Finite Type Surfaces of Revolution

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Abstract

In this paper, it is shown that the plane, the sphere and the circular cylinder are the only finite type surfaces of revolution whose generating curves are also of finite type.

0. Introduction.

The research into Riemannian submanifolds has grown up since 1983, when B.-Y.Chen introduces the notion of Euclidean immersion of finite type [1], [2], and from that time on it has become a useful tool in the study of submanifolds, as one can see in the recent literature in this field. That concept is the natural extension of minimal submanifolds on which many mathematicians have devoted in the last years.
Although the rigorous definition will be given in the next section, the finite type submanifolds are those whose immersion into Euclidean space E^m is constructed with a finite number of E^m-valuated eigenfunctions of their Laplacian. The simplest immersions of this kind are built with one or two eigenfunctions getting the 1-type and 2-type submanifolds, respectively. For example, when the eigenfunction is associated to the zero eigenvalue we find a minimal submanifold. Nevertheless, as far as we know all papers concern with spherical submanifolds and recently that condition is going to be removed, as one can see in papers by Chen [3], Garay [5] and the authors (jointly with Garay) [4].
In this way, B.-Y.Chen, [3], laid out the problem to find finite type surfaces in E^3 other than ordinary spheres and minimal ones. With the aim of getting an answer to that question, he studied an important family of surfaces, the tubes in E^3, finding that the circular cylinders are the only tubes of finite type. One year later, Garay, [5], pays attention in the case of complete surfaces of revolution whose coordinate functions are eigenfunctions of their Laplacian. Observe that this condition implies that the surface is at most of 3-type. Then Garay finds that the plane, the catenoid, the sphere and the circular cylinder are the only surfaces of this kind. Now, notice that the type of all of them is less than or equal to two.
At this moment, it may be worth to stand out some differences between the above papers. Whereas Chen considers the finite type tubes with no restriction on the type, Garay studies the surfaces of revolution of type less than three. It seems reasonable to generalize the problem in order to poke about finite type surfaces of revolution with no restriction on the type. The problem so stated does not look easy to solve and in a first attempt we are going to assume that the generating curve is also of finite type, even if it needs not to be the same type as the surface. Indeed, in this paper we ask the following geometric question:

PROBLEM: “*To what extent the finite type character of a plane curve affects the finite type condition of the surface of revolution built on it?*”

Evidently, neither a surface of revolution of finite type needs to be generated by a curve of finite type, nor a curve of finite type has to generate a surface of revolution of finite type. To show that we will consider the following examples giving all posibilities:

(a) a minimal surface, the catenoid, generated by a curve of infinite type, the catenary (see section 2);
(b) a surface of 1-type, the ordinary sphere, generated by a curve of 1-type, the circle;
(c) a surface of infinite type, the cone, generated by a straight line which is minimal (see section 3); and
(d) a surface of infinite type, the ellipsoid, generated by a curve of infinite type, the ellipse.
In order to solve that problem, we first study finite type curves in E^2 and we show that a plane circle (or an open piece of a plane circle) and a straight line are the only plane finite type curves. From that result one obtains that the circle is the only closed curve of finite type in E^2. In a second step, we will prove that a plane, a sphere and a circular cylinder are the only finite type surfaces of revolution whose generating curves are also of finite type. That result is the answer to the problem stated before and leads immediately to the sphere as the only finite type closed surface of revolution which is also generated by a finite type curve. Then one obtains a partial answer to the Chen's conjecture, which states that the sphere is the only closed finite type surface in E^3.
Finally, in view of our Theorem 3.1 (see section 3) we may hazard the following:

> <u>CONJECTURE:</u> *"The minimal surfaces, the ordinary spheres and the circular cylinders are the only surfaces of finite type in E^3."*

Notice that saying yes to our conjecture will imply, in fact, the Chen's conjecture.

1. Preliminaries.

Let $x : M^n \longrightarrow E^m$ be an isometric immersion of a connected (needs not to be compact) n-dimensional Riemannian manifold M^n into the Euclidean space E^m. We represent by Δ the Laplacian operator of M^n (with respect to the induced metric) acting on the space of smooth functions $\mathcal{C}^\infty(M)$. The manifold M^n is said to be of k-type if the position vector x of M^n can be decomposed in the following form:

$$x = x_0 + x_{i_1} + \cdots + x_{i_k},$$

where

$$\Delta x_{i_j} = \lambda_{i_j} x_{i_j},$$

$\lambda_{i_1} < \cdots < \lambda_{i_k}$, x_0 represents the center of mass of M^n in E^m and Δ is the extension of the Laplace operator to E^m-valuated smooth functions on M^n in a natural way. A manifold M^n is said to be of finite type if it is of k-type for some natural number k; otherwise, M^n is said to be of infinite type.
So by regarding the decomposition (1), finite type means that one constructs the immersion x by making use only of a finite number of eigenvalues of M^n. From this point of view, the easiest decomposition (1) corresponds to the manifolds of 1-type, which were characterized by Takahashi, [6], as the minimal submanifold in E^m or minimal in some hypersphere of E^m centered at x_0 and whose radius is determined by the associated eigenvalue giving the 1-type character.

In general, if M^n is a Riemannian manifold of finite type, for example of k-type, from (1) there exists a monic polynomial, say $Q(t)$, such that $Q(\Delta)(x - x_0) = 0$. If we suppose that $Q(t) = t^k + d_1 t^{k-1} + \cdots + d_{k-1} t + d_k$ then coefficients d_i are given by

$$d_1 = -\sum_{t=1}^{k} \lambda_{i_t} \ ; \quad d_2 = \sum_{t<t'} \lambda_{i_t} \lambda_{i_{t'}} \ ; \ \cdots \ ; \ d_k = (-1)^k \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k},$$

where $\{\lambda_{i_1}, \lambda_{i_2} \ldots, \lambda_{i_k}\}$ are the associated eigenvalues giving the k-type character. Therefore, the immersion x satisfies the following equation:

$$\Delta^k x + d_1 \Delta^{k-1} x + \cdots + d_{k-1} \Delta x + d_k (x - x_0) = 0.$$

It is also known that in terms of local coordinates $\{y_1, \ldots, y_n\}$ of M^n the Laplacian can be written as:

$$\Delta = \frac{-1}{\sqrt{g}} \sum_{i,j} \frac{\partial}{\partial y_i} (\sqrt{g} g^{ij} \frac{\partial}{\partial y_j}),$$

where $g = det(g_{ij})$, $(g^{ij}) = (g_{ij})^{-1}$ and (g_{ij}) are the components of the metric of M^n with respect to $\{y_1, \ldots, y_n\}$.
Let us represent by $\alpha(t) = (f(t), 0, g(t)),\ t \in I$, any $\mathcal{C}^\infty$ curve in the xz-plane whose domain of definition I is an open real interval of finite length. The surface of revolution M in E^3 is defined by

$$x(t, \theta) = (f(t)cos\theta, f(t)sin\theta, g(t)),\ t \in I,\ \theta \in (-\pi, \pi).$$

Now we are going to see some examples:

(a) *Sphere.* We consider $f(t) = cost$ and $g(t) = sint,\ t \in (-\pi, \pi)$. Then $\alpha(t)$ is a circle which is of 1-type, and M is the ordinary sphere S^2 which is also of 1-type.
(b) *Cone.* We take $f(t) = at + b$ and $g(t) = ct + d$ with a and c nonzero. Then $\alpha(t)$ is a straight line which is minimal, and so of 1-type, whereas M is a cone which is of infinite type (see section 3 and also [4]).
(c) *Cylinder.* We take $f(t) = b$ and $g(t) = ct + d$. Now $\alpha(t)$ is again a straight line and M is a circular cylinder, which are both of finite type.
(d) *Catenoid.* We consider $f(t) = acosh(t)$ and $g(t) = at$. Then $\alpha(t)$ is a catenary, which is of infinite type (see section 2) and M is a catenoid, which is of 1-type.
(e) *Anchor Ring.* Finally, we consider $f(t) = a + rcost$ and $g(t) = rsint,\ a > r > 0$. In this example, $\alpha(t)$ is a circle and M is an Anchor Ring, which is of infinite type (see [3]).

2. Finite Type Curves in E^2.

Let $\alpha : (0, l) \longrightarrow E^2$ be a parametrization of a smooth unit speed curve C which is topologically imbedded in E^2, and let us suppose, without loss of generality, that the center of mass of C is the origen of E^2. Finally, since we want to find out plane finite type curves, let us assume C is of k-type.
On the other hand, because C is parametrized by arc length s, one has $\Delta = -\frac{d^2}{ds^2}$, where Δ denotes the Laplacian operator of C. Now, if we put $\alpha(s) = (\alpha_1(s), \alpha_2(s))$, an easy computation yields:

$$\Delta^j \alpha_i = (-1)^j \frac{d^{(2j)} \alpha_i}{ds^{(2j)}},\ i = 1, 2,\ j = 1, \ldots, k.$$

Now, we may use (3) to obtain the following ordinary differential equation:

$$(-1)^k \frac{d^{(2k)}\alpha_i}{ds^{(2k)}} + (-1)^{k-1} d_1 \frac{d^{(2k-2)}\alpha_i}{ds^{(2k-2)}} + \cdots - d_{k-1}\frac{d^2\alpha_i}{ds^2} + d_k\alpha_i = 0,$$

whose characteristic polynomial equation is:

$$(-\mu^2)^k + d_1(-\mu^2)^{k-1} + \cdots + d_{k-1}(-\mu^2) + d_k = 0,$$

i.e., $Q(-\mu^2) = 0$, being Q the unique monic polynomial giving the k-type character. Since the roots of Q are the eigenvalues appearing in (2), one knows that the solutions of (2) are given by

$$\alpha_i(s) = a_i s + b_i + \sum_{t=1}^{p}\{F_{it}cos(\mu_t s) + G_{it}sin(\mu_t s)\} + \\ + \sum_{t=1}^{q}\{A_{it}e^{\xi_t s} + B_{it}e^{-\xi_t s}\},\ i = 1, 2, \tag{3}$$

where $a_i,\ b_i,\ F_{it},\ G_{it},\ \mu_t$ and ξ_t are constant (recall that μ_t are the square roots of the positive eigenvalues and ξ_t are the square roots of the opposite sign negative eigenvalues).
Then we have the following:

Proposition 2.1 *Let α be a smooth curve of finite type in E^2. Then:*

$$\alpha(s) = As + B + \sum_{t=1}^{p}\beta_t(s) + \sum_{t=1}^{q}\gamma_t(s),$$

where $\beta_t(s) = F_t cos(\mu_t s) + G_t sin(\mu_t s)$, $\gamma_t(s) = A_t e^{\xi_t s} + B_t e^{-\xi_t s}$, F_t, G_t, A_t, B_t, A and B are vectors in E^2.

Now we wish to prove that the only finite type curves are of the 1-type ones. To do so, we can assume that $0 < \mu_1 < \cdots < \mu_p,\ \ 0 < \xi_1 \cdots < \xi_q$ and the pairs (F_t, G_t) and (A_t, B_t) do not vanish. First, the functions $\gamma_t(s)$ can be rewritten as:

$$\gamma_t(s) = C_t cosh(\xi_t s) + D_t sinh(\xi_t s),$$

where vectors $C_t = A_t + B_t$ and $D_t = A_t - B_t$.
Using now that $< \alpha', \alpha' >= 1$, bearing in mind the linear independence of circular and hyperbolic functions, and the mixed products between them, we pay attention, after an standard and messy computation, to the components in $cosh$ to get

$$\begin{aligned} &4\sum_{t=1}^{q}\xi_t < A, D_t > \cosh(\xi_t s) + \sum_{t=1}^{1}\xi_t^2\{|C_t|^2 + |D_t|^2\}\cosh(2\xi_t s) \\ &+2\sum_{t'<t}\xi_t\xi_{t'}\{< C_t, C_{t'} > + < D_t, D_{t'} >\}\cosh(\xi_t + \xi_{t'})s \\ &+2\sum_{t'<t}\xi_t\xi_{t'}\{- < C_t, C_{t'} > + < D_t, D_{t'} >\}\cosh(\xi_t - \xi_{t'})s \\ &\equiv \sum_{\xi\in\Lambda} M_\xi \cosh(\xi s) = 0, \end{aligned} \tag{5}$$

where $\Lambda = \{\xi_t : 1 \leqslant t \leqslant q\} \cup \{\xi_t + \xi_{t'} : 1 \leqslant t' \leqslant t \leqslant q\} \cup \{\xi_t - \xi_{t'} : 1 \leqslant t' < t \leqslant q\}$.
Take $\xi = 2\xi_q$ in (5) to find

$$0 = \xi_q^2 \{|C_q|^2 + |D_q|^2\} = M_{2\xi_q},$$

from which we have $|C_q| = |D_q| = 0$. Therefore $q = 0$. A similar argument, jointly with that in [1, Theorem 5.3, p. 289], shows that $p \leqslant 1$. If $p = 0$, then α is a straight line. If $p = 1$, then

$$\alpha(s) = As + B + F\cos\mu s + G\sin\mu s,$$

such that F and G do not vanish simultaneously. Writing down again the condition $|\alpha'| = 1$, one gets

$$|F|^2 = |G|^2, \; < F, G >= 0,$$
$$< A, F >=< A, G >= 0.$$

Hence $A = 0$ and therefore $\alpha(s)$ is a circle.
So we have proved the following:

Theorem 2.2 *Let C be a plane curve. If C is of finite type, then one of the following two cases holds:*
(a) C is an open piece of a straight line;
(b) C is an open piece of a plane circle.

From this theorem one immediately obtains the following consequences:

Corollary 2.3 *Let C be a plane curve. If C is of finite type, then C must be either:*
(a) minimal in E^2, and so C is a straight line; or
(b) of 1-type not minimal, and so C is an arc of a circle.

Corollary 2.4 *Let C be a closed plane curve of finite type. Then C is of 1-type and so C is a circle.*

Remark. Corollary 2.2 has been proved by B.-Y.Chen [1].

3. Finite Type Surfaces of Revolution in E^3.

In order to state and prove the main result, it might be interesting recalling the situation set out in $\mathbb{S}^1$. Let $\alpha(t) = (f(t), 0, g(t))$ be a $\mathcal{C}^\infty$ curve in the xz-plane. Then $\alpha(t)$ generates by rotation around the z-axis a connected (not necessarily compact) surface M which is usually giving by

$$M = \{(f(t)cos\theta, f(t)sin\theta, g(t)) : t \in I, \theta \in (-\pi, \pi)\},$$

that we are going to assume of finite type.
If we suppose that the generating curve $\alpha(t)$ is also of finite type, we can use Theorem 2.2 to obtain that $\alpha(t)$ is an open piece of a straight line or an open piece of a circle. We will study separatedly these two posibilities.
CASE 1. Let us suppose that $\alpha(t)$ is an open piece of a circle. Without loss of generality, we may assume that $r > 0$ and $(a, 0, 0)$, $a + r > 0$, are, respectively, its radius and center. Then M can be parametrized by:

$$x(t, \theta) = ((a + rcos\theta)cost, (a + rcos\theta)sint, rsin\theta),$$

where $t \in (-\pi, \pi)$ and $\theta \in (-\beta, \beta)$. Here β is the largest real number in $(-\pi, \pi)$ such that $a + rcos\beta > 0$.
If we use the formula (4), we obtain for $u \in \mathcal{C}^\infty(M)$:

$$\Delta u = \frac{sin\theta}{rf}\frac{\partial u}{\partial \theta} - \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} - \frac{1}{f^2}\frac{\partial^2 u}{\partial t^2},$$

where $f(\theta) = a + rcos\theta$. Let x_1, x_2 and x_3 be the three component functions of the immersion $x(t, \theta)$. Then by using (1) we have:

$$\Delta x_3 = \frac{sin\theta\ cos\theta}{f} + \frac{sin\theta}{r}.$$

Moreover, by a direct computation, we obtain:

$$\Delta^2 x_3 = -\frac{sin^3\theta\ cos\theta}{f^3} - \frac{1}{rf^2}[2sin\theta\ cos^2\theta - sin^3\theta] + \frac{5sin\theta\ cos\theta}{r^2 f} + \frac{sin\theta}{r},$$

which can be rewritten as:

$$\Delta^2 x_3 = -\frac{sin^3\theta\ cos\theta}{f^3} + \frac{1}{f^2}P_2(sin\theta, cos\theta),$$

where $P_2(u, v)$ is a polynomial of degree 3 in u, v.
Now we may use (1) through (3) to prove by induction that

$$\Delta^{n+1} x_3 = (-1)^n \frac{1^2 3^2 \cdots (2n-1)^2 sin^{2n+1}\theta\ cos\theta}{f^{2n+1}} + \frac{1}{f^{2n}}P_{n+1}(sin\theta, cos\theta),$$

for $n \geqslant 1$, where $P_{n+1}(u, v)$ is a polynomial of degree 2n+1 in u, v.
Then, since we are supposing that M is of finite type, from (3) we get:

$$\Delta^k x_3 + d_1 \Delta^{k-1} x_3 + \cdots + d_{k-1}\Delta x_3 + d_k x_3 = 0,$$

for some natural number k and constants d_i.
From (2), (4) and (5) we conclude that there exists some polynomial $S(u, v)$ of degree 2k-1 such that

$$\frac{sin^{2k-1}\theta\ cos\theta}{a + rcos\theta} = S(sin\theta, cos\theta).$$

But that holds good if and only if $a = 0$ or $a = r$. If $a = 0$ then M is the ordinary sphere of radius r centered at the origin of E^3, which is of 1-type. Assume, without loss of generality, $a = r = 1$.
Now, we use (1) to obtain:

$$\Delta x_1 = \left(\frac{cos^2\theta}{f} + cos\theta\right) cost$$

and

$$\Delta^2 x_1 = \left(\frac{cos^4\theta}{f^3} + \frac{1}{f^2}P_2(cos\theta)\right) cost,$$

where $f(\theta) = 1 + cos\theta$ and $P_2(u)$ is a polynomial of degree 3 in u.

Thus we may use (1), (7) and (8) to prove by induction the formula:

$$\Delta^{n+1}x_1 = \left(\frac{cos^{2n+2}\theta}{f^{2n+1}} + \frac{1}{f^{2n}}P_{n+1}(cos\theta)\right)cost,$$

for $n \geqslant 0$, where $P_{n+1}(v)$ is a polynomial of degree 2n+1 in v.
In the same way, we get:

$$\Delta^{n+1}x_2 = \left(\frac{cos^{2n+2}\theta}{f^{2n+1}} + \frac{1}{f^{2n}}P_{n+1}(cos\theta)\right)sint,$$

for $n \geqslant 0$.
At this point, if we use (3) and (7) through (10) we may assert that there exists a polynomial $P(u)$ of degree 2k-1 such that

$$\frac{cos^{2k}\theta}{1+cos\theta} = P(cos\theta),$$

which is a contradiction.
CASE 2. Now, let us suppose that $\alpha(t)$ is an open piece of a straight line. Then M is one of the following three surfaces:
(a) an open piece of a plane;
(b) an open piece of a circular cylinder;
(c) an open piece of a circular cone.
Case (2a) is of 1-type and case (2b) is of 2-type. Now we are going to study the third posibility. Without loss of generality, M can be parametrized by

$$x(t,\theta) = (atcos\theta, atsin\theta, bt),\ a^2+b^2=1,\ a \text{ and } b \text{ nonzero},$$

with $t \in I$ and $\theta \in (-\pi,\pi)$.
If we use the formula (4) we obtain that the Laplacian of M can be written as

$$\Delta = -\left\{\frac{1}{t}\frac{\partial}{\partial t} + \frac{\partial^2}{\partial t^2} + \frac{1}{a^2t^2}\frac{\partial^2}{\partial\theta^2}\right\},$$

and, if x_3 denotes the third coordinate function of M, we get

$$\Delta x_3 = -\frac{b}{t}$$

and

$$\Delta^2 x_3 = \frac{b}{t^3}.$$

Now, we can use (12) and (14) to prove by induction:

$$\Delta^n x_3 = (-1)^n\frac{1^2 3^2\cdots(2n-3)^2 b}{t^{2n-1}},$$

for all natural number $n \geqslant 2$. If we suppose that M is of finite type, there exists a natural number $k \geqslant 1$ and constants d_i, someone nonzero, for which (5) holds good. Then we obtain:

$$(-1)^k\frac{1^2\cdots(2k-3)^2 b}{t^{2k-1}} + d_1(-1)^{k-1}\frac{1^2\cdots(2k-5)^2 b}{t^{2k-3}} + \cdots - d_{k-1}\frac{b}{t} + d_k bt = 0$$

and therefore $b=0$, which is a contradiction.
Then we have shown the following:

Theorem 3.1 *Let M be a surface of revolution in E^3 whose generating curves are finite type curves. If M is of finite type, then M must be a plane, a circular cylinder or a sphere.*

From this result we easily deduce the following consequences:

Corollary 3.2 *Let M be a finite type surface of revolution in E^3 which is also generated by a plane finite type curve. Then M must be either*
(a) minimal in E^3, and so it is a plane; or
(b) of 1-type not minimal, and then it is a sphere; or
(c) of 2-type in E^3, and in this case it will be a circular cylinder.

Corollary 3.3 *Let M be a finite type closed surface of revolution in E^3 which is generated by a finite type curve. Then M is of 1-type and so M is an ordinary sphere.*

Remarks.
(a) Corollary 3.2 can be viewed as a partial answer to Chen's conjecture.
(b) The fact that the cones are of infinite type can be also shown using [4,Theorem 4.1], nevertheless we would rather give an alternative proof with the aim of making a self-contained paper.

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