Null 2-type hypersurfaces in a Lorentz space

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Canad. Math. Bull. 35 (1992), 354–360

(Partially supported by DGICYT grant PS87-0115)

Abstract

In this paper, under certain hypothesis, we characterize generalized hyperbolic cylinders as the only null 2-type hypersurfaces in a Lorentz space.

1. Introduction

In last years, the problem of characterizing or classifying null 2-type hypersurfaces or, in general, submanifolds appears to be quite interesting. In [3], B.Y. Chen has given a classification theorem of null 2-type surfaces in the Euclidean three-space and he has proved that they are circular cylinders. In a later paper [4], he shows that helical cylinders are the only surfaces of null 2-type and constant mean curvature of the Euclidean four-space. Thus, it seems that cylinders are an important family in the classification of finite type submanifolds and, in particular, null finite type submanifolds. In fact, the authors (jointly with O.J. Garay) [5], have shown that for a particular class of ruled manifolds M^* over a given compact spherical submanifold M, generalized cylinders over a finite type submanifold M are the only finite type manifolds.
In this paper, we discuss a pseudo-Riemannian version of that problem and obtain a result similiar to Chen's. Now, in this context, hyperbolic cylinders play the same role as circular cylinders in the Euclidean case. Actually, in this paper, we generalize this theorem to n-dimensional case and with the additional assumption of having at most two distinct principal curvatures, we prove that a space-like hypersurface of the Lorentz space $\mathbb{R}^{n+1}_1$ is of null 2-type if and only if it is locally isometric to a generalized hyperbolic cylinder.

2. Preliminaries

Let $\mathbb{R}^m_s$ be the m-dimensional pseudo-Euclidean space with metric tensor

$$<,>= -\sum_{j=1}^{s} dx^j \otimes dx^j + \sum_{j=s+1}^{m} dx^j \otimes dx^j,$$

where $(x_1,\ldots,x_m)$ are usual coordinates in $\mathbb{R}^m_s$. $(\mathbb{R}^m_s,<,>)$ is a flat pseudo-Riemannian manifold of signature $(s,m-s)$. When $s=1$, $\mathbb{R}^m_1$ is called the Lorentz space. Let $x: M^n_t \longrightarrow \mathbb{R}^m_s$ be an isometric immersion of a connected n-dimensional manifold M^n_t of index t in $\mathbb{R}^m_s$. We represent by Δ the Laplacian operator of M^n_t (with respect to the induced metric) acting on the space of smooth functions $\mathcal{C}^\infty(M^n_t)$. The manifold M^n_t is said to be of k-type if its position vector x can be decomposed in the following form:

$$x = x_0 + x_{i_1} + \cdots + x_{i_k},$$

where

$$\Delta x_{i_j} = \lambda_{i_j} x_{i_j},$$

$\lambda_{i_1} < \cdots < \lambda_{i_k}$, x_0 is a constant vector in $\mathbb{R}^{\mathrm{m}}_{\mathrm{s}}$ and Δ is the extension of the Laplace operator to $\mathbb{R}^{\mathrm{m}}$-valued smooth functions on M_t^n in a natural way. The manifold is said to be of finite type if it is of k-type for some natural number k; otherwise, it is said to be of infinite type. When some $\lambda_{i_j} = 0$ then M_t^n is called of null k-type or null finite type.

If M_t^n is of finite type, for example of k-type, from (1) there exists a monic polynomial, say $Q(t)$, such that $Q(\Delta)(x - x_0) = 0$. If we suppose that $Q(t) = t^k + d_1 t^{k-1} + \cdots + d_{k-1} t + d_k$ then, by the formula $\Delta x = -nH$, where H is the mean curvature vector field of M_t^n in $\mathbb{R}^{\mathrm{m}}_{\mathrm{s}}$, we have the following differential equation:

$$\Delta^{k-1} H + d_1 \Delta^{k-2} H + \cdots + d_{k-1} H - \frac{d_k}{n}(x - x_0) = 0.$$

We note that $d_k = 0$ when the manifold is of null k-type and therefore (3) only contains terms involving the mean curvature vector H. For the general knowledge on finite type submanifolds in pseudo-Euclidean spaces, see for instance [1, 2].

If M_t^n is a hypersurface of the Lorentz space, then either $t = 0$ and we will write M^n by M_0^n, M^n inherits a Riemannian metric and M^n is said to be a space-like hypersurface; or $t = 1$, the induced metric on M_1^n is Lorentzian and M_1^n is said to be a Lorentzian hypersurface. Throught this paper we will deal only with space-like hypersurfaces. To fix the notation will be used later on, let $\bar{\nabla}$ be the flat connection on the Lorentz space and let ∇ be the Levi-Civita connection on the hypersurface. Let X, Y be two vector fields tangent to M^n and ξ a vector field normal to M^n. The second fundamental form σ of M^n acting on X and Y is defined as the normal component of $\bar{\nabla}_X Y$ and the Weingarten map A_ξ in the direction of ξ as $A_\xi X = -(\bar{\nabla}_X \xi)$. The well known relation between σ and A_ξ is given by

$$< A_\xi X, Y > = < \sigma(X, Y), \xi > .$$

Let N be a unit vector field normal to M^n and write A by A_N. Then since M^n is space like, $< N, N > = -1$, i.e., N is time-like, and therefore

$$\sigma(X, Y) = - < \sigma(X, Y), N > N = - < AX, Y > N.$$

We notice that the mean curvature vector field H of M^n is defined as $\frac{1}{n}\mathrm{tr}(\sigma)$, that can also be written as $H = -\frac{1}{n}\mathrm{tr(A)N}$, so that the function α given by $\alpha = -\frac{1}{n}\mathrm{tr(A)}$ is the mean curvature of M^n (in the direction of N) and we usually write $H = \alpha N$.

Let $\{E_1, \ldots, E_n, E_{n+1}\}$ be an adapted local orthonormal frame of the Lorentz space, i.e., $\{E_1, \ldots, E_n\}$ is a local orthonormal frame tangent to M^n and E_{n+1} is a unit time-like vector field normal to M^n. Let ω^i be the 1-forms defined by $\omega^i(Z) = < E_i, Z >$, for $i = 1, \ldots, n+1$, and any vector field Z on the Lorentz space. As it is well known $\omega^{n+1} = 0$ on the hypersurface. Now the connection 1-forms $\{\omega_i^j\}$, $i, j = 1, \ldots, n+1$ are defined by means of the expression

$$\bar{\nabla}_X E_i = \sum_{j=1}^{n+1} \omega_i^j(X) E_j.$$

An easy and standard computation yields to structure equations

$$\begin{aligned} \mathrm{d}\omega^i &= -\sum_{k=1}^{n} \omega_k^i \wedge \omega^k, \qquad i = 1, \ldots, n \\ \mathrm{d}\omega_i^j &= -\sum_{k=1}^{n+1} \omega_k^j \wedge \omega_i^k, \qquad i, j = 1, \ldots, n+1, \end{aligned}$$

with $\omega_i^j = -\omega_j^i$ for $i, j = 1, \ldots, n$, and $\omega_i^{n+1} = \omega_{n+1}^i$ for $i = 1, \ldots, n$.

3. Basic Results

In order to make a study of null 2-type hypersurfaces of the Lorentz space, we start with a formula for ΔH.

Lemma 3.1 *Let $x : M^n \longrightarrow \mathrm{IR}_1^{n+1}$ be a space-like orientable hypersurface. Then*

$$\Delta H = 2A(\nabla\alpha) - \frac{n}{2}\nabla\alpha^2 + \{\Delta\alpha - \alpha|A|^2\}N,$$

where N is a global unit normal vector field and $|A|^2$ stands for tr(A^2).

Proof. Let p be in M^n, $\{E_1, \ldots, E_n\}$ a local orthonormal frame tangent to M^n such that $\nabla_{E_i} E_j(p) = 0$. From the formula

$$\bar{\nabla}_{E_i} \bar{\nabla}_{E_i} H = E_i E_i(\alpha) N - 2E_i(\alpha) A E_i - \alpha\{(\nabla_{E_i} A)E_i + \sigma(AE_i, E_i)\}$$

we have

$$\Delta H = 2A(\nabla\alpha) + \alpha\,\mathrm{tr}(\nabla A) + \{\Delta\alpha - \alpha|A|^2\}N,$$

where $\mathrm{tr}(\nabla A) = \sum_{i=1}^{n} (\nabla_{E_i} A)E_i$.

To compute $\mathrm{tr}(\nabla A)$, let $\{X_1, \ldots, X_n\}$ be the local orthonormal frame of eigenvectors of the Weingarten map, i.e., $AX_i = \mu_i X_i$. Then, using the well-known connection equations, we have

$$\mathrm{tr}(\nabla A) = \sum_{j=1}^{n} X_j(\mu_j) X_j + \sum_{i \neq j} (\mu_i - \mu_j)\omega_i^j(X_i) X_j.$$

Now, from Codazzi's equation $(\nabla_{X_i} A)X_j = (\nabla_{X_j} A)X_i$, one gets

$$X_j(\mu_i) = (\mu_i - \mu_j)\omega_i^j(X_i).$$

Then, since $\alpha = -\frac{1}{n}\sum_{i=1}^{n} \mu_i$, we obtain

$$\mathrm{tr}(\nabla A) = -n\nabla\alpha$$

and the lemma follows from here and (1).

Now, if M^n is of 2-type, i.e., $\Delta H = bH + cx$ (where we assume without loss of generality that x_0 is the origin of $\mathbb{R}^{n+1}_1$), from the above lemma we have

$$cx = 2A(\nabla\alpha) - \frac{n}{2}\nabla\alpha^2 + \{\Delta\alpha - \alpha|A|^2 - b\alpha\}N.$$

This formula allow us to get the following easy and interesting consequence.

Corollary 3.2 *If M^n is a space-like hypersurface of null 2-type in the Lorentz space, then*

$$A(\nabla\alpha^2) = \frac{n}{2}\alpha\nabla\alpha^2$$

in the open set $\mathcal{V} = \{p \in M : \nabla\alpha^2(p) \neq 0\}$.

The problem of characterizing space-like hypersurfaces of null 2-type does not seem an easy task without additional hypothesis. The constancy of the mean curvature does not even provide, in principle, enough information to get such characterization. Nevertheless, we have the following result. Let $\mathcal{C}$ denote the family of space-like hypersurfaces of the Lorentz space with at most two distinct principal curvatures.

Proposition 3.3 *Let $M^n \in \mathcal{C}$. Then M^n is of null 2-type and constant mean curvature if and only if it is locally isometric to a hyperbolic cylinder $\mathbb{R}^p \times H^{n-p}(r)$.*

Proof. If M^n is of null 2-type and has constant mean curvature, by using (5) we have $|A|^2$ is a constant. Furthermore, the hypothesis on principal curvatures yields to M^n has exactly two constant principal curvatures. From [6, Section 4], M^n is an open piece of $\mathbb{R}^p \times H^{n-p}(r)$. The converse is trivial.
In the proof of the above proposition, it has been crucial to deduce that M^n is isoparametric and for that to be possible we have needed the hypothesis on principal curvatures. To get down to work in a more general situation we need a previous lemma.

Lemma 3.4 *Let $M^n \in \mathcal{C}$. Then $\mathcal{V}$ is empty or, at the points of $\mathcal{V}$, $\frac{n}{2}\alpha$ is a principal curvature with multiplicity one.*

Proof. Let us suppose $\mathcal{V}$ is not empty. At the points of $\mathcal{V}$, by using Corollary 3.2, $\frac{n}{2}\alpha$ is a principal curvature with associated principal direction $\nabla\alpha^2$. Let V_1 be a connected component of $\mathcal{V}$. Then V_1 is not empty and on V_1 there are exactly two distinct principal curvatures, say $\mu_1 = \frac{n}{2}\alpha$ and μ_2. Choose the local orthonormal frame $\{E_1, \ldots, E_n\}$ of principal directions such that E_1 is parallel to $\nabla\alpha^2$ and let $D_i = \{X \in T\mathcal{V} : AX = \mu_i X\}$, $i = 1, 2$, be the distribution associated with the eigenvalue μ_i, which is differentiable in the open set V_1. If we assume dim$D_1 > 1$, we can take two linearly independent vector fields X and Y in D_1 and working as in the proof of [7, Proposition 2.3] we have $(A - \mu_1)[X, Y] = X(\mu_1)Y - Y(\mu_1)X$. The hypothesis on the principal curvatures implies that $p(t) = (t - \mu_1)(t - \mu_2)$ is the minimal polynomial of A and therefore the left hand side of that equation lies in D_1 and D_2. But $D_1 \cap D_2 = \{0\}$, so $X(\mu_1) = 0$ for any vector field $X \in D_1$. In particular, $E_1(\alpha) = 0$ on V_1, so that being E_1 and $\nabla\alpha^2$ parallel, we get α is a constant on V_1, which is a contradiction. Therefore, dim$D_1 = 1$ and $\frac{n}{2}\alpha$ has multiplicity one.

4. The Characterization Theorem

This section is devoted to prove the following theorem.

Theorem 4.1 *Let $M^n \in \mathcal{C}$. Then M^n is of null 2-type if and only if it is locally isometric to a hyperbolic cylinder* $\mathbb{R}^p \times H^{n-p}(r)$.

Proof. Suppose M^n is a space-like hypersurface. Our goal is to prove that M^n has constant mean curvature. If α were not constant, then by the Lemma 3.4 we have that $\mathcal{V}$ is not empty and the vector $\nabla\alpha^2$ is an eigenvector of A corresponding to the eigenvalue $\frac{n}{2}\alpha$ with multiplicity 1. Choose a local orthonormal frame $\{E_1, \ldots, E_{n+1}\}$, in an open set of $\mathcal{V}$, satisfying that $\{E_1, \ldots, E_n\}$ are eigenvectors of A, E_1 is parallel to $\nabla\alpha^2$ and E_{n+1} is normal to M.
Now by hypothesis $\Delta H = bH$ so that from Lemma 3.1 we have

$$\Delta\alpha = (b + |A|^2)\alpha; \qquad A(\nabla\alpha) - \frac{n}{2}\alpha\nabla\alpha = 0.$$

Let $\{\omega^1, \ldots, \omega^n\}$ and $\{\omega_i^j\}, i, j = 1, \ldots, n+1$, the dual frame and the connection forms of the chosen frame. Then we have

$$\omega_{n+1}^1 = -\frac{n}{2}\alpha\omega^1; \qquad \omega_{n+1}^j = \frac{3}{2}\frac{n}{n-1}\alpha\omega^j, j = 2, \ldots, n.$$

$$d\alpha = E_1(\alpha)\omega^1.$$

From the first equation of (2) we have

$$d\omega_{n+1}^1 = -\frac{n}{2}\alpha d\omega^1.$$

Using now the second equation of (2) and the structure equations, one has

$$d\omega_{n+1}^1 = \frac{3}{2}\frac{n}{n-1}\alpha d\omega^1.$$

These two last equations mean that

$$d\omega^1 = 0.$$

Therefore one locally has $\omega^1 = du$, for a certain function u, which along with (3) imply that $d\alpha \wedge du = 0$. Thus α depends on u, $\alpha = \alpha(u)$. Then $d\alpha = \alpha' du = \alpha'(u)\omega^1$ and so $E_1(\alpha) = \alpha'$.
Taking differentiation in the second equation of (2) we have

$$d\omega_{n+1}^j = \frac{3}{2}\frac{n}{n-1}\alpha'\omega^1 \wedge \omega^j + \frac{3}{2}\frac{n}{n-1}\alpha d\omega^j,$$

and, also by the structure equations:

$$d\omega_{n+1}^j = \frac{3}{2}\frac{n}{n-1}\alpha d\omega^j + \frac{n(n+2)}{2(n-1)}\alpha\omega_1^j \wedge \omega^1.$$

From both equations we get

$$\{(n+2)\alpha\omega_1^j + 3\alpha'\omega^j\} \wedge \omega^1 = 0.$$

Then we can write

$$(n+2)\alpha\omega_1^j + 3\alpha'\omega^j = f\omega^1,$$

where f is the function given by

$$f = (n+2)\alpha\omega_1^j(E_1).$$

Now from (6) and the structure equations we find

$$0 = \mathrm{d}\omega^1(E_1, E_j) = -\frac{1}{2}\omega_j^1(E_1),$$

and thus $f = 0$. Consequently (9) implies that

$$(n+2)\alpha\omega_1^j + 3\alpha'\omega^j = 0,\ j = 2, \dots, n.$$

Now, differentiating (10) we have

$$(n+2)\{\alpha'\omega^1 \wedge \omega_1^j + \alpha d\omega_1^j\} + 3\{\alpha''\omega^1 \wedge \omega^j + \alpha' d\omega^j\} = 0,$$

and using (2), (10) and the structure equations we get

$$d\omega_1^j = -\frac{3}{n+2}\frac{\alpha'}{\alpha}d\omega^j - \left\{\frac{9}{(n+2)^2}\frac{(\alpha')^2}{\alpha^2} + \frac{3}{4}\frac{n^2}{n-1}\alpha^2\right\}\omega^1 \wedge \omega^j.$$

Bringing (12) into (11) we find

$$\left[3\alpha'' - \frac{3(n+5)}{n+2}\frac{(\alpha')^2}{\alpha} - \frac{3n^2(n+2)}{4(n-1)}\alpha^3\right]\omega^1 \wedge \omega^j = 0,$$

and therefore we have the following differential equation

$$4\alpha\alpha'' - \frac{4(n+5)}{n+2}(\alpha')^2 - \frac{n^2(n+2)}{n-1}\alpha^4 = 0.$$

If we put $y = (\alpha')^2$, the above equation turns into

$$2\alpha\frac{dy}{d\alpha} - \frac{4(n+5)}{n+2}y = \frac{n^2(n+2)}{n-1}\alpha^4,$$

and then

$$y = (\alpha')^2 = C\alpha^{\frac{2(n+5)}{n+2}} + \left(\frac{n(n+2)}{2(n-1)}\right)^2\alpha^4,$$

with C a constant.
Now we use the definition of $\Delta\alpha$, the fact that E_1 is parallel to $\nabla\alpha^2$ and equation (10) to obtain

$$(n+2)\alpha\Delta\alpha = -(n+2)\alpha\alpha'' + 3(n-1)(\alpha')^2.$$

Since $|A|^2 = \dfrac{n^2(n+8)}{4(n-1)}\alpha^2$, combining (16) and the first equation of (1), we have

$$\alpha\alpha'' - \frac{3(n-1)}{(n+2)}(\alpha')^2 + \left(b + \frac{n^2(n+8)}{4(n-1)}\alpha^2\right)\alpha^2 = 0.$$

Thus, putting together (13) and (17) one has

$$\frac{2(n-4)}{n+2}(\alpha')^2 = \frac{n^2(n+5)}{2(n-1)}\alpha^4 + b\alpha^2.$$

We deduce, using (15) and (18) that α is locally constant on $\mathcal{V}$, which is a contradiction with the definition of $\mathcal{V}$. Hence α is constant on M^n and the result follows from Proposition 3.3. The converse is trivial and the proof finishes.

Remark 4.2 If $\mathbb{R}^{n+1}_1$ is the Euclidean-space $\mathbb{R}^3$, Theorem 4.1 has been proved by B.Y. Chen in [3].

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