# Submanifolds in pseudo-Euclidean spaces satisfying the condition $\Delta x=A x+B$ 

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#### Abstract

In this paper we study pseudo-Riemannian submanifolds in $\mathbb{R}_{t}^{n+k}$ satisfying the condition $\Delta x=A x+B$, where $A$ is an endomorphism of $\mathbb{R}_{t}^{n+k}$ and $B$ is a constant vector in $\mathbb{R}_{t}^{n+k}$. We give a characterization theorem when $A$ is a self-adjoint endomorphism. As for hypersurfaces we are able to obtain a classification theorem for any endomorphism $A$.


## 0. Introduction

An old and celebrated result due to T. Takahashi, [13], gives a characterization of the minimal submanifolds in an ordinary sphere $\mathbb{S}^{m}(r)$ by means of a nonzero eigenvalue of the Laplacian, getting also an explicit expression of that eigenvalue in terms of the dimension $n$ of the submanifold and the radius $r$ of the sphere. This can be seen in some sense as a starting point of an ambitious programme drawn up by B.Y. Chen (see, for example, [2] and [4]) directed to classify submanifolds by the spectrum of its Laplacian.

As for Euclidean hypersurfaces an extension of Takahashi's Theorem has been obtained by O.J. Garay, [7], where he studies the hypersurfaces whose coordinate functions are eigenfunctions of their Laplacian, with not necessarily the same eigenvalue. By considering the hypersurface as the graph of a differentiable function, he has got that the family of those Euclidean hypersurfaces is restricted to open pieces of minimal hypersurfaces, ordinary hyperspheres or generalized circular cylinders. Recently, T. Hasanis and T. Vlachos, [8], generalize this study by considering submanifolds of arbitrary codimension.

A little later, Dillen, Pas and Verstraelen, in [6], pointed out that in order to get Garay's condition to be coordinate invariant, it must be slightly modified as $\Delta x=A x+B$, for some $A \in \operatorname{End}\left(\mathbb{R}^{m}\right)$ and $B \in \mathbb{R}^{m}$. Getting back Garay's idea into this context and working on surfaces in $\mathbb{R}^{3}$, they find no other surfaces appart from those given from Garay's condition. The work of those authors has been generalized, independently and using different tecniques, to the n-dimensional case by Chen, Dillen, Verstraelen and Vrancken, [5], and Hasanis and Vlachos, [9].

The indefinite Riemannian case deserves a special care. First, the shape operator needs not be diagonalizable, condition which plays a chief role in the definite Riemannian case. On the other hand, all of results already obtained have been found, even implicitly, by achieving the isoparametricity of the hypersurface. It seems reasonable thinking of a richer classification can be hoped in working on pseudo-Riemannian submanifolds in an indefinite Euclidean space. The first attempt in this line has been made by the authors in [1], where they classify the surfaces in the 3-dimensional Lorentz-Minkowski space $\mathbb{L}^{3}$ satisfying the quoted condition. Now we are going to generalize that work not only by considering hypersurfaces but also taking them in any pseudo-Euclidean space.

The main result of this paper is a classification theorem for hypersurfaces $M_{s}^{n}$ in pseudoEuclidean space $\mathbb{R}_{t}^{n+1}$. Actually, it is shown that $M_{s}^{n}$ satisfies $\Delta x=A x+B$ if and only if it is an open piece of a minimal hypersurface, a totally umbilical hypersurface or a pseudo-Riemannian product of a totally umbilical and a totally geodesic submanifold.

## 1. Some results on submanifolds

Let $M_{s}^{n}$ be a pseudo-Riemannian submanifold of $\mathbb{R}_{t}^{n+k}$. In order to set up the notation to be used later on, we will denote by $\sigma, H, S_{H}, \nabla$ and $\bar{\nabla}$ the second fundamental form, the mean curvature vector field, the Weingarten map with respect to $H$, the Levi-Civita connection on $M_{s}^{n}$ and the usual flat connection on $\mathbb{R}_{t}^{n+k}$, respectively.

Let $x: M_{s}^{n} \longrightarrow \mathbb{R}_{t}^{n+k}$ be an isometric immersion satisfying the equation

$$
\Delta x=A x+B
$$

where $A$ is an endomorphism of $\mathbb{R}_{t}^{n+k}$ and $B$ is a constant vector in $\mathbb{R}_{t}^{n+k}$. By taking covariant derivative in (1), using the well-known equation $\Delta x=-n H$ and applying the Weingarten formula we get

$$
A X=n S_{H} X-n D_{X} H
$$

for any vector field $X$ tangent to $M_{s}^{n}$. Since $S_{H}$ is a self-adjoint endomorphism, from (2) we have

$$
<A X, Y>=<X, A Y>
$$

for any tangent vector fields $X$ and $Y$. By taking covariant derivative in last equation we obtain

$$
\begin{align*}
& <A \sigma(X, Z), Y>-<A \sigma(Y, Z), X>=  \tag{4}\\
& \quad<\sigma(X, Z), A Y>-<\sigma(Y, Z), A X>
\end{align*}
$$

We need, for later use, the following formula for $\Delta H$, which has been already obtained by B.Y. Chen in [3]:

$$
\begin{equation*}
\Delta H=\Delta^{D} H+\frac{n}{2} \nabla<H, H>+2 \operatorname{tr}\left(S_{D H}\right)+\sum_{r=n+1}^{n+k} \varepsilon_{r} \operatorname{tr}\left(S_{H} S_{r}\right) E_{r} \tag{5}
\end{equation*}
$$

where $\Delta^{D}$ is the Laplacian of the normal bundle, $\nabla$ also will denote the gradient on $M_{s}^{n}, \operatorname{tr}\left(S_{D} H\right)=$ $\sum_{i=1}^{n} \varepsilon_{i} S_{D_{E_{i}} H} E_{i}$ and $\left\{E_{1}, \ldots, E_{n}, E_{n+1}, \ldots, E_{n+k}\right\}$ is an adapted local orthonormal frame.

Let us start by assuming that A is a self-adjoint endomorphism of $\mathbb{R}_{t}^{n+k}$ and let $f: \mathbb{R}_{t}^{n+k} \longrightarrow$ $\mathbb{R}$ be the quadratic function defined by

$$
f(y)=<A y, y>+2<B, y>
$$

Since $X(f)=0$, for any vector field $X$ tangent to $M_{s}^{n}$, then f is constant on $M_{s}^{n}$, say $c$. Let us write $N=f^{-1}(c)$. Now from (1) and (6) we know that $\bar{\nabla} f=-2 n H$, so that if we suppose $M_{s}^{n}$ is a non-minimal submanifold in $\mathbb{R}_{t}^{n+k}$, it follows that $N$ is a hypersurface of $\mathbb{R}_{t}^{n+k}$. If $<H, H>=0$, then $M_{s}^{n}$ is a quasi-minimal submanifold (in the sense of R. Rosca [11]) in $\mathbb{R}_{t}^{n+k}$.

Finally we analize the case $<H, H>\neq 0$. Then $N$ is a pseudo-Riemannian hypersurface of $\mathbb{R}_{t}^{n+k}$ being $\bar{\nabla} f$ a normal vector field to $N$. Therefore, by using the formula

$$
H=H_{1}+\frac{1}{n} \sum_{i=1}^{n} \varepsilon_{i} \sigma_{1}\left(E_{i}, E_{i}\right)
$$

where $H_{1}$ is the mean curvature vector field of $M_{s}^{n}$ in $N$ and $\sigma_{1}$ is the second fundamental form of $N$ in $\mathbb{R}_{t}^{n+k}$, and the fact that $H$ is normal to $N$, we obtain $H_{1}=0$, i.e., $M_{s}^{n}$ is a minimal submanifold of $N$. Summarizing, we have got the following result.

Proposition 1.1 Let $x: M_{s}^{n} \longrightarrow \mathbb{R}_{t}^{n+k}$ be an isometric immersion. Then $\Delta x=A x+B$, where $A$ is a self-adjoint endomorphism of $\mathbb{R}_{t}^{n+k}$ and $B$ a constant vector, if and only if one of the following statements holds:

1) $M_{s}^{n}$ is a minimal submanifold of $\mathbb{R}_{t}^{n+k}$;
2) $M_{s}^{n}$ is a quasi-minimal submanifold (in the sense of $R$. Rosca) of $\mathbb{R}_{t}^{n+k}$ lying in a quadratic hypersurface given by $f(y)=<A y, y>+2<B, y>=c$, which also satisfies $\bar{\nabla} f=-2 n H$;
3) $M_{s}^{n}$ is a minimal submanifold of a pseudo-Riemannian quadratic hypersurface given by $f(y)=<A y, y>+2<B, y>=c$, which also satisfies $\bar{\nabla} f=-2 n H$.

As a first consequence, we get Theorem 1 given by B.Y. Chen in [3], where he gave a pseudoRiemannian version of Takahashi's Theorem.

Corollary 1.2 Let $x: M_{s}^{n} \longrightarrow \mathbb{R}_{t}^{n+k}$ be an isometric immersion. Then $\Delta x=\lambda x$, for a real constant $\lambda$, if and only if one of the following statements holds:

1) $\lambda=0$ and $M_{s}^{n}$ is a minimal submanifold of $\mathbb{R}_{t}^{n+k}$;
2) $\lambda>0$ and $M_{s}^{n}$ is a minimal submanifold of $\mathrm{S}_{t}^{n+k-1}(r)$ with $r=\sqrt{n / \lambda}$;
3) $\lambda<0$ and $M_{s}^{n}$ is a minimal submanifold of $\mathrm{H}_{t-1}^{n+k-1}(r)$ with $r=\sqrt{-n / \lambda}$.

Proof. By assuming that $\lambda \neq 0$, take $A=\lambda I$ and $B=0$. Then from (6) we obtain $M_{s}^{n}$ is a submanifold of a quadratic hypersurface with equation $f(y)=\lambda<y, y>=c$, for some real constant $c$. Now from the equation

$$
\Delta<x, x>=-2 n-2 n<H, x>
$$

we get $c=n$, so that $M_{s}^{n}$ cannot be a quasi-minimal submanifold of $\mathbb{R}_{t}^{n+k}$ and the proof finishes.
In the Riemannian case, the self-adjoint endomorphism $A$ can always be diagonalized. Then the Theorem 2.1 in [8] given by Hasanis and Vlachos can be easily obtained from the following.

Corollary 1.3 Let $x: M^{n} \longrightarrow \mathbb{R}^{n+k}$ be an isometric immersion. Then $\Delta x=A x+B$, with $A=\operatorname{diag}\left[\lambda_{1}, \ldots, \lambda_{n+k}\right]$ and $B=\left(b_{1}, \ldots, b_{n+k}\right) \in \mathbb{R}^{n+k}$, if and only if either $M$ is minimal in $\mathbb{R}^{n+k}$ or $M$ is minimal in a quadratic hypersurface given by $f\left(y_{1}, \ldots, y_{n+k}\right)=\sum_{i=1}^{n+k} \lambda_{i} y_{i}^{2}+$ $2 \sum_{i=1}^{n+k} b_{i} y_{i}=c$, which also satisfies $\bar{\nabla} f=-2 n H$.

## 2. The classification theorem for hypersurfaces

We begin this section by giving some examples of hypersurfaces $M_{s}^{n}$ in $\mathbb{R}_{t}^{n+1}$ which satisfy the condition $\Delta x=A x+B$. To do that, let $N, \alpha$ and $S$ be a unit normal vector field with
$<N, N\rangle=\varepsilon$, the mean curvature with respect to $N(H=\alpha N)$ and the shape operator of $M_{s}^{n}$, respectively. As usual, the metric tensor of $\mathbb{R}_{t}^{n+1}$ is given by

$$
d s^{2}=\sum_{i=1}^{n+1} \varepsilon_{i} d x^{i} \otimes d x^{i},
$$

where $t=\#\left\{i: \varepsilon_{i}=-1\right\}$.
Take $k \in\{1, \ldots, n-1\}$ and let $f: \mathbb{R}_{t}^{n+1} \longrightarrow \mathbb{R}$ be the function defined by

$$
f\left(x_{1}, \ldots, x_{n+1}\right)=\delta_{1} \sum_{i=1}^{k} \varepsilon_{i} x_{i}^{2}+\varepsilon_{k+1} x_{k+1}^{2}+\delta_{2} \sum_{j=k+2}^{n+1} \varepsilon_{j} x_{j}^{2},
$$

where $\delta_{1}$ and $\delta_{2}$ belong to the set $\{0,1\}$ and they do not vanish simultaneously. Taking $r>0$ and $\delta= \pm 1$, the set $M=f^{-1}\left(\delta r^{2}\right)$ is a hypersurface of $\mathbb{R}_{t}^{n+1}$ for appropriate choices of $k, \delta_{1}, \delta_{2}$ and $\delta$.

A straighforward computation shows that the unit normal vector field is given by

$$
N=(1 / r)\left(\delta_{1} x_{1}, \ldots, \delta_{1} x_{k}, x_{k+1}, \delta_{2} x_{k+2}, \ldots, \delta_{2} x_{n+1}\right)
$$

and the principal curvatures are $\mu_{1}=-\delta_{1} / r$ and $\mu_{2}=-\delta_{2} / r$ with multiplicities $k$ and $n-k$, respectively. Then the mean curvature is given by $\alpha=-\delta /(n r)\left(\delta_{1} k+\delta_{2}(n-k)\right)$ and by using the equation $\Delta x=-n H=-n \alpha N$ we have $\Delta x=A x$ where

$$
A=\frac{\delta}{r^{2}}\left(\delta_{1} k+\delta_{2}(n-k)\right)\left(\begin{array}{ccccccc}
\delta_{1} & & & & & & \mathbf{O} \\
& \ddots & & & & & \\
& & \delta_{1} & & & & \\
& & & 1 & & & \\
& & & & \delta_{2} & & \\
\mathbf{O} & & & & & \ddots & \\
& & & & & \delta_{2}
\end{array}\right) \text {, }
$$

appearing $k$ times $\delta_{1}$ and $n-k$ times $\delta_{2}$.
These examples are nothing but pseudo-Riemannian spheres $\mathrm{S}_{t}^{n}(r)$, pseudo-Riemannian hyperbolic spaces $\mathrm{H}_{t-1}^{n}(r)$ and pseudo-Riemannian products $\mathbb{R}_{s}^{k} \times \mathrm{S}_{t-s}^{n-k}(r)$ and $\mathbb{R}_{s}^{k} \times \mathrm{H}_{t-s-1}^{n-k}(r)$. It is easy to see that all of them have constant mean curvature, actually they are pseudo-Riemannian isoparametric hypersurfaces having at most one non-zero constant principal curvature. Nevertheless, it seems natural thinking of hypersurfaces in $\mathbb{R}_{t}^{n+1}$, appart from those ones, having non constant mean curvature satisfying the asked condition. The next theorem allows us to give a negative answer to that question.

Theorem 2.1 Let $x: M_{s}^{n} \longrightarrow \mathbb{R}_{t}^{n+1}$ be an isometric immersion satisfying the condition $\Delta x=$ $A x+B$. Then $M_{s}^{n}$ has constant mean curvature.

Proof. Take in $M_{s}^{n}$ the open set $\mathcal{U}=\left\{p \in M_{s}^{n}: \nabla \alpha^{2}(p) \neq 0\right\}$. Our goal is to show $\mathcal{U}$ is empty; otherwise, we get

$$
\sigma(X, Y)=\varepsilon \frac{<S X, Y>}{\alpha} H,
$$

for any tangent vector fields on $\mathcal{U}$.
Now by applying the Laplacian on both sides of (1) and taking into account (5) we have

$$
A H=2 S(\nabla \alpha)+n \varepsilon \alpha \nabla \alpha+\left\{\Delta \alpha+\varepsilon \alpha \operatorname{tr}\left(S^{2}\right)\right\} N
$$

Then from (7) and (8) we obtain

$$
<A \sigma(X, Y), Z>=\frac{<S X, Y>}{\alpha}(2 \varepsilon S Z(\alpha)+n \alpha Z(\alpha))
$$

that jointly with (2) and (4) gives

$$
T X(\alpha) S Y=T Y(\alpha) S X
$$

where $T$ is the self-adjoint operator given by $T X=n \alpha X+\varepsilon S X$.
Case 1: $T(\nabla \alpha) \neq 0$ on $\mathcal{U}$. Then there exists a tangent vector field $X$ such that $T X(\alpha) \neq 0$, which implies from (10) that $S$ has rank one on $\mathcal{U}$. Therefore we can choose a local orthonormal frame $\left\{E_{1}, \ldots, E_{n+1}\right\}$ with $S E_{1}=n \varepsilon \alpha E_{1}, S E_{i}=0, i=2, \ldots, n$ and $E_{n+1}=N$. From (10) we deduce $E_{1}$ is parallel to $\nabla \alpha$ and then we use (2) and (8) to get

$$
\begin{aligned}
A E_{1} & =\varepsilon n^{2} \alpha^{2} E_{1}-n E_{1}(\alpha) N \\
A E_{i} & =0, \quad i=2, \ldots, n \\
A N & =3 n \varepsilon \varepsilon_{1} E_{1}(\alpha) E_{1}+\left\{\frac{\Delta \alpha}{\alpha}+\varepsilon n^{2} \alpha^{2}\right\} N
\end{aligned}
$$

Thus the characteristic polynomial $P_{A}(t)$ of $A$ is given by

$$
\begin{aligned}
P_{A}(t)= & (-1)^{n-1} t^{n-1}\left\{t^{2}-\left(2 \varepsilon n^{2} \alpha^{2}+\frac{\Delta \alpha}{\alpha}\right) t\right. \\
& \left.+\varepsilon n^{2} \alpha \Delta \alpha+n^{4} \alpha^{4}+3 n^{2} \varepsilon \varepsilon_{1} E_{1}(\alpha)^{2}\right\}
\end{aligned}
$$

from which we can find two real constants $\lambda_{1}$ and $\lambda_{2}$ such that

$$
\begin{gather*}
\Delta \alpha=\lambda_{1} \alpha-2 \varepsilon n^{2} \alpha^{3}  \tag{11}\\
n^{2} \alpha \Delta \alpha=\lambda_{2} \varepsilon-\varepsilon n^{4} \alpha^{4}-3 n^{2} \varepsilon_{1} E_{1}(\alpha)^{2}
\end{gather*}
$$

Let $\left\{\omega^{1}, \ldots, \omega^{n+1}\right\}$ and $\left\{\omega_{i}^{j}\right\}_{i, j}$ be the dual frame and the connection forms, respectively, of the chosen frame. Then we can write

$$
\begin{align*}
\omega_{n+1}^{1} & =-n \varepsilon \alpha \omega^{1}  \tag{13}\\
\omega_{n+1}^{i} & =0, \quad i=2, \ldots, n  \tag{14}\\
d \alpha & =\varepsilon_{1} E_{1}(\alpha) \omega^{1} \tag{15}
\end{align*}
$$

Take exterior differentiation in (13) and use (15), then it can be deduced $d \omega^{1}=0$. Then we locally have $\omega^{1}=d u$, for some function $u$, and again from (15) we get $d \alpha \wedge d u=0$. Thus $\alpha$ depends on $u, \alpha=\alpha(u)$, and therefore $E_{1}(\alpha)=\varepsilon_{1} \alpha^{\prime}(u)$.

Taking again exterior differentiation in (14) and using $d \omega^{1}=0$ we have $\omega_{i}^{1}=0, i=1, \ldots, n$. Then we get

$$
\begin{equation*}
\Delta \alpha=-\varepsilon_{1} E_{1} E_{1}(\alpha)=-\varepsilon_{1} \alpha^{\prime \prime} \tag{16}
\end{equation*}
$$

and this equation allows us to rewrite equations (11) and (12) as

$$
\begin{gathered}
\alpha^{\prime \prime}=2 \varepsilon \varepsilon_{1} n^{2} \alpha^{3}-\lambda_{1} \varepsilon_{1} \alpha, \\
n^{2} \alpha \alpha^{\prime \prime}=\varepsilon \varepsilon_{1} n^{4} \alpha^{4}+3 n^{2}\left(\alpha^{\prime}\right)^{2}-\lambda_{2} \varepsilon \varepsilon_{1} .
\end{gathered}
$$

Let us take $\beta=\left(\alpha^{\prime}\right)^{2}$. Then $\frac{d \beta}{d \alpha}=2 \alpha^{\prime \prime}$ and from (17) we obtain

$$
\beta=\varepsilon \varepsilon_{1} n^{2} \alpha^{4}-\varepsilon_{1} \lambda_{1} \alpha^{2}+C
$$

where $C$ is a constant.
On the other hand, by comparing (17) and (18) we deduce

$$
3 n^{2} \beta=\lambda_{2} \varepsilon \varepsilon_{1}+\varepsilon \varepsilon_{1} n^{4} \alpha^{4}-\varepsilon_{1} \lambda_{1} n^{2} \alpha^{2}
$$

and finally from (19) and (20) we have $\alpha$ is locally constant on $\mathcal{U}$, which is a contradiction.
Case 2: There is a point $p$ in $\mathcal{U}$ such that $T(\nabla \alpha)(p)=0$. Then from (2) and (8) we have

$$
<A H, X>(p)=-n \varepsilon \alpha(p) X(\alpha)(p)=<H, A X>(p)
$$

which implies, jointly with (3), that $A$ is a self-adjoint endomorphism in $\mathbb{R}_{t}^{n+1}$. Thus the above equation remains valid everywhere on $\mathcal{U}$ and therefore $\nabla \alpha$ is a principal direction of $S$ with associated principal curvature given by $-n \varepsilon \alpha$. First, we claim $\nabla \alpha$ is a non-null vector field. Otherwise, from (2) we would have $-\varepsilon n^{2} \alpha^{2}$ should be an eigenvalue of $A$ and therefore $\alpha$ must be constant on $\mathcal{U}$, which cannot hold. Now, we can choose a local orthonormal frame $\left\{E_{1}, \ldots, E_{n+1}\right\}$ such that $E_{1}$ is parallel to $\nabla \alpha$ and $E_{n+1}=N$. Then from (2) and (8) we have

$$
\begin{aligned}
A E_{1} & =-\varepsilon n^{2} \alpha^{2} E_{1}-n E_{1}(\alpha) N \\
A E_{i} & =n \alpha S E_{i}, \quad i=2, \ldots, n \\
A N & =-n \varepsilon \varepsilon_{1} E_{1}(\alpha) E_{1}+\left\{\frac{\Delta \alpha}{\alpha}+\varepsilon \operatorname{tr}\left(S^{2}\right)\right\} N
\end{aligned}
$$

Writing down $S^{*}$ for the endomorphism $S$ restricted to $\operatorname{span}\left\{E_{2}, \ldots, E_{n}\right\}$ and working on characteristic polynomials we deduce

$$
\begin{aligned}
P_{A}(t) & =q(t)(n \alpha)^{n-1} P_{S^{*}}\left(\frac{t}{n \alpha}\right) \\
P_{S}(t) & =-(n \varepsilon \alpha+t) P_{S^{*}}(t)
\end{aligned}
$$

where $q(t)$ is a polynomial of degree two. Let $\left\{r_{1}, \ldots, r_{n}\right\}$ be the possibly complex roots of $P_{S}$, so that $r_{1}=-n \varepsilon \alpha$ and $\left\{r_{2}, \ldots, r_{n}\right\}$ are the roots of $P_{S^{*}}$. From the relation between $P_{A}$ and $P_{S^{*}}$ we get $n \alpha r_{i}, i=2, \ldots, n$, is a constant eigenvalue of $A$. On the other hand, since $\operatorname{tr}(S)=\sum_{i=1}^{n} r_{i}=n \varepsilon \alpha$ we deduce $\sum_{i=2}^{n} r_{i}=2 n \varepsilon \alpha$ and therefore $\alpha$ is locally constant on $\mathcal{U}$, which is a contradiction with the definition of $\mathcal{U}$.

Anyway, we have found $\mathcal{U}$ is empty, i.e., $M_{s}^{n}$ has constant mean curvature.
Looking for the asked classification, the next theorem becomes the main result of this paper.
Theorem 2.2 Let $x: M_{s}^{n} \longrightarrow \mathbb{R}_{t}^{n+1}$ be an isometric immersion. Then $\Delta x=A x+B$ if and only if one of the following statements holds:

1) $M_{s}^{n}$ is a minimal hypersurface of $\mathbb{R}_{t}^{n+1}$;
2) $M_{s}^{n}$ is an open piece of one of the following hypersurfaces: $\mathrm{S}_{t}^{n}(r), \mathrm{H}_{t-1}^{n}(r), \mathbb{R}_{u}^{k} \times \mathrm{S}_{t-u}^{n-k}(r)$, $\mathbb{R}_{u}^{k} \times \mathrm{H}_{t-u-1}^{n-k}(r)$.

Proof. Assume $\alpha$ is a nonzero constant. Then from (2) and (8) we find

$$
\begin{align*}
& A X=n \alpha S X  \tag{21}\\
& A N=\operatorname{str}\left(S^{2}\right) N \tag{22}
\end{align*}
$$

from which we deduce $\varepsilon \operatorname{tr}\left(S^{2}\right)$ is an eigenvalue of $A$ and therefore $\operatorname{tr}\left(S^{2}\right)$ is constant.
On the other hand, as $A x+B$ is normal to $M_{s}^{n}$, from (22) we have

$$
\left(A^{2}-\varepsilon \operatorname{tr}\left(S^{2}\right) A\right) x+\left(A B-\varepsilon \operatorname{tr}\left(S^{2}\right) B\right)=0
$$

Since $M_{s}^{n}$ is non-minimal, last equation yields $A^{2}-\varepsilon \operatorname{tr}\left(S^{2}\right) A=0$ and taking into account (21) we get

$$
S(S-\lambda I)=0, \quad \lambda=\frac{\varepsilon \operatorname{tr}\left(S^{2}\right)}{n \alpha}
$$

which implies $\lambda \neq 0$. Otherwise, $S$ would be a self-adjoint endomorphism with $S^{2}=0$ and then $\operatorname{tr}(S)$ must be zero (this can be easily seen by using the canonical forms of $S$ ), which cannot hold because $\alpha \neq 0$. Hence $M_{s}^{n}$ is an isoparametric hypersurface whose shape operator is diagonalizable having as principal curvatures zero, with multiplicity at most $n-1$, and $\lambda$, with multiplicity at least one. If $M_{s}^{n}$ is not totally umbilical, then by using similar arguments as in Theorem 2.5 of [12] and Lemma 2 of [10], $M_{s}^{n}$ is an open piece of a pseudo-Riemannian product of a totally umbilical and a totally geodesic submanifold. This completes the proof.

As a consequence of this theorem we obtain the following Riemannian version, which has been shown by Dillen-Pas-Verstraelen [6] when $n=2$ and by Chen-Dillen-Verstraelen-Vrancken [5] and Hasanis-Vlachos [9], independently, in the n-dimensional case.

Corollary 2.3 Let $x: M^{n} \longrightarrow \mathbb{R}^{n+1}$ be an isometric immersion. Then $\Delta x=A x+B$ if and only if one of the following statements holds:

1) $M$ is a minimal hypersurface;
2) $M$ is an open piece of a hypersphere;
3) $M$ is an open piece of a generalized circular cylinder.

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