \frac{n}{2}\alpha d\omega^1.$$

Using now the second equation of (2) and the structure equations, one has

$$d\omega^1_{n+1} = -\frac{3}{2}\frac{n}{n-1}\alpha d\omega^1.$$

These two last equations mean that

$$d\omega^1 = 0.$$

Therefore one locally has $\omega^1 = du$, for a certain function u, which along with (3) imply that $d\alpha \wedge du = 0$. Thus α depends on u, $\alpha = \alpha(u)$. Then $d\alpha = \alpha' du = \alpha'(u)\omega^1$ and so $E_1(\alpha) = \alpha'$. Taking differentiation in the second equation of (2) we have

$$d\omega^j_{n+1} = -\frac{3}{2}\frac{n}{n-1}\alpha'\omega^1 \wedge \omega^j - \frac{3}{2}\frac{n}{n-1}\alpha d\omega^j,$$

and, also by the structure equations:

$$d\omega^j_{n+1} = -\frac{3}{2}\frac{n}{n-1}\alpha d\omega^j - \frac{n(n+2)}{2(n-1)}\alpha\omega^j_1 \wedge \omega^1.$$

Consequently

$$\omega^1_j = \frac{3}{n+2}\frac{\alpha'}{\alpha}\omega^j,\ j = 2, \dots, n,$$

that is

$$(n+2)\alpha\omega^1_j = 3\alpha'\omega^j,\ j = 2, \dots, n.$$

Differentiating (10) and using (2) and (9) we have

$$d(\alpha\omega^1_j) = \frac{3}{n+2}\frac{(\alpha')^2}{\alpha}\omega^1 \wedge \omega^j + \alpha d\omega^1_j,$$

$$d\omega^1_j = -\frac{3}{4}\frac{n^2}{n-1}\alpha^2\omega^1 \wedge \omega^j + \frac{3}{n+2}\frac{\alpha'}{\alpha}(d\omega^j + \frac{3}{n+2}\frac{\alpha'}{\alpha}\omega^1 \wedge \omega^j).$$

On the other hand

$$d(\alpha'\omega^j) = \alpha''\omega^1 \wedge \omega^j + \alpha' d\omega^j.$$

Hence from (10) to (13) we obtain

$$4\alpha\alpha'' - \frac{4(n+5)}{n+2}(\alpha')^2 + \frac{n^2(n+2)}{n-1}\alpha^4 = 0.$$

Putting $y = (\alpha')^2$ the above equation turns into

$$2\alpha y' - \frac{4(n+5)}{n+2}y = -\frac{n^2(n+2)}{n-1}\alpha^4,$$

and then

$$y=(\alpha')^2=C\alpha^{\frac{2(n+5)}{n+2}}-\left(\frac{n(n+2)}{2(n-1)}\right)^2\alpha^4,$$

with C a constant.
Now we use the definition of $\Delta\alpha$, the fact that E_1 is parallel to $\nabla\alpha^2$ and equation (9) to obtain

$$(n+2)\alpha\Delta\alpha=-(n+2)\alpha\alpha''+3(n-1)(\alpha')^2.$$

As we know $\Delta^D H=(\Delta\alpha)\xi$, hence from (1) we get

$$\alpha\Delta\alpha=(\lambda-|\sigma|^2)\alpha^2.$$

Since $|\sigma|^2=\dfrac{n^2(n+8)}{4(n-1)}\alpha^2$, combining (17) and (18), we have

$$\alpha\alpha''-\frac{3(n-1)}{(n+2)}(\alpha')^2+\left(\lambda-\frac{n^2(n+8)}{4(n-1)}\alpha^2\right)\alpha^2=0.$$

Thus, putting together (14) and (19) one has

$$\frac{2(4-n)}{n+2}(\alpha')^2=\frac{n^2(n+5)}{2(n-1)}\alpha^4-\lambda\alpha^2.$$

We deduce, using (16) and (20) that α is locally constant on $\mathcal{U}$, which is a contradiction with the definition of $\mathcal{U}$. Hence α is constant on M^n. Taking again (1) into consideration, we have $(\Delta\alpha)\xi=(\lambda-|\sigma|^2)H$, so that either $\alpha=0$ and M^n is minimal or $|\sigma|^2=\lambda$ and therefore $|\sigma|^2$ is constant. But we had at most two different eigenvalues, then because α and $|\sigma|^2$ are constant, such eigenvalues are also constant. We have therefore that M^n is in fact isoparametric.
A classical result of B. Segre [9] states that the isoparametric hypersurfaces in $\mathbb{R}^{n+1}$ are $\mathbb{R}^n$, $S^n(r)$ and $S^p(r)\times\mathbb{R}^{n-p}$, where $S^p(r)$ is the p-sphere of radius r in the Euclidean subspace $\mathbb{R}^{p+1}$ perpendicular to $\mathbb{R}^{n-p}$. On the other hand, if M^n is minimal and conformally flat with $n\geqslant 4$, a result of Blair, [1], states that $x(M)$ is contained in a catenoid, see also [2]. Taking into account these results and Theorem 3.1 one has the following.

Theorem 3.2 *Let M^n be a complete conformally flat orientable hypersurface of $\mathbb{R}^{n+1}$, $n>3$. Then M^n is in the family $\mathcal{C}_\lambda$ if and only if it is one of the following hypersurfaces:*
1) a hyperplane $\mathbb{R}^n$,
2) a catenoid,
3) a round sphere $S^n(r)$,
4) a cylinder over a circle $\mathbb{R}^{n-1}\times S^1(r)$,
5) a cylinder over a round $(n-1)$-sphere $\mathbb{R}\times S^{n-1}(r)$.

Prof. Chen kindly pointed out to us that this result can also be considered under the viewpoint of the finite type theory (see [4]). In fact, it can be shown that a Euclidean immersion satisfying $\Delta H=\lambda H$ is either minimal or of infinite type if $\lambda=0$, and either of 1-type or of null 2-type if $\lambda\neq 0$.
One should observe that conditions $n>3$ and conformally flat have been used in order to guarantee the existence of at most two distinct eigenvalues of multiplicities 1 and $n-1$. This is automatically satisfied by a surface in $\mathbb{R}^3$. It means that the above computations are also correct in the case of surfaces of $\mathbb{R}^3$ (see [6]). Then one obtains

Proposition 3.3 *Let M^2 be a surface of $\mathbb{R}^3$ in $\mathcal{C}_\lambda$. Then either M^2 is minimal or it is a piece of one of the following surfaces: a 2-sphere $S^2(r)$ or a right circular cylinder $S^1(r) \times \mathbb{R}$.*

Remark 3.4 From this result we see that the only surfaces of $\mathbb{R}^3$ satisfying $\Delta H = 0$ are the minimal ones.

Bibliography

[1] D. BLAIR, *A generalization of the catenoid.* Canadian J. of Math., **27** (1975), 231-236.

[2] M. do CARMO and M. DAJCZER, *Rotation hypersurfaces in spaces of constant curvature.* Transactions of the A.M.S., **277** (1983), 685-709.

[3] T.E. CECIL and P.J. RYAN, *Conformal geometry and the cyclides of Dupin.* Canadian J. of Math., **32** (1980), 767-782.

[4] B.-Y. CHEN, *Total mean curvature and submanifolds of Finite Type.* World Scientific. Singapore and New Jersey, 1984.

[5] B.-Y. CHEN, *Finite type submanifolds and generalizations.* Instituto “Guido Castelnuovo”. Roma, 1985.

[6] B.-Y. CHEN, *Null 2-type surfaces in $\mathbb{R}^3$ are circular cylinders.* Kodai Math. J., **11** (1988), 295-299.

[7] R.S. KULKARNI, *Conformally flat manifolds.* Proc. Nat. Acad. Sci. U.S.A., **69** (1972), 2675-2676.

[8] S. NISHIKAWA and Y. MAEDA, *Conformally flat hypersurfaces in a conformally flat manifold.* Tohoku Math. J., **26** (1974), 159-168.

[9] B. SEGRE, *Famiglie di ipersuperficie isoparametrische negli spazi euclidei ad un qualunque numero di dimensioni.* Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., **27** (1938), 203-207.

[10] T. TAKAHASHI, *Minimal immersions of Riemannian manifolds.* J. Math. Soc. Japan, **18** (1966), 380-385.