Classifying pseudo-Riemannian hypersurfaces by means of certain characteristic differential equations

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1. Introduction

The study of minimal surfaces has a long and rich history, and the connection between them and soap films motivated the celebrated Plateau's problem, which has remain completely unsolved for any non-planar contour until the last third of the nineteenth century. However, in the late twenties of this century, Jesse Douglas ([**18**], [**19**] and [**20**]) and Tibor Radó ([**41**], [**42**] and [**43**]) have, quite independently of each other, been successful in developing new methods for solving Plateau's problem. Douglas's work was important both for the simplicity of the method of proof, using calculus of variations of a certain functional, and for the result itself, since the desired minimal surface is nothing but that where the minimun of the above functional is achieved.

It is well known that minimal immersions of a differentiable manifold M in the Euclidean sphere $\mathbb{S}^n$ are just those immersions whose coordinate functions in the ambient Euclidean space are eigenfunctions of the Laplacian operator in the induced metric with eigenvalue $\lambda = -\dim(M)$. Moreover, Takahashi's result, [**45**], is particularly useful in studing isometric minimal immersions of Riemannian symmetric spaces into spheres, since it shows that such immersions correspond precisely to the isometric immersions into $\mathbb{R}^n$ which can be achieved by eigenfunctions of the Laplacian operator with the same non-zero eigenvalue. This will be the starting viewpoint of our study in order to obtain further natural extensions, all of them showing minimal immersions as trivial solutions.

The quoted theorem of Takahashi gives a characterization of minimally immersed submanifolds in nonnegatively curved space forms. That is given in terms of the coordinate eigenfunctions of the isometric immersion $x : M^n \longrightarrow \mathbb{R}^m$. Actually, Takahashi's result is dealing with the eigenvalue equation

$$\Delta x = \lambda x,$$

being Δ the Laplacian on M coming from the induced metric and λ a real constant. Then either $\lambda = 0$ and M is minimal or $\lambda > 0$ and M is minimal in $S^{m-1}(r) \subset \mathbb{R}^m$, where $r = \sqrt{n/\lambda}$.

Takahashi's theorem can be seen as a result of classifying submanifolds satisfying a certain differential equation in the Laplacian of the immersion. Then the following general problem comes out in a natural way:

> *Classify submanifolds by means of some Laplacian differential equation involving the isometric immersion.*

On the other hand, the equation ($*$) says that minimal submanifolds in nonnegatively curved space forms are the only ones whose immersion is associated to exactly one eigenvalue of its Laplacian.

Then, from this viewpoint and considering a first extension of Takahashi's theorem, B.Y. Chen, based on the equation ($*$), built up and developed a fruitful and interesting technique, the so-called *finite type submanifolds* (see [**9**]), chiefly directed to characterize certain families of Euclidean submanifolds. For instance, a Chen-type question states as follows:

> *Could you characterize Euclidean submanifolds whose isometric immersion is associated to two distinct eigenvalues of its Laplacian?*

In particular, if M^n is a compact hypersurface of the sphere S^{n+1} in $\mathbb{R}^{n+2}$ having constant mean curvature α and constant scalar curvature τ, then either M is a small hypersphere in S^{n+1} constructed in $\mathbb{R}^{n+2}$ by using eigenfunctions associated to only one eigenvalue of its Laplacian or M lies in $\mathbb{R}^{n+2}$ by means of eigenfunctions associated to exactly two distinct eigenvalues, which in addition completely determine the geometric quantities α and τ of M. Therefore, the family of Euclidean submanifolds which can be built by using only two eigenvalues of the Laplacian is large enough to pay attention on it, since it contains, among others, those spherical hypersurfaces with constant principal curvatures.

A second extension of Takahashi's theorem can be viewed as follows. For any isometric immersion $x : M^n \longrightarrow \mathbb{R}^m$ it is well known the formula $\Delta x = -nH$, that along with ($*$) yields to

$$\Delta H = \lambda H,$$

where H states for the mean curvature vector field of the immersion. Let us denote by $\mathcal{C}_\lambda$ the family of submanifolds satisfying equation (†). It is not dificult to see that cylinders are in $\mathcal{C}_\lambda$ but they do not satisfy ($*$), so that $\mathcal{C}_\lambda$ contains Takahashi's family as a proper one. However, if M is compact, both equations define the same family. Then it seems natural to ask for the following geometric question:

> *Which is the size of $\mathcal{C}_\lambda$?*

One hopes to find in $\mathcal{C}_\lambda$ other submanifolds apart from cylinders and those of Takahashi's family. Notice that this problem is closely related to those of Chen, because an immersion satisfying (†) is (i) either minimal or of infinite type, if $\lambda = 0$, or (ii) either of 1-type or of null 2-type, provided λ does not vanish.

Furthermore, it is worth exploring the existence of non-minimal submanifolds having harmonic mean curvature vector field.

As a third attempt to generalize Takahashi's condition, O.J. Garay, [**26**], pointed out that if you extend the Laplacian in a natural way to $\mathbb{R}^m$-valued functions on M^n, then equation ($*$) characterizes those submanifolds whose coordinate functions in $\mathbb{R}^m$ restricted to M^n are eigenfunctions of its Laplacian, all of them associated to the same eigenvalue. There he deals with Euclidean hypersurfaces whose coordinate functions are eigenfunctions of its Laplacian but not necessarily for the same eigenvalue, expecting for enough examples apart from those given by Takahashi. Garay's condition can be written as a Laplacian coordinate equation as follows

$$\Delta x_i = \lambda_i x_i, \qquad i = 1, \ldots, m,$$

where $x = (x_1, \ldots, x_m)$, being x_i the coordinate functions; or even, as a matricial equation

$$\Delta x = Ax,$$

where $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$. Nevertheless, Dillen, Pas and Verstraelen, [**16**], pointed out that Garay's condition is not coordinate invariant as a circular cylinder in $\mathbb{R}^3$ shows. Then they study and classify those surfaces in $\mathbb{R}^3$ satisfying the new following equation

$$\Delta x = Ax + B,$$

where A is an endomorphism of $\mathbb{R}^3$ and B is a constant vector in $\mathbb{R}^3$.

One immediately asks for the geometric meaning of equation (‡). Before giving an answer we first notice that both equations (†) and (‡) are equivalent for surfaces in $\mathbb{R}^3$ (see [**16**] and [**21**]), but that situation is quite different for surfaces in the 3-dimensional Lorentz-Minkowski space. In this ambient space those equations show its power, bringing out the B-scrolls as an interesting and own family which cannot be given in the definite case and in his turn satifies (†) but not (‡). Furthermore, it is worthwhile to set off that both equations work as constant mean curvature conditions, so that we are on the track of an isoparametric problem which allows us to reach the asked classification.

In the past few years we have made some contributions to each one of the problems above stated, so that this paper will be a sort of survey in the following sense. We will revisit our recent results concerning to the quoted extensions of Takahashi's theorem rather emphasizing on those we have got about hypersurfaces in the realm of Lorentzian geometry and that revision will include not only published or accepted papers for publication, but also unpublished and others in preparation results in order to make a self-contained article. Finally, we should like to take this opportunity to propose some open problems.

2. Spherical 2-type hypersurfaces

This section is devoted to get a nice characterization of those submanifolds that can be constructed by using exactly two eigenvalues of its Laplacian in terms of its mean and scalar curvatures, which in his turn allows us to solve a series of problems stated by B.Y. Chen in [**8**]. As an interesting consequence, provided the number of principal curvatures is bounded above, a classification of spherical Dupin hypersurfaces constructed in S^n by means of two eigenvalues is given.

A connected (not necessarily compact) submanifold M^n of a pseudo-Euclidean m-space Q^m is called *of finite type* if its position vector field x can be written as a finite sum of eigenfunctions of its Laplacian; more precisely, M^n is said to be *of finite k-type* if its position vector field x admits the following spectral decomposition

$$x = x_0 + \sum_{t=1}^{k} x_t,$$

where $\Delta x_t = \lambda_t x_t$, $t = 1, \ldots, k$, $\lambda_1 < \cdots < \lambda_k$, x_0 is a constant vector in Q^m and x_t $(t = 1, \ldots, k)$ are non-constant Q^m-valued maps on M^n. Otherwise, M^n is said to be *of infinite type*. In particular, if one of the eigenvalues λ_t vanishes, then M^n is said to be *of null k-type* (see [**9**]).

Let M be a hypersurface of the unit hypersphere $\mathbb{S}^{n+1}$ in $\mathbb{R}^{n+2}$ which we will assume (without loss of generality) centred at the origin of $\mathbb{R}^{n+2}$. Denote by x the position vector of M in $\mathbb{R}^{n+2}$ and by ∇ and D the Levi-Civita connection of M and the normal connection of M in $\mathbb{R}^{n+2}$, respectively. We also denote by σ, S and H (H') the second fundamental form of M in $\mathbb{R}^{n+2}$, the shape operator of M in $\mathbb{S}^{n+1}$ and the mean curvature vector field of M in $\mathbb{R}^{n+2}$ ($\mathbb{S}^{n+1}$, respectively). If Δ denotes the Laplacian of M, then the following formula for ΔH was computed in [**12**]:

$$\Delta H = \frac{n}{2}\nabla\alpha^2 + 2\,\mathrm{tr}S_{DH'} + (\Delta\alpha + \alpha|\sigma|^2)N - (n\alpha^2 + n)x,$$

where $H' = \alpha N$, being N the unit normal vector field of M in $\mathbb{S}^{n+1}$. Here $\nabla\alpha^2$ denotes the gradient of α^2 and $\mathrm{tr}S_{DH'} = \sum_{i=1}^n S_{D_{E_i}H'}E_i$, where $\{E_1, \ldots, E_n\}$ is a local orthonormal frame tangent to M.

Now, assume that M is of 2-type. Then its position vector in $\mathbb{R}^{n+2}$ can be written as

$$x = x_0 + x_1 + x_2, \quad \text{with} \quad \Delta x_1 = \lambda_1 x_1 \text{ and } \Delta x_2 = \lambda_2 x_2,$$

where x_0 is a constant vector in $\mathbb{R}^{n+2}$ and $x_1,\ x_2$ are $\mathbb{R}^{n+2}$–valued non-constant differentiable functions on M.

From (22) and the well known fact $\Delta x = -nH$, we have

$$\Delta H = bH + c(x - x_0),$$

where $b = \lambda_1 + \lambda_2$ and $c = \frac{1}{n}\lambda_1\lambda_2$.

Remark 2.1 Through this section, we can assume that $c \neq 0$, otherwise last two authors have proved in [**22**] the non-existence of such hypersurfaces. Of course, if M is compact then $c \neq 0$.

From (21) and (23) one gets the following formulae:

$$n\alpha^2 + n = b - c + c < x, x_0 >$$

and

$$< \Delta H, X >= -c < x_0, X >,$$

for any vector field X tangent to M.

By using (24) and (25) a nice expression for the tangential component of ΔH is found:

$$(\Delta H)^T = -n\nabla\alpha^2.$$

On the other hand, from (21) one has

$$(\Delta H)^T = \frac{n}{2}\nabla\alpha^2 + 2\,\mathrm{tr}S_{DH'}.$$

Finally, an easy computation involving (26), (27) and Codazzi equation gives

$$S(\nabla\alpha) = \mathrm{tr}S_{DH'} = -\frac{3n}{4}\nabla\alpha^2.$$

Therefore, the following lemma is proved.

Lemma 2.2 [**12**] *Let M be a 2-type hypersurface of $\mathbb{S}^{n+1}$. Then $\nabla\alpha^2$ is a principal direction with principal curvature $-\frac{3n}{2}\alpha$ on the open set $\mathcal{U} = \{p \in M : \nabla\alpha^2(p) \neq 0\}$.*

Next lemma, which can also be found in [**12**], allows us to get a good information about the above quoted open set $\mathcal{U}$.

Lemma 2.3 *Let M be a 2-type hypersurface of $\mathbb{S}^{n+1}$. Then either M has constant mean curvature or $\mathcal{U}$ is dense in M.*

2.1. Main results

Before going any further, some computations are needed. For short, we write $h = (b-|\sigma|^2)\alpha - \Delta\alpha$ and $g = n\alpha^2 + n + c - b$, and use (21), (23) and (26) to get

$$cx_0 = n\nabla\alpha^2 + hN + gx.$$

Now, working on $\mathcal{U}$, choose a local orthonormal frame of principal directions $\{E_1, \dots, E_n\}$ with associated principal curvatures $\{\mu_1, \dots, \mu_n\}$, being E_1 in the direction of $\nabla\alpha^2$, so that $\mu_1 = -\frac{3n}{2}\alpha$. By using (29) we find the following auxiliar result.

Lemma 2.4 *Let M be a 2-type hypersurface of $\mathbb{S}^{n+1}$. Then the following formulae hold on $\mathcal{U}$:*

$$E_1(h) = \frac{3n^2}{2}\alpha E_1(\alpha^2), \tag{210}$$

$$E_j(h) = 0, \quad j = 2, \dots, n, \tag{211}$$

$$nE_1E_1(\alpha^2) + \frac{3n}{2}\alpha h + g = 0. \tag{212}$$

Finally, an easy computation from (210), (211) and Lemma 2.3 gives

$$h = n^2\alpha^3 + k, \tag{213}$$

for a constant k, holding anywhere on M.

We are going to compute $\Delta\alpha^2$ in two different ways. First, by using (24) we find

$$n\Delta\alpha^2 = \Delta < cx_0, x >= -n < cx_0, H' > +n < cx_0, x >$$

and then, from (29), we get

$$\Delta\alpha^2 = -\alpha h + g.$$

On the other hand,

$$\begin{aligned} \Delta\alpha^2 &= 2\alpha\Delta\alpha - 2|\nabla\alpha|^2 \\ &= 2(b - |\sigma|^2)\alpha^2 - 2\alpha h - 2|\nabla\alpha|^2. \end{aligned} \tag{215}$$

Now, a straightforward computation (see [**4**]) yields to

Proposition 2.5 *Let M be a 2-type hypersurface of $\mathbb{S}^{n+1}$. Then the mean curvature α does not vanish anywhere on M.*

Next, we are going to prove one of the chief results of this section, which gives an affirmative answer to an open problem stated by B.Y. Chen [**8**, I.6].

Theorem 2.6 *Let M be a 2-type hypersurface of $\mathbb{S}^{n+1}$. Then M has constant mean curvature if and only if M has constant scalar curvature.*

Proof. If α is a constant, then h so is because (213) and then $|\sigma|^2$ is also a constant. As a consequence, we use the Gauss equation

$$|\sigma|^2 = n^2\alpha^2 - n(n-1)\tau + n \tag{216}$$

to get M has constant scalar curvature.

Conversely, suppose now M has constant scalar curvature. From (29) we find

$$|\nabla\alpha|^2 = \frac{1}{4n^2\alpha^2}\{c^2|x_0|^2 - h^2 - g^2\},$$

that jointly with (214) and (215) leads to

$$4n^2(b - |\sigma|^2)\alpha^4 + (h - 2n^2\alpha^3)h + (g - 2n^2\alpha^2)g - c^2|x_0|^2 = 0.$$

Finally, from here, (213) and Gauss equation α must be a root of a polynomial with constant coefficients and therefore α is a constant. ■

Let M be a 2-type hypersurface of $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$. Consider again the open set $\mathcal{U}$ which is dense in M unless it was empty and so M has constant mean curvature (see Lemma 2.3). Let *p* be any point of $\mathcal{U}$ and denote by $\gamma(t)$ the integral curve of $\nabla\alpha^2$ through the point $p \in \mathcal{U}$. Now, (213) allows us to rewrite (212) along $\gamma(t)$ as follows:

$$\frac{d^2}{dt^2}(\alpha^2) + \frac{3}{2}n^2\alpha^4 + \alpha^2 + \frac{3}{2}k\alpha + \frac{1}{n}(n + c - b) = 0.$$

Let $\beta = \left(\frac{d\alpha}{dt}\right)^2$. Then it is easy to see that equation (219) can be reduced to the following first order differential equation:

$$\alpha\frac{d\beta}{d\alpha} + 2\beta = -\frac{3}{2}n^2\alpha^4 - \alpha^2 - \frac{3}{2}k\alpha - \frac{1}{n}(n + c - b).$$

From this equation we obtain the following solution:

$$\begin{aligned} 4n^2\alpha^2\beta &= -\frac{3}{2}n^4\alpha^4 - 2n^2\alpha^2 - 6kn^2\alpha \\ &\quad - 4n(n + c - b)\ln(\alpha) + C_1, \end{aligned} \tag{221}$$

where C_1 is some constant.

On the other hand, from (29) one has

$$4n^2\alpha^2\beta = c^2|x_0|^2 - (n^2\alpha^3 + k)^2 - (n\alpha^2 + n + c - b)^2. \tag{222}$$

Therefore, (221) and (222) prove the following

Theorem 2.7 *Let M be a 2-type hypersurface of $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$. Then M has constant mean curvature.*

The following result gives a nice characterization of compact 2-type hypersurfaces in the hypersphere $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ and partially solves an open problem stated by B.Y. Chen [**8**, I.4].

Corollary 2.8 *Let M be a compact hypersurface of $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ which is not a small hypersphere of $\mathbb{S}^{n+1}$. Then M is of 2-type if and only if M has non-zero constant mean curvature α and constant scalar curvature τ. Moreover, if M is of 2-type, α and τ are completely determined for the eigenvalues $\{\lambda_1, \lambda_2\}$ involved in the 2-type condition.*

Proof. The necessary condition follows automatically from Theorems 2.7 and 2.6. Now, if α and τ are constant, then $|S|^2$ is also constant and so (21) allows us to write

$$\Delta H = (|S|^2 + n)H + (|S|^2 - n\alpha^2)x,$$

where we have used $H = H' - x$. As a consequence there exist two constants, say r and s, such that $\Delta H = rH + sx$, with $s \neq 0$ because M is not a small hypersphere of $\mathbb{S}^{n+1}$. Therefore, we use Theorem 2.2 of [**9**, p. 257] to get that M is of 2-type. Last claim of the statement follows from Theorem 4.2 of [**9**, p. 276]. ■

Next result gives a partial answer to another open problem stated by B.Y. Chen [**8**, I.1].

Corollary 2.9 *Let M be a compact 2-type hypersurface of $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$. Then M is mass-symmetric in $\mathbb{S}^{n+1}$.*

Proof. First, we use Theorem 2.7 to have (223), where both coefficients $|S|^2 + n$ and $|S|^2 - n\alpha^2$ are constant. Moreover, $|S|^2 - n\alpha^2 \neq 0$ because M is assumed to be of 2-type in $\mathbb{R}^{n+2}$ (notice that $|S|^2 = n\alpha^2$ implies M is a small hypersphere and so of 1-type in some hyperplane of $\mathbb{R}^{n+2}$ and then of 1-type in $\mathbb{R}^{n+2}$). Thus we have

$$0 = \int_M \Delta H dv = (|S|^2 + n)\int_M H dv + (|S|^2 - n\alpha^2)\int_M x dv,$$

and so

$$\int_M x dv = 0,$$

this means, the center of mass of M is nothing but the origin of $\mathbb{R}^{n+2}$. ■

Remark 2.10 We would like to point out that Theorem 2.7 and Corollaries 2.8 and 2.9 have been also obtained, simultaneously and independently, by Hasanis and Vlachos in [**31**], where they use a different method of proof.

2.2. Applications

A hypersurface M of $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ is called a Dupin hypersurface if the multiplicity of each principal curvature is constant on M and each principal curvature is constant along its associated principal directions. In [**6**] it is proved that compact embedded Dupin hypersurfaces are conformal images of isoparametric hypersurfaces when the number g of principal curvatures is $g \leqslant 2$, but this is not the case when $g \geqslant 3$. In [**46**], G. Thorbergsson proves that, in cohomology level, compact embedded Dupin hypersurfaces are isoparametric. That result leads to the Cecil-Ryan's conjecture [**7**]: *A compact embedded Dupin hypersurface is Lie equivalent to an isoparametric hypersurface.* That holds when $g \leqslant 3$, see [**6**] and [**37**]; otherwise, it can be found counterexamples to the conjecture in [**38**] and [**40**]. These facts suggest a close relation between compact embedded Dupin hypersurfaces and isoparametric ones.

It is a well-known fact that isoparametric hypersurfaces of $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ with $g \leqslant 2$ are spheres and Riemannian products of spheres. When $g = 3$, they were completely classified by E. Cartan [**5**]. They are all homogeneous spaces and the multiplicities of principal curvatures (m_1, m_2, m_3) and dimensions n are listed in the adjoint table:

M^n	(m_1, m_2, m_3)	n
$SO(3)/Z_2 + Z_2$	$(1,1,1)$	3
$SU(3)/T^2$	$(2,2,2)$	6
$SP(3)/SP(1)^3$	$(4,4,4)$	12
$F_4/Spin(8)$	$(8,8,8)$	24

Now, we are going to state and prove the following classification result.

Theorem 2.11 *Let M be a Dupin hypersurface of $\mathbb{S}^{n+1}$ with at most three distinct principal curvatures which is not a small hypersphere of $\mathbb{S}^{n+1}$. Then M is of 2-type if and only if one the following statements holds:*

1) M is an open piece of a Riemannian product $\mathbb{S}^p \times \mathbb{S}^{n-p}$.

2) M is an open piece of one of the hypersurfaces exhibited in the above table.

Proof. The sufficient condition follows easily from above results in this section. Now, let us suppose M is a 2-type hypersurface of $\mathbb{S}^{n+1}$. Then from Theorems 2.7 and 2.6 we know that M has constant mean curvature and constant scalar curvature. Since M is a Dupin hypersurface it is not difficult to see that M is, in fact, an isoparametric hypersurface. Thus, we obtain the desired conclusion, because M cannot have only one principal curvature. ■

As a consequence, we obtain the following.

Corollary 2.12 *Let M be a Dupin hypersurface of $\mathbb{S}^4$ which is not a small hypersphere. Then M is of 2-type if and only if M is an open piece of one of the following hypersurfaces: $\mathbb{S}^1 \times \mathbb{S}^2$, $SO(3)/Z_2 + Z_2$.*

3. Hypersurfaces with a characteristic eigenvector field

In this section we will tackle the second extension of Takahashi's condition set in the Lorentz-Minkowski ambient. Before starting this task, it will be convenient recalling the pseudo-Riemannian version of Takahashi's theorem, which can be found in [**10**] and [**36**]. Let x be an isometric immersion of a submanifold M in a pseudo-Euclidean space $\mathbb{R}^m_s$. Then M satisfies equation $(*)$ if and only if M is either minimal in $\mathbb{R}^m_s$, or minimal in a pseudo-hyperbolic space $H^{m-1}_{s-1}(r)$, or minimal in a pseudo-sphere $S^{m-1}_s(r)$. Furthermore, from here we obtain that minimal submanifolds of $\mathbb{R}^m_s$ are the only ones having harmonic coordinate functions and therefore there can be characterized by the equation $\Delta x = 0$.

As we pointed out in the Introduction, the condition (†) means that the coordinate functions of the mean curvature vector field H are eigenfunctions of the Laplacian associated to the same eigenvalue, thus that equation connects again with the spectral geometry of the submanifold.

We have already mentioned that information furnished by the equation (†) is different from that of $(*)$. However, we wish to give an example in the Lorentzian ambient to ratify this fact.

Let $\mathbb{L}^{n+1}$ be the $(n+1)$-dimensional Lorentz-Minkowski space with the usual coordinates $(x_1, \ldots, x_{n+1})$ and the standard flat metric given by $ds^2 = -dx_1^2 + dx_2^2 + \cdots + dx_{n+1}^2$. Let us consider the differentiable function $f : \mathbb{L}^{n+1} \longrightarrow \mathbb{R}$ defined by

$$f(x_1, \ldots, x_{n+1}) = -x_1^2 + x_2^2 + \cdots + x_{k+1}^2 - r^2,$$

where $r > 0$ and $k \in \{1, 2, \ldots, n-1\}$. Then $M = f^{-1}(0)$ is a spacial hypersurface of $\mathbb{L}^{n+1}$, i.e., it is endowed with a Riemannian metric and furthermore isometric to the Riemannian product $H^k(r) \times \mathbb{R}^{n-k}$. Bearing in mind the relation between the second fundamental forms of $H^k(r) \times \mathbb{R}^{n-k}$ and $H^k(r)$, it is easy to see that both submanifolds satisfy equation (†), but we know that cylinders do not fall into Takahashi's family.

In analysing condition (†) we will study two cases separatedly, according to the curvature of the ambient space. To do that we would like to notice that no restriction on the causal character of the hypersurface is made.

3.1. Flat ambient space

Let M_s^n be a hypersurface, with index $s = 0, 1$, in $\mathbb{L}^{n+1}$ and let ∇f denote the gradient of a diferentiable function f. An easy computation allows us to get the following formula for the Laplacian of the mean curvature vector field H ([**10**],[**24**]):

$$\Delta H = 2S(\nabla\alpha) + \frac{n\varepsilon}{2}\nabla\alpha^2 + \{\Delta\alpha + \varepsilon\alpha \mathrm{tr}(S^2)\}N,$$

where S stands for the shape operator of M, α the mean curvature, N the unit normal vector field and $\varepsilon =< N, N >$.

Assuming that M satisfies the equation (†), we easily get from (31) the following equations:

$$S(\nabla\alpha) + \frac{n\varepsilon}{2}\alpha\nabla\alpha = 0, \tag{32}$$

$$\Delta\alpha + (\varepsilon \mathrm{tr}(S^2) - \lambda)\alpha = 0. \tag{33}$$

Now we wish to deduce some easy consequences from there. If α is a non-vanishing constant then (33) implies that tr(S^2) is constant and therefore M also has constant scalar curvature. On the other hand, if M has constant mean and scalar curvatures then equation (†) holds for the real constant $\lambda = \varepsilon \mathrm{tr}(S^2)$. Hence, the following problem arises in a natural way:

> **(P1)** *Are the non-vanishing constant mean curvature and constant scalar curvature hypersurfaces of the Lorentz-Minkowski space characterized by the equation $\Delta H = \lambda H$?*

Before beginning the study of this problem, we would like to remark that equation (32) can be obtained by supposing only that ΔH is a vector field normal to M. In this way, Garay and Romero, [**27**], have recently studied those hypersurfaces in $\mathbb{L}^{n+1}$ satisfying the condition $\Delta H = C$, where C is a constant vector in $\mathbb{L}^{n+1}$ which is normal to M at every point, and they show that C should vanish. Bearing in mind that minimal submanifolds are the only ones whose immersion is harmonic, i.e., $\Delta x = 0$, it seems natural to ask for the following geometric question:

> **(P2)** *Does the equation $\Delta H = 0$ characterize the vanishing mean curvature hypersurfaces of $\mathbb{L}^{n+1}$?*

Submanifolds satisfying the condition $\Delta H = 0$ are called *biharmonic*, because they satisfy $\Delta^2 x = 0$, and they have been handled, among others, in [**11**], [**17**], [**21**], in the Euclidean case, and in [**14**], in the pseudo-Euclidean case.

In dealing with problem (P1), we are going to find surfaces in $\mathbb{L}^3$ satisfying (†). At a first stage we know that minimal surfaces, hyperbolic $H^2(r)$ and de Sitter $S_1^2(r)$ planes are trivial solutions of (†), as well as the three cylinders appearing in this ambient space: $H^1(r) \times \mathbb{R}$, $\mathbb{L} \times S^1(r)$ and $S_1^1(r) \times \mathbb{R}$. Observe that all quoted examples, as spacial as Lorentzian ones, have diagonalizable shape operators. Therefore, there arises the following question: *Are there Lorentzian surfaces satisfying (†) and having non-diagonalizable shape operators?* To get an affirmative answer we present an example which was first given by Graves, [**28**].

Let $x(s)$ be a null curve in $\mathbb{L}^3$ with Cartan frame $\{A, B, C\}$, i.e., A, B and C are vector fields along $x(s)$ such that

$$\begin{aligned}
\dot{x}(s) &= A(s), \\
\dot{A}(s) &= k(s)C(s), \\
\dot{B}(s) &= w_0 C(s), \\
\dot{C}(s) &= w_0 A(s) + k(s)B(s),
\end{aligned}$$

where $k(s) \neq 0$ and w_0 is a nonzero constant. Then the map $\Phi : (s, u) \longrightarrow x(s) + uB(s)$ parametrizes a Lorentzian surface in $\mathbb{L}^3$, which Graves called a B-scroll. An easy computation shows that $\alpha = w_0$ and $\mathrm{tr}(S^2) = 2w_0^2$, and then from (31) we have $\Delta H = 2w_0^2 H$. Moreover, the shape operator S can be put, in the usual frame $\{\frac{\partial \Phi}{\partial s}, \frac{\partial \Phi}{\partial u}\}$, as

$$\begin{pmatrix} w_0 & 0 \\ k(s) & w_0 \end{pmatrix},$$

showing that the minimal polynomial $p_S(t)$ of S is given by $p_S(t) = (t - w_0)^2$ and S is not diagonalizable.

All above examples illustrating equation (†) satisfy a chiefly interesting geometric property: all of them have constant mean curvature. Then it is reasonable to ask for *surfaces in $\mathbb{L}^3$ satisfying (†) and having non-constant mean curvature.* However we are able to get a negative answer in [**23**]:

Theorem 3.1 *Every surface M_s^2 in $\mathbb{L}^3$ satisfying the condition $\Delta H = \lambda H$ has constant mean curvature.*

As a consequence, from here and (33), it is easy to see that the only non-minimal surfaces in $\mathbb{L}^3$ satisfying (†) are the isoparametric ones, i.e., those whose shape operators have constant characteristic polynomial. Now bringing here the classification of such surfaces, given in [**35**] and [**39**], we get in [**23**] a complete answer to problem (P1):

Theorem 3.2 *Let M_s^2 be a surface in $\mathbb{L}^3$. Then $\Delta H = \lambda H$, for a real constant λ, if and only if M_s^2 is either minimal or an open piece of one of the following surfaces: $H^2(r)$, $S_1^2(r)$, $H^1(r) \times \mathbb{R}$, $\mathbb{L} \times S^1(r)$, $S_1^1(r) \times \mathbb{R}$ and a B-scroll.*

Some consequences can be deduced from this theorem. On one hand, we find that minimal surfaces in $\mathbb{L}^3$ are characterized as the only ones having harmonic mean curvature vector field, solving problem (P2). On the other hand, paying attention on the causal character of the surface, we get that the only spacial surfaces satisfying (†) are either those having zero mean curvature (the so-called *maximal* ones) or open pieces of one of the following surfaces: a hyperbolic plane $H^2(r)$, a hyperbolic cylinder $H^1(r) \times \mathbb{R}$.

As for the n-dimensional case, problems (P1) and (P2) remain open and we have only found partial solutions. In [**24**] we deal with hypersurfaces in $\mathbb{L}^{n+1}$ satisfying (†) and such that the minimal polynomial of the shape operator is at most of degree two. *Under this additional hypothesis*, we are in a position to show the following.

Proposition 3.3 *All hypersurfaces M_s^n in $\mathbb{L}^{n+1}$ satisfying $\Delta H = \lambda H$ have constant mean curvature.*

This result allows us to get an affirmative answer to (P1), under the above additional condition. Taking into account that theorem and the solution obtained for surfaces, we dare to state the following conjecture.

(C1) *The answer to problem (P1) is affirmative.*

Finally, we would like to remark that problem (P1) also involves the Euclidean case and thus we also guess that conjecture (C1) can be applied to Euclidean hypersurfaces.

3.2. Non-flat ambient space

Let $\bar{M}_\nu^{n+1}(c)$ denote the pseudo-Riemannian space form with index ν and constant sectional curvature sgn(c)/c^2. Without loss of generality, we can assume $c = \pm 1$ and in what follows $\bar{M}_\nu^{n+1}$ will denote the pseudo-sphere S_ν^{n+1} or the pseudo-hyperbolic space H_ν^{n+1}, according to $c = 1$ or $c = -1$, respectively. Let x be an isometric immersion of a hypersurface M_s^n in $\bar{M}_\nu^{n+1}$ and let H denote the mean curvature vector field of M_s^n in the pseudo-Euclidean space $\mathbb{R}_t^{n+2}$ where $\bar{M}_\nu^{n+1}$ is lying. Then it is easy to show that H is given by

$$H = \alpha N - cx,$$

where N denotes a unit vector field normal to M_s^n in $\bar{M}_\nu^{n+1}$ and α the mean curvature of M_s^n in $\bar{M}_\nu^{n+1}$. An easy computation from (34) yields to the following nice formula for ΔH, [**10**]:

$$\begin{aligned} \Delta H &= 2S(\nabla\alpha) + n\varepsilon\alpha\nabla\alpha + \{\Delta\alpha + \varepsilon\alpha \operatorname{tr}(S^2) + nc\alpha\}N \\ &\quad - n(1 + \varepsilon c\alpha^2)x, \end{aligned} \tag{35}$$

where S stands for the shape operator of the hypersurface.

If M_s^n is a hypersurface satisfying the condition (†), we can use equations (34) and (35) to obtain the following formulae:

$$\begin{aligned} &2S(\nabla\alpha) + n\varepsilon\alpha\nabla\alpha = 0, \\ &\Delta\alpha + (\varepsilon \operatorname{tr}(S^2) + nc - \lambda)\alpha = 0, \\ &n(1 + \varepsilon c\alpha^2) - \lambda c = 0. \end{aligned}$$

Now we can combine those equations in order to get the following result, which can be considered as a first approximation to the solution of problem (P1) in the new ambient space.

Theorem 3.4 *A hypersurface M_s^n of $\bar{M}_\nu^{n+1}$ satisfies the condition $\Delta H = \lambda H$ if and only if M_s^n is minimal in $\bar{M}_\nu^{n+1}$ or has nonzero constant mean curvature α and* $\operatorname{tr}(S^2) = (1/n)\operatorname{tr}(S)^2$.

As a first consequence we get the following. Let us suppose M_s^n has diagonalizable shape operator, for example provided M_s^n is a spacial hypersurface. Then the above result implies M_s^n is a totally umbilical hypersurface and we can use [**34**, Theorem 1.4] to find that M_s^n is an open piece either of $S_s^n(r)$, or $H_s^n(r)$, or $\mathbb{R}_s^n$ (in the last case, the immersion is constructed from a quadratic function). As a second consequence we deduce, taking into account the Gauss equation, that non-minimal hypersurfaces of $\bar{M}_\nu^{n+1}$ satisfying (†) are characterized by having constant mean α and scalar τ curvatures which satify the equation $\tau = n(n-1) < H, H >= n(n-1)(\varepsilon\alpha^2 + c)$.

With the aim of studying in depth the condition (†), we are going to deal with surfaces. Following [**15**], we construct B-scrolls over null curves to obtain some surfaces in $\bar{M}_1^3$ satisfying (†) and whose shape operators are non-diagonalizable, with minimal polynomials having only real roots. Nevertheless, it seems natural thinking of surfaces in $\bar{M}_1^3$ satisfying (†) and whose shape operator has a minimal polynomial with complex roots. However, that cannot happen because of the condition $\mathrm{tr}(S^2) = \frac{1}{2}\mathrm{tr}(S)^2$.

Let us suppose M_s^2 is a surface satisfying (†). Then M_s^2 has constant mean and scalar curvatures and thus M_s^2 is an isoparametric surface of $\bar{M}_1^3$. Now we may carry on an standard reasoning to obtain the following.

Theorem 3.5 *Let M_s^2 be a non-minimal surface of $\bar{M}_1^3$ satisfying the condition $\Delta H = \lambda H$. Then M_s^2 is an open piece either of a totally umbilical surface or a B-scroll.*

This result leads to the characterization of biharmonic surfaces of $\bar{M}_1^3$.

Corollary 3.6 *A surface M_s^2 in $\bar{M}_1^3$ is biharmonic if and only if it is either a flat totally umbilical one or a flat B-scroll.*

In order to complete our study, we must consider hypersurfaces M_s^n in $\bar{M}_1^{n+1}$, where we guess more promising prospects than in the flat ambient space. We approach the problem by analising separatedly the different shape operators locally allowed for the hypersurface. In the diagonalizable case, the problem has been already solved. Let us suppose that the minimal polynomial of S is given, in an open set of M_s^n, by $p_S(t) = (t-\beta)^2(t-\mu_1)\cdots(t-\mu_k)$. Then the equatility $\mathrm{tr}(S^2) = \frac{1}{n}\mathrm{tr}(S)^2$ implies that $\mu_1 = \cdots = \mu_k = \beta$ is constant and therefore M_s^n is an isoparametric hypersurface of $\bar{M}_1^{n+1}$ with $p_S(t) = (t-\beta)^2$. A standard reasoning on integral submanifolds leads us to an explicit description of the hypersurface M_s^n. The case $p_S(t) = (t-\beta)^3\,(t-\mu_1)\cdots(t-\mu_k)$ can be treated in a similar way. Finally, the situation when the minimal polynomial has complex roots, i.e., $p_S(t) = [(t-\beta)^2+\gamma^2](t-\mu_1)\cdots(t-\mu_k)$ with $\gamma \neq 0$, becomes more complicated, but at the present we think that cannot hold. In this way, we state the following conjecture.

> **(C2)** *There are no hypersurfaces in M_1^{n+1} satisfying (†) and whose shape operators have minimal polynomials with complex roots.*

To finish this section, we would like telling of that the results of this subsection, i.e., the non-flat ambient space case, are being purified in order to be published elsewhere.

4. Hypersurfaces satisfying the condition $\Delta x = Ax + B$

As we have pointed out in the Introduction, both conditions $\Delta x = Ax$, original from Garay, and (‡), due to Dillen, Pas and Verstraelen, were only established for submanifolds and, particularly, hypersurfaces in the Euclidean space and, in this context, they have been recently studied by some authors, [**13**], [**16**], [**25**], [**26**], [**30**], [**32**]. However, the pseudo-Riemannian case presents an own behaviour, mainly because the shape operator need not be diagonalizable, which plays a chief role in the Riemannian case.

In this section, we will study those pseudo-Riemannian hypersurfaces in pseudo-Riemannian space forms which are characterized by the matricial condition (‡) in the Laplacian of the isometric immersion. A first step in this way was given by the authors in [**2**], where surfaces in the 3-dimensional Lorentz-Minkowski space satisfying the equation (‡) were classified. The interesting changes found here with regard to the Euclidean case leaded us to consider that condition not only for hypersurfaces in a pseudo-Euclidean space, but also for hypersurfaces in a pseudo-spherical or pseudo-hyperbolic space.

4.1. Flat ambient space

Let $\mathbb{R}^{n+1}_\nu$ be the $(n+1)$-dimensional pseudo-Euclidean space of index ν with metric tensor, in the usual coordinates $(x_1, \ldots, x_{n+1})$, given by

$$ds^2 = -\sum_{i=1}^{\nu} dx_i \otimes dx_i + \sum_{j=\nu+1}^{n+1} dx_j \otimes dx_j.$$

Let M^n_s be a pseudo-Riemannian hypersurface in $\mathbb{R}^{n+1}_\nu$ with index $s = \nu - 1, \nu$ and let us write by H and N the mean curvature and the unit normal vector fields of M^n_s in $\mathbb{R}^{n+1}_\nu$, respectively, so that $H = \alpha N$, being α the mean curvature in the direction of N.

Let $x : M^n_s \longrightarrow \mathbb{R}^{n+1}_\nu$ be an isometric immersion satisfying (‡), where A is now an endomorphism of $\mathbb{R}^{n+1}_\nu$ and B is a constant vector. From here, the formula for ΔH given in Section 3 and the well known formula $\Delta x = -nH$, it is not difficult to see that

$$AX = n\alpha SX - nX(\alpha)N,$$

for any vector field X tangent to M^n_s, and

$$AH = 2S(\nabla\alpha) + n\varepsilon\alpha\nabla\alpha + \{\Delta\alpha + \varepsilon\alpha \mathrm{tr}(S^2)\}N.$$

If we suppose now M^n_s has non-zero constant mean curvature, then from (41) and (42) we have

$$AX = n\alpha SX, \tag{43}$$
$$AN = \varepsilon \mathrm{tr}(S^2)N, \tag{44}$$

from which we deduce tr(S^2) is also a constant and, taking covariant differentiation in (44), we find that the shape operator satisfies the polynomial equation

$$S(S - \lambda I) = 0,$$

where λ is the non-vanishing real constant given by $\lambda = \dfrac{\varepsilon \mathrm{tr}(S^2)}{n\alpha}$. That equation means that M^n_s is an isoparametric hypersurface in $\mathbb{R}^{n+1}_\nu$ with diagonalizable shape operator and having as principal curvatures zero, with multiplicity at most $n-1$, and $\lambda \neq 0$, with multiplicity at least one. Therefore, if M^n_s is totally umbilical in $\mathbb{R}^{n+1}_\nu$ then M^n_s is an open piece of a pseudo-sphere

$S^n_\nu(r)$ or a pseudo-hyperbolic space $H^n_{\nu-1}(r)$. Otherwise, by using similar arguments as those in [**44**, Theorem 2.5] and [**33**, Lemma 2], M^n_s is an open piece of one of the pseudo-Riemannian products $\mathbb{R}^k_u \times S^{n-k}_{\nu-u}(r)$ and $\mathbb{R}^k_u \times H^{n-k}_{\nu-u-1}(r)$. We will refer to these four classes as the standard examples in $\mathbb{R}^{n+1}_\nu$.

On the other hand, besides the trivial case of minimal hypersurfaces, it is not difficult to show that the standard examples also satisfy equation (‡).

Then it seems reasonable to state the following question.

> **(P3)** *Does the equation $\Delta x = Ax + B$ characterize to the family of minimal hypersurfaces and standard examples in $\mathbb{R}^{n+1}_\nu$?*

We have just obtained an affirmative answer to this question when the mean curvature is constant. The following result completely solves that problem, [**1**].

Theorem 4.1 *The hypersurfaces in $\mathbb{R}^{n+1}_\nu$ satisfying the condition $\Delta x = Ax + B$ have constant mean curvature.*

Our result generalize those given, when the ambient space is $\mathbb{R}^{n+1}$, in [**16**], [**13**] and [**32**]. On the other hand, for hypersurfaces in the Lorentz-Minkowski space we have the following proposition.

Proposition 4.2 *Let $x : M^n_s \longrightarrow \mathbb{L}^{n+1}$ be an isometric immersion. Then $\Delta x = Ax + B$ if and only if M^n_s is either minimal or an open piece of one of the following hypersurfaces: $S^n_1(r)$, $H^n(r)$, $S^k_1(r) \times \mathbb{R}^{n-k}$, $\mathbb{L}^k \times S^{n-k}(r)$, $H^k(r) \times \mathbb{R}^{n-k}$.*

Remark 4.3 We wish to pointed out that both equations $\Delta H = \lambda H$ and $\Delta x = Ax + B$ characterize the same family of surfaces in the Euclidean case, but they make notably differences in Lorentzian ambient. In fact, we have seen that a B-scroll, which has constant mean curvature but non-diagonalizable shape operator, satisfies the former but not the latter.

4.2. Non-flat ambient space

Through this section, we will keep the notation fixed in Section 3.2. A hypersurface M^n_s in $\bar{M}^{n+1}_\nu$ can also be viewed as a codimension two submanifold in the corresponding pseudo-Euclidean space $\mathbb{R}^{n+2}_t$, where $\bar{M}^{n+1}_\nu$ is canonically immersed, and therefore we can ask ourselves for those hypersurfaces in $\bar{M}^{n+1}_\nu$ whose isometric immersion $x : M^n_s \longrightarrow \bar{M}^{n+1}_\nu \subset \mathbb{R}^{n+2}_t$ satisfies the condition (‡), being A and endomorphism of $\mathbb{R}^{n+2}_t$ and B a constant vector in $\mathbb{R}^{n+2}_t$.

In order to guide our study, we are going to give some examples. A first trivial one is provided by minimal hypersurfaces in $\bar{M}^{n+1}_\nu$. Consider now a totally umbilical hypersurface M^n_s in $\bar{M}^{n+1}_\nu$. Bearing in mind the classification given in [**34**], we know that, according to the causal character of H, M^n_s is an open piece of a pseudo-Riemannian space form. It is not difficult to see that both $S^n_s(r)$ and $H^n_s(r)$ satisfy the asked condition, so that the most interesting situacion comes up in the flat case, where H is a null vector. Here the isometric immersion $x : \mathbb{R}^n_s \longrightarrow \bar{M}^{n+1}_\nu \subset \mathbb{R}^{n+2}_{s+1}$ is given by $x = f - x_0$, being x_0 a fixed vector in $\mathbb{R}^{n+2}_{s+1}$ and $f : \mathbb{R}^n_s \longrightarrow \mathbb{R}^{n+2}_{s+1}$ the function defined by $f(u_1, \ldots, u_n) = (q(u), u_1, \ldots, u_n, q(u))$, where $q(u) = a < u, u > + < b, u > +c$, $a \neq 0$. Now we have $\Delta x = (-2na, 0, \ldots, 0, -2na)$, showing that this hypersurface also satisfies equation (‡) with $A = 0$ and $B = (-2na, 0, \ldots, 0, -2na)$. Finally, a straightforward computation shows that those hypersurfaces in $\bar{M}^{n+1}_\nu$ built up as the following pseudo-Riemannian products

$S_u^k(r_1) \times S_{s-u}^{n-k}(r_2)$, $H_u^k(r_1) \times H_{s-u}^{n-k}(r_2)$ and $S_u^k(r_1) \times H_{s-u}^{n-k}(r_2)$ also satisfy that condition (see [**3**]). These hypersurfaces will be called the standard pseudo-Riemannian products in $\bar{M}_\nu^{n+1}$.

Now, let $x : M_s^n \longrightarrow \bar{M}_\nu^{n+1} \subset \mathbb{R}_t^{n+2}$ be an isometric immersion satisfying the condition (‡). Then the well known formula $\Delta x = -nH$ and the expresions for H and ΔH obtained in Section 3.2 lead to

$$\begin{aligned} AX &= n\alpha SX + ncX - nX(\alpha)N, \\ \alpha AN &= 2S(\nabla\alpha) + n\varepsilon\alpha\nabla\alpha + \{\Delta\alpha + \varepsilon\alpha\,\mathrm{tr}(S^2)\}N - nc\varepsilon\alpha^2 x - cB, \\ Ax &= -n\alpha N + ncx - B, \end{aligned} \tag{45}$$

for any tangent vector field X.

Then we have $< AX, x >= 0$ and taking covariant derivative we deduce that

$$< A\sigma(X,Y), x >= - < AX, Y >,$$

that along with (45) leads to

$$< SX - \varepsilon\alpha X, Y >< B, x >= 0.$$

In a first aproximation to our problem, let us suppose that the mean curvature of M_s^n in $\bar{M}_\nu^{n+1}$ is a non-zero constant. If M_s^n is not totally umbilical (46) implies $< B, x >= 0$ and, reasoning as in [**3**, Lemma 3.1], $B = 0$. Now, (45) can be rewritten as

$$\begin{aligned} AX &= n\alpha SX + ncX, \\ AN &= \varepsilon\mathrm{tr}(S^2)N - nc\varepsilon\alpha x, \\ Ax &= -n\alpha N + ncx. \end{aligned} \tag{47}$$

From these equations we deduce M_s^n is an isoparametric hypersurface in $\bar{M}_\nu^{n+1}$ whose shape operator S satisfies the polynomial equation

$$S^2 + \frac{nc - \varepsilon\mathrm{tr}(S^2)}{n\alpha} S - c\varepsilon I = 0. \tag{48}$$

The last equation plays a key role in the following reasoning. If S is diagonalizable and M_s^n is not totally umbilical in $\bar{M}_\nu^{n+1}$ from (48) we get M_s^n is isoparametric with two principal curvatures and, by using similar arguments as in [**44**, Theorem 2.5] and [**33**, Lemma 2], it is an open piece of one of the standard pseudo-Riemannian products. In particular, we have got a first solution to the problem when the ambient space is either spherical or hyperbolic, that is, when $\nu = 0$. Otherwise, M_s^n could be endowed with an indefinite metric and then the shape operator needs not be diagonalizable.

Now, we will consider the simplest situation where one can find a non-diagonalizable shape operator, that is, a Lorentzian surface M_1^2 in $\bar{M}_1^3$, satisfying condition (‡) and having non-zero constant mean curvature. To do that, we know M_1^2 is an isoparametric surface in $\bar{M}_1^3$ and the characteristic polynomial of S is given by (48), being $n = 2$ and $\varepsilon = 1$. From here we find M_1^2 is a flat surface in $\bar{M}_1^3 \subset \mathbb{R}_t^4$ with non diagonalizable shape operator and parallel second fundamental form in $\mathbb{R}_t^4$. Therefore, by using [**34**, Theorem 1.15 and Theorem 1.17] we deduce such a situation only appears when $M_1^3 = H_1^3$ and, in that case, M_1^2 is an open piece either of a complex circle, [**34**, Example 1.12], or of the surface exhibited in [**34**, Example 1.13]. Now it is not difficult to see that both surfaces satisfy the asked condition.

In conclusion, we have found a first significant difference in studying the condition (‡) in non-flat pseudo-Riemannian space forms with respect to the similar one in the flat case. In fact, we have seen in Section 4.1 that hypersurfaces in $\mathbb{R}^{n+1}_\nu$ satisfying that condition must have diagonalizable shape operators. However we have just obtained some examples of surfaces in H^3_1 satisfying (‡) and having non-diagonalizable shape operators.

In order to generalize these examples to any $\bar{M}^{n+1}_\nu$ we profit by Hahn's ideas, [**29**]. Let L be a self-adjoint endomorphism of $\mathbb{R}^{n+2}_t$, consider the quadratic function $f : \bar{M}^{n+1}_\nu \longrightarrow \mathbb{R}$ defined by $f(x) =< Lx, x >$ and assume that the minimal polynomial of L is given by $p_L(t) = t^2 + at + b$, $a, b \in \mathbb{R}$. Then the level set $M = f^{-1}(r)$, where r is a real constant such that $p_L(cr) \neq 0$, is an isoparametric hypersurface in $\bar{M}^{n+1}_\nu$. A straightforward computation shows that the mean curvature vector field of M^n_s in $\bar{M}^{n+1}_\nu$ is given by

$$H' = \frac{a + \mathrm{tr}(L) - cnr}{cnp_L(cr)}(Lx - crx),$$

from which we deduce that $\Delta x = Ax$, A being the following matrix

$$A = \frac{cnr - a - \mathrm{tr}(L)}{cp_L(cr)} L + \frac{r\mathrm{tr}(L) + (n+1)ar + cnb}{p_L(cr)} I_{n+2}.$$

We will refer this example as a *quadratic hypersurface*. It is worth noticing that in the above family all possibilities for the shape operator can appear, depending on the sign of $a^2 - 4b$.

At this point, it seems reasonable to ask for non-constant mean curvature hypersurfaces satisfying (‡). In this case, $\mathcal{U} = \{p \in M^n_s : \nabla\alpha^2(p) \neq 0\}$ is a non-empty open set and the equation (46) leads to $< B, x >= 0$ on $\mathcal{U}$. Taking covariant derivative here we deduce B should vanish. From equation (45) we have $< AX, Y >=< X, AY >$ which yields

$$\begin{aligned} &< A\sigma(X, Z), Y > - < A\sigma(Y, Z), X >= \\ &< \sigma(X, Z), AY > - < \sigma(Y, Z), AX > . \end{aligned}$$

Finally, from (45) we obtain

$$TX(\alpha)SY = TY(\alpha)SX, \tag{49}$$

where T denotes the self-adjoint operator defined by $TX = n\alpha X + \varepsilon SX$. This equation is the key to show the following result, [**3**].

Theorem 4.4 *Let $x : M^n_s \longrightarrow \bar{M}^{n+1}_\nu$ be an isometric immersion such that $\Delta x = Ax + B$. Then M^n_s has constant mean curvature.*

Proof. (Outline) We consider two cases.

(A) $T(\nabla\alpha) \neq 0$ on $\mathcal{U}$. Then the shape operator S has rank one on $\mathcal{U}$ and thus we can choose a local orthonormal frame $\{E_1, \ldots, E_n\}$ such that $SE_1 = n\varepsilon\alpha E_1$, $SE_i = 0, i = 2, \ldots, n$. Working on the characteristic polynomials of both A and S it can be deduced that α is a root of a polynomial with constant coefficients and therefore it is locally constant on $\mathcal{U}$, which is a contradiction.

(B) There exists a point $p \in \mathcal{U}$ such that $T(\nabla\alpha)(p) = 0$. Then from (45) we have A is a self-adjoint endomorphism of $\mathbb{R}^{n+2}_t$ and $T(\nabla\alpha) = 0$ on $\mathcal{U}$. Moreover $\nabla\alpha$ is a non-null vector which allows us to take a local orthonormal frame $\{E_1, \ldots, E_n\}$ with E_1 parallel to $\nabla\alpha$. Working again on the characteristic polynomials we obtain the same contradiction as in case (A). ∎

Now we are ready to state the main result of this subsection.

Theorem 4.5 *Let $x : M_s^n \longrightarrow \bar{M}_\nu^{n+1}$ be an isometric immersion. Then $\Delta x = Ax + B$ if and only if one of the following statements holds:*
a) M_s^n is a minimal hypersurface,
b) M_s^n is a totally umbilical hypersurface,
c) M_s^n is one of the standard pseudo-Riemannian products.
d) M_s^n is a quadratic hypersurface with non-diagonalizable shape operator.

Proof. We know α is constant. If $\alpha = 0$ there is nothing to prove, so we can assume $\alpha \neq 0$. When S is diagonalizable, we have seen that either (b) or (c) holds. Otherwise, from (46) and (47) we get

$$A^2 - (\varepsilon \mathrm{tr}(S^2) + cn)A + n\varepsilon c(\mathrm{tr}(S^2) - n\alpha^2)I_{n+2} = 0,$$

and therefore the minimal polynomial $p_A(t)$ of A is given by $p_A(t) = t^2 + at + b$, where $a = -(\varepsilon \mathrm{tr}(S^2) + cn)$ and $b = n\varepsilon c(\mathrm{tr}(S^2) - n\alpha^2)$. From (45) we know A is a self-adjoint endomorphism of $\mathbb{R}_t^{n+2}$ and $< Ax, x >$ is constant on M_s^n. Hence, it is an open piece of a quadratic hypersurface with non-diagonalizable S. ∎

Bibliography

[1] L. J. Alías, A. Ferrández and P. Lucas. Submanifolds in pseudo-Euclidean spaces satisfying the condition $\Delta x = Ax + B$. *Geom. Dedicata*, **42** (1992), 345–354.

[2] L. J. Alías, A. Ferrández and P. Lucas. Surfaces in the 3-dimensional Lorentz-Minkowski space satisfying $\Delta x = Ax + B$. *Pacific J. Math.*, **156** (1992), 201–208.

[3] L. J. Alías, A. Ferrández and P. Lucas. Hypersurfaces in space forms satisfying the condition $\Delta x = Ax + B$. Trans. Amer. Math. Soc. 347 (1995), 1793–1801.

[4] M. Barros, A. Ferrández and P. Lucas. Hipersuperficies esféricas de tipo 2. Publicaciones del Departamento de Matemáticas (Trabajo de Investigación) 1, Universidad de Murcia, 1991. In English.

[5] E. Cartan. Sur des familles remarquables d'hypersurfaces isoparamétriques dans les espaces sphériques. *Math. Z.*, **45** (1939), 335–367.

[6] T. Cecil and P. Ryan. Focal sets, taut embeddings and the cyclides of Dupin. *Math. Ann.*, **236** (1978), 177–190.

[7] T. Cecil and P. Ryan. *Tight and Taut Immersions of Manifolds*, volume 107 of *Research Notes in Math.* Pitman, London, 1985.

[8] B. Y. Chen. Some open problems and conjectures on finite type submanifolds. Preprint (revised version, 1989).

[9] B. Y. Chen. *Total Mean Curvature and Submanifolds of Finite Type*, volume 1 of *Series in Pure Math.* World Scientific, Singapur, 1984.

[10] B. Y. Chen. Finite-type pseudo-Riemannian submanifolds. *Tamkang J. of Math.*, **17** (1986), 137–151.

[11] B. Y. Chen. Some open problems and conjectures on submanifolds of finite type. *Soochow J. Math.*, **17** (1991), 169–188.

[12] B. Y. Chen, M. Barros and O. J. Garay. Spherical finite type hypersurfaces. *Algebras, Groups and Geometries*, **4** (1987), 58–72.

[13] B. Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken. Submanifold of restricted type. Preprint, 1990.

[14] B. Y. Chen and S. Ishikawa. Biharmonic surfaces in pseudo-euclidean spaces. Special issue dedicated to Prof. T. Otsuki on the occasion of his 75th birthday.

[15] M. Dajczer and K. Nomizu. On flat surfaces in $\mathbb{S}^3_1$ and $\mathbb{H}^3_1$. In *Manifolds and Lie Groups*, pages 71–108. Univ. Notre Dame, Indiana, Birkhäuser, 1981.

[16] F. Dillen, J. Pas and L. Verstraelen. On surfaces of finite type in euclidean 3-space. *Kodai Math. J.*, **13** (1990), 10–21.

[17] I. Dimitric. *Quadric Representation and Submanifolds of Finite Type*. PhD thesis, Michigan State University, 1989.

[18] J. Douglas. A general formulation of the problem of plateau. *Bull. Amer. Math. Soc.*, **36** (1930), p. 50.

[19] J. Douglas. Solution of the problem of plateau. *Trans. Amer. Math. Soc.*, **33** (1931), 263–321.

[20] J. Douglas. The problem of plateau. *Bull. Amer. Math. Soc.*, **39** (1933), 227–251.

[21] A. Ferrández, O. J. Garay and P. Lucas. On a certain class of conformally flat Euclidean hypersurfaces. In Ferus, Pinkall, Simon and Wegner, editors, *Global Differential Geometry and Global Analysis, Berlin 1990*, pages 48–54, 1991. Lecture Notes in Mathematics, n. 1481.

[22] A. Ferrández and P. Lucas. Null finite type hypersurfaces in space forms. *Kodai Math. J.*, **14** (1991), 406–419.

[23] A. Ferrández and P. Lucas. On surfaces in the 3-dimensional Lorentz-Minkowski space. *Pacific J. Math.*, **152** (1992), 93–100.

[24] A. Ferrández and P. Lucas. Classifying hypersurfaces in the Lorentz-Minkowski space with a characteristic eigenvector. Tokyo J. Math. 17 (1994), 447–454.

[25] O. J. Garay. On a certain class of finite type surfaces of revolution. *Kodai Math. J.*, **11** (1988), 25–31.

[26] O. J. Garay. An extension of Takahashi's theorem. *Geometriae Dedicata*, **34** (1990), 105–112.

[27] O. J. Garay and A. Romero. An isometric embedding of the complex hyperbolic space in a pseudo- euclidean space and its applications to the study of real hypersurfaces. *Tsukuba J. Math.*, **14** (1990), 293–313.

[28] L. Graves. Codimension one isometric immersions between Lorentz spaces. *Trans. A.M.S.*, **252** (1979), 367–392.

[29] J. Hahn. Isoparametric hypersurfaces in the pseudo-riemannian space forms. *Math. Z.*, **187** (1984), 195–208.

[30] T. Hasanis and T. Vlachos. Coordinate finite-type submanifolds. *Geom. Dedicata*, **37** (1991), 155–165.

[31] T. Hasanis and T. Vlachos. Spherical 2-type hypersurfaces. *J. of Geom.*, (1991), 82–94.

[32] T. Hasanis and T. Vlachos. Hypersurfaces of $\mathbb{R}^{n+1}$ satisfying $\Delta x = Ax + B$. *J. Austral. Math. Soc., Series A*, **53** (1992), 377–384.

[33] H. Lawson. Local rigidity theorems for minimal hypersurfaces. *Ann. of Math.*, **89** (1969), 187–197.

[34] M. A. Magid. Isometric immersions of Lorentz space with parallel second fundamental forms. *Tsukuba J. Math.*, **8** (1984), 31–54.

[35] M. A. Magid. Lorentzian isoparametric hypersurfaces. *Pacific J. Math.*, **118** (1985), 165–197.

[36] S. Markvorsen. A characteristic eigenfunction for minimal hypersurfaces in space forms. *Math. Z.*, **202** (1989), 375–382.

[37] R. Miyaoka. Compact Dupin hypersurfaces with three principal curvatures. *Math. Z.*, **187** (1984), 433–452.

[38] R. Miyaoka and T. Ozawa. Construction of taut embeddings and Cecil-Ryan conjecture. In *Proc. of Symp. in Diff. Geom.*, 1988.

[39] K. Nomizu. On isoparametric hypersurfaces in the lorentzian space forms. *Japan J. Math.*, **7** (1981), 217–226.

[40] U. Pinkall and G. Thorbergsson. Deformations of Dupin hypersurfaces. *Proc. A.M.S.*, **107** (1989), 1037–1043.

[41] T. Radó. On plateau's problem. *Ann. of Math.*, **31** (1930), 457–469.

[42] T. Radó. The problem of the least area and the problem of plateau. *Math. Z.*, **32** (1930), 763–796.

[43] T. Radó. On the functional of mr. douglas. *Ann. of Math.*, **32** (1931), 785–803.

[44] P. J. Ryan. Homogeneity and some curvature conditions for hypersurfaces. *Tôhoku Math. J.*, **21** (1969), 363–388.

[45] T. Takahashi. Minimal immersions of Riemannian manifolds. *J. Math. Soc. Japan*, **18** (1966), 380–385.

[46] G. Thorbergsson. Dupin hypersurfaces. *Bull. London Math. Soc.*, **15** (1983), 493–498.