Hypersurfaces in the non-flat Lorentzian space forms with a characteristic eigenvector field

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1. Introduction

In a series of early papers, with the aim of knowing of the shape of a pseudo-Riemannian hypersurface satisfying a certain differential equation in the induced Laplacian, we found a remarkable family of hypersurfaces in the Lorentz-Minkowski space whose mean curvature vector is an eigenvector of the Laplacian. Actually, the last two authors showed in [**8**] that the equation $\Delta H = \lambda H$, for a real constant λ, characterizes the family of surfaces in $\mathbb{L}^3$ made up of the quite interesting B-scrolls and the so-called standard examples, as well as minimal surfaces.

Looking at those results obtained for surfaces in $\mathbb{L}^3$, the following geometric question was stated in [**9**] for hypersurfaces in $\mathbb{L}^{n+1}$ ($n > 2$): *Does the equation $\Delta H = \lambda H$ mean that both the mean and the scalar curvatures of the hypersurface are constant?* We were able to give a partial solution to that problem, since we had needed to do an additional hypothesis on the degree of the minimal polynomial of the shape operator.

It is worth pointing out that the additional assumption was mainly made to control the position vector field of the hypersurface into $\mathbb{L}^{n+1}$. Now when the ambient space is a non-flat pseudo-Riemannian space form, $\mathbb{S}^{n+1}_\nu(r)$ or $\mathbb{H}^{n+1}_\nu(r)$, then the hypersurface is of codimension two in $\mathbb{R}^{n+2}_\nu$ or $\mathbb{R}^{n+2}_{\nu+1}$, respectively, but $\mathbb{S}^{n+1}_\nu(r)$ and $\mathbb{H}^{n+1}_\nu(r)$ being both totally umbilical hypersurfaces in the corresponding pseudo-Euclidean space, it seems reasonable to hope for a richer classification of hypersurfaces into those spaces by means of the equation $\Delta H = \lambda H$. Or even, one looks for getting a complete answer to the stated problem in non-flat ambient spaces.

In this paper we give a classification of surfaces in the 3-dimensional non-flat Lorentzian space forms satisfying the equation $\Delta H = \lambda H$. We show that the family of such surfaces consists of minimal, totally umbilical and B-scroll surfaces. As for hypersurfaces we suppose that their shape operators have no complex eigenvalues. This condition does not seem as restrictive as one could think, in view of examples and results given in section 5. Actually, we find that family is set up by minimal, totally umbilical and so-called generalized umbilical hypersurfaces, which are nothing but a natural generalization of B-scrolls.

2. Preliminaries

Let $\mathbb{R}^{n+2}_t$ be the $(n+2)$-dimensional pseudo-Euclidean space with index t endowed with the indefinite inner product given by

$$\langle x, y\rangle = -\sum_{i=1}^{t} x_i y_i + \sum_{j=t+1}^{n+2} x_j y_j,$$

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where $(x_1, \ldots, x_{n+2})$ is the usual coordinate system. As is usual, let $\mathbb{S}^{n+1}_\nu = \{x \in \mathbb{R}^{n+2}_\nu : \langle x, x\rangle = 1\}$ and $\mathbb{H}^{n+1}_\nu = \{x \in \mathbb{R}^{n+2}_{\nu+1} : \langle x, x\rangle = -1\}$ be the unit pseudo-sphere and the unit pseudo-hyperbolic space, respectively. They are pseudo-Riemannian hypersurfaces of index ν in $\mathbb{R}^{n+2}_\nu$ and $\mathbb{R}^{n+2}_{\nu+1}$, respectively, with constant sectional curvature $c = +1$ and $c = -1$, respectively. Throughout this paper, $\bar{M}^{n+1}_\nu$ will denote the unit pseudo-sphere or the unit pseudo-hyperbolic space, according to $c = +1$ or $c = -1$, and $\mathbb{R}^{n+2}_t$ will stand for the corresponding pseudo-Euclidean space where $\bar{M}^{n+1}_\nu$ is lying.

Let $x : M^n_s \longrightarrow \bar{M}^{n+1}_\nu \subset \mathbb{R}^{n+2}_t$ be an isometric immersion of a pseudo-Riemannian hypersurface M^n_s in $\bar{M}^{n+1}_\nu$. Let ∇, $\overline{\nabla}$ and $\tilde{\nabla}$ be the Levi-Civita connection on M^n_s, $\bar{M}^{n+1}_\nu$ and $\mathbb{R}^{n+2}_t$, respectively, and let H and H' denote the mean curvature vector fields of M^n_s in $\mathbb{R}^{n+2}_t$ and $\bar{M}^{n+1}_\nu$, respectively. Thus we may write

$$H = H' - cx = \alpha N - cx,$$

N and α being a unit vector field normal to M^n_s in $\bar{M}^{n+1}_\nu$ and the mean curvature with respect to N, respectively.

Let us write down the Laplacian operator Δ on M^n_s , extended in a natural way to $\mathbb{R}^{n+2}$-valuated functions, as

$$\Delta = -\sum_{i=1}^{n} \varepsilon_i \left(\tilde{\nabla}_{E_i} \tilde{\nabla}_{E_i} - \tilde{\nabla}_{\nabla_{E_i} E_i} \right),$$

where $\{E_1, \ldots, E_n\}$ is a local orthonormal frame on M^n_s and $\varepsilon_i = \langle E_i, E_i\rangle$. Then the formula below is well known

$$\Delta x = -nH,$$

as well as the following one, which is given in [**1**]:

$$\Delta H = 2S(\nabla\alpha) + n\varepsilon\alpha\nabla\alpha + \{\Delta\alpha + \varepsilon\alpha \mathrm{tr}(S^2) + nc\alpha\}N - nc(c + \varepsilon\alpha^2)x,$$

where S denotes the shape operator of the hypersurface M^n_s in $\bar{M}^{n+1}_\nu$, $\varepsilon = \langle N, N\rangle$ and $\nabla\alpha$ stands for the gradient of α.

3. First characterization results

Let M^n_s be a hypersurface in $\bar{M}^{n+1}_\nu$ satisfying the condition

$$\Delta H = \lambda H,$$

for a real constant λ. Then from equations (1) and (3) we see that the above condition is equivalent to the following set of equations:

$$\begin{aligned} &2S(\nabla\alpha) + n\varepsilon\alpha\nabla\alpha = 0, \\ &\Delta\alpha + \alpha(\varepsilon \mathrm{tr}(S^2) + nc - \lambda) = 0, \\ &n(c + \varepsilon\alpha^2) - \lambda = 0. \end{aligned}$$

Therefore we obtain that $\lambda = n(c + \varepsilon\alpha^2) = n\langle H, H\rangle$ and α is a constant satisfying $\alpha(\mathrm{tr}(S^2) - n\alpha^2) = 0$. Then if $\alpha \neq 0$ we deduce that $\mathrm{tr}(S^2) = n\alpha^2 = (1/n)\mathrm{tr}(S)^2$. Thus we have proved the following proposition.

Proposition 3.1 *Let M_s^n be a hypersurface in $\overline{M}_\nu^{n+1}$. Then $\Delta H = \lambda H$ if and only if one of the following statements holds:*

(1) *M_s^n is minimal in $\overline{M}_\nu^{n+1}$.*

(2) *M_s^n has nonzero constant mean curvature α and tr(S^2)=($1/n$)tr(S)^2.*

Moreover, the constant λ is always given by $\lambda = n\langle H, H\rangle = n(c + \varepsilon\alpha^2)$.

The statement (2) in the above proposition hides a remarkable fact. Indeed, from the Gauss equation one knows that

$$\tau = n^2\langle H, H\rangle - nc - \varepsilon \mathrm{tr}(S^2),$$

where τ stands for the scalar curvature. Then $\mathrm{tr}(S^2) = (1/n)\mathrm{tr}(S)^2$ if and only if $\tau = n(n-1)\langle H, H\rangle$, which shows that the constant λ is surprisingly an intrinsic quantity, because one has $\tau = (n-1)\lambda$. Therefore the Proposition 3.1 can be rephrased as follows.

Proposition 3.2 *Let M_s^n be a non-minimal hypersurface in $\overline{M}_\nu^{n+1}$. Then $\Delta H = \lambda H$ if and only if M_s^n has both constant mean and constant scalar curvatures, and they are related by the equation $\tau = n(n-1)(c + \varepsilon\alpha^2)$.*

A first interesting consequence of Proposition 3.1 is based on the following fact. If we suppose that M_s^n has diagonalizable shape operator, then $\mathrm{tr}(S^2) = (1/n)\mathrm{tr}(S)^2$ if and only if M_s^n is a totally umbilical hypersurface in $\overline{M}_\nu^{n+1}$. Then bearing in mind the classification theorem for such hypersurfaces given in [**10**, Theorem 1.4], we know that M_s^n is an open piece of either a pseudo-sphere $\mathbb{S}_s^n(r)$, or a pseudo-hyperbolic space $\mathbb{H}_s^n(r)$ or $\mathbb{R}_s^n$, according to $\langle H, H\rangle$ is positive, negative or zero, respectively. In the last case, the isometric immersion $x : \mathbb{R}_s^n \longrightarrow \overline{M}_\nu^{n+1} \subset \mathbb{R}_{s+1}^{n+2}$ is given by $x = f - x_0$, x_0 being a fixed vector in $\mathbb{R}_{s+1}^{n+2}$ and $f : \mathbb{R}_s^n \longrightarrow \mathbb{R}_{s+1}^{n+2}$ the function defined by $f(u_1, \ldots, u_n) = (q(u), u_1, \ldots, u_n, q(u))$, where $q(u) = a\langle u, u\rangle + \langle b, u\rangle + c$, $a \neq 0$. We will refer this example as a *flat totally umbilical hypersurface*. Now the following result is clear.

Proposition 3.3 *Let M_s^n be a non-minimal hypersurface in $\overline{M}_\nu^{n+1}$ with diagonalizable shape operator. Then $\Delta H = \lambda H$ if and only if one of the following statements holds:*

(1) *$\lambda > 0$ and M_s^n is an open piece of a pseudo-sphere $\mathbb{S}_s^n(\sqrt{n/\lambda})$.*

(2) *$\lambda < 0$ and M_s^n is an open piece of a pseudo-hyperbolic space $\mathbb{H}_s^n(\sqrt{-n/\lambda})$.*

(3) *$\lambda = 0$ and M_s^n is an open piece of a flat totally umbilical hypersurface.*

A pseudo-Euclidean submanifold is said to be *biharmonic* if it has harmonic mean curvature vector field, i.e., $\Delta H = 0$ (see [**2**], [**3**], [**5**] and [**6**]). It is well known, [**7**], that there are no biharmonic hypersurfaces in $\mathbb{S}^{n+1}$. As for hypersurfaces in $\mathbb{H}^{n+1}$, Proposition 3.3 yields the following result.

Corollary 3.4 *A hypersurface M^n in $\mathbb{H}^{n+1}$ is biharmonic if and only if is a flat totally umbilical hypersurface.*

In dealing with the problem of characterizing hypersurfaces M_s^n in $\overline{M}_\nu^{n+1}$ satisfying equation (1), we have solved the problem in the diagonalizable case. Nevertheless, for a pseudo-Riemannian hypersurface the shape operator can be reduced to another canonical forms and thus it seems natural thinking of hypersurfaces satisfying (1) and having non-diagonalizable shape operators. Therefore, in what follows we will study Lorentzian hypersurfaces, and we will start that analysis in the next section trying out surfaces in $\overline{M}_1^3$.

4. On surfaces in $\overline{M}_1^3$

We are looking for surfaces M_s^2 in $\overline{M}_1^3$ satisfying the condition $\Delta H = \lambda H$. Now, the new and interesting situation arises provided the surface inherits a Lorentzian metric ($s = 1$), since the shape operator of M_1^2 need not be diagonalizable. To elucidate this case, we begin, following [**4**], by giving an example of such a surface.

Example 4.1 *B-scroll over a null curve.*
Let $x(s)$ be a null curve in $\overline{M}_1^3 \subset \mathbb{R}_t^4$ with an associated Cartan frame $\{A, B, C\}$, i.e., $\{A, B, C\}$ is a pseudo-orthonormal frame of vector fields along $x(s)$

$$\begin{aligned} \langle A, A\rangle = \langle B, B\rangle = 0, \qquad & \langle A, B\rangle = -1, \\ \langle A, C\rangle = \langle B, C\rangle = 0, \qquad & \langle C, C\rangle = 1, \end{aligned}$$

satisfying

$$\begin{aligned} \dot{x}(s) &= A(s), \\ \dot{C}(s) &= -\tau_0 A(s) - k(s)B(s), \end{aligned}$$

where τ_0 is a nonzero constant and $k(s) \neq 0$ for all s. Then the map $\Psi : (s, u) \longrightarrow x(s) + uB(s)$ parametrizes a Lorentzian surface in $\overline{M}_1^3$ called a B-scroll (see [**4**]).

It is not difficult to see that a unit normal vector field is given by

$$N(s, u) = -\tau_0 uB(s) + C(s),$$

and the shape operator can be put in the usual frame $\left\{\dfrac{\partial\Psi}{\partial s}, \dfrac{\partial\Psi}{\partial u}\right\}$ as

$$S = \begin{pmatrix} \tau_0 & 0 \\ k(s) & \tau_0 \end{pmatrix}.$$

Thus the B-scroll has non-diagonalizable shape operator with minimal polynomial $P_S(t) = (t - \tau_0)^2$ and constant mean curvature $\alpha = \tau_0$. Therefore by using (3) we deduce $\Delta H = 2(c + \tau_0^2)H$.

Let M_1^2 be a Lorentzian surface in $\overline{M}_1^3$ with non-diagonalizable shape operator. Then the minimal polynomial of S is given by $P_S(t) = (t - \beta)^2$, β being a differentiable function on M_1^2, and S can be put in a pseudo-orthonormal frame as

$$\text{I.} \quad \begin{pmatrix} \beta & 0 \\ 1 & \beta \end{pmatrix},$$

or the minimal polynomial is $P_S(t) = (t - \beta)^2 + \gamma^2$, β and $\gamma \neq 0$ being differentiable functions on M_1^2, and S can be written in an orthonormal frame as

$$\text{II.} \quad \begin{pmatrix} \beta & \gamma \\ -\gamma & \beta \end{pmatrix}.$$

Suppose now M_1^2 is a non-minimal surface in $\overline{M}_1^3$ satisfying (1). By using Proposition 3.1 we deduce that the shape operator S of the surface M_1^2 can only be put in the form (I), β being a nonzero constant, since the possibility (II) cannot happen because of the condition $\mathrm{tr}(S^2) = (1/2)\mathrm{tr}(S)^2$. Now we are going to show the following useful result.

Theorem 4.2 *Let M_1^2 be a Lorentzian surface in $\overline{M}_1^3 \subset \mathbb{R}_t^4$ and let $(t-\tau_0)^2$, τ_0 being a nonzero constant, be the minimal polynomial of its shape operator. Then, in a neighborhood of any point, M_1^2 is a B-scroll over a null curve.*

Proof. Pick a point p in M_1^2, choose a pseudo-orthonormal frame $\{A, B\}$ of tangent vector fields in a neighborhood of p such that

$$\begin{aligned} SA &= \tau_0 A + kB, \\ SB &= \tau_0 B, \end{aligned}$$

where $k \neq 0$, and let N be a unit vector field normal to M_1^2 in $\overline{M}_1^3$. Considering M_1^2 as an embedded surface in $\overline{M}_1^3$, we can take an integral curve $x(s)$ of A starting from p. For short, let us write $A(s) = A(x(s))$, $B(s) = B(x(s))$, $C(s) = N(x(s))$ and $k(s) = k(x(s))$. Then

$$\dot{C}(s) = \frac{\tilde{D}C}{ds}(s) = -\tau_0 A(s) - k(s)B(s).$$

For each s, let $\gamma_s(t)$ denote an integral curve of B starting from $x(s)$. Then taking covariant derivate we get

$$\frac{\tilde{D}B}{dt}(\gamma_s(t)) = \tilde{\nabla}_{\dot{\gamma}_s(t)} B(\gamma_s(t)) = \tilde{\nabla}_B B(\gamma_s(t)) = \nabla_B B(\gamma_s(t)).$$

By using now Codazzi's equation we have $\nabla_B B$ is in span$\{B\}$, and then the above equation yields

$$\frac{\tilde{D}B}{dt}(\gamma_s(t)) = f(\gamma_s(t))B(\gamma_s(t)),$$

for a certain diferentiable function f. It is not difficult to see that the solution of that differential equation is given by

$$B(\gamma_s(t)) = g_s(t)B(s),$$

for a certain positive function $g_s(t)$ with $g_s(0) = 1$. Then we get

$$\gamma_s(t) = x(s) + \int_0^t g_s(v)dv\ B(s),$$

and M_1^2 is, in a neighborhood of p, a B-scroll as in Example 4.1.

Theorem 4.2, along with Proposition 3.3, leads to the following main results.

Theorem 4.3 *Let M_s^2 be a surface in $\mathbb{S}_1^3$. Then $\Delta H = \lambda H$ if and only if one of the following statements holds:*

(1) *$\lambda > 0$ and M_s^2 is either minimal or an open piece of one of the following surfaces: $\mathbb{S}^2(\sqrt{2/\lambda})$, $\mathbb{S}_1^2(\sqrt{2/\lambda})$ and a B-scroll over a null curve.*

(2) *$\lambda < 0$ and M_0^2 is an open piece of a hyperbolic plane $\mathbb{H}^2(\sqrt{-2/\lambda})$.*

(3) *$\lambda = 0$ and M_0^2 is a flat totally umbilical surface.*

Theorem 4.4 *Let M_s^2 be a surface in $\mathbb{H}_1^3$. Then $\Delta H = \lambda H$ if and only if one of the following statements holds:*

(1) *$\lambda > 0$ and M_1^2 is an open piece of either $\mathbb{S}_1^2(\sqrt{2/\lambda})$ or a B-scroll with $\tau_0^2 < 1$.*

(2) *$\lambda < 0$ and M_s^2 is either minimal or an open piece of one of the following surfaces: $\mathbb{H}^2(\sqrt{-2/\lambda})$, $\mathbb{H}_1^2(\sqrt{-2/\lambda})$ and a B-scroll with $\tau_0^2 > 1$.*

(3) *$\lambda = 0$ and M_1^2 is either a flat totally umbilical surface or a flat B-scroll ($\tau_0^2 = 1$).*

To finish this section we are going to characterize the biharmonic surfaces in $\overline{M}_1^3$, which is a consequence of the above two theorems.

Corollary 4.5 *A surface M_s^2 in $\mathbb{S}_1^3$ is biharmonic if and only if is a flat totally umbilical surface.*

Corollary 4.6 *A surface M_s^2 in $\mathbb{H}_1^3$ is biharmonic if and only if is either a flat totally umbilical surface or a flat B-scroll.*

5. The n-dimensional case

With the aim of completing our study we must consider non-minimal hypersurfaces M_s^n in $\overline{M}_1^{n+1}$ satisfying $\Delta H = \lambda H$ with $n > 2$. In view of Proposition 3.3 we can assume that it is endowed with a Lorentzian metric ($s = 1$) and the shape operator of M_1^n in $\overline{M}_1^{n+1}$ is not diagonalizable. In this case S can be put in one of the following two forms:

$$\text{I.}\quad \begin{pmatrix} \beta & 0 & & & & \\ 1 & \beta & & & & \\ & & \mu_3 & & & \\ & & & \ddots & & \\ & & & & \mu_n \end{pmatrix} \qquad \text{II.}\quad \begin{pmatrix} \beta & 0 & 0 & & & \\ 0 & \beta & 1 & & & \\ -1 & 0 & \beta & & & \\ & & & \mu_4 & & \\ & & & & \ddots & \\ & & & & & \mu_n \end{pmatrix}$$

in a pseudo-orthonormal frame, and

$$\text{III.}\quad \begin{pmatrix} \beta & \gamma & & & & \\ -\gamma & \beta & & & & \\ & & \mu_3 & & & \\ & & & \ddots & & \\ & & & & \mu_n \end{pmatrix}, \qquad \gamma \neq 0$$

in an orthonormal frame.

Assume now that S has no complex eigenvalues. Then from Proposition 3.1 the minimal polynomial of S is given by $P_S(t) = (t-\alpha)^2$ or $P_S(t) = (t-\alpha)^3$, α being a nonzero constant. Examples of those hypersurfaces in the $(n+1)$-dimensional Lorentz-Minkowski space can be found in [**11**]. Before going any further, we wish to show some examples of such a kind of hypersurfaces in the non-flat Lorentzian space forms.

Example 5.1 *Generalized umbilical hypersurface of degree 2.*
Let $x : I \subset \mathbb{R} \longrightarrow \mathbb{H}_1^{n+1} \subset \mathbb{R}_2^{n+2}$ be a null curve with an associated pseudo-orthonormal frame $\{A, B, Z_1, \ldots, Z_{n-2}, C\}$ along $x(s)$ such that

$$\begin{aligned} \dot{x} &= A(s), \\ \dot{C} &= -\tau_0 A(s) - k(s)B(s), \end{aligned}$$

where $\tau_0^2 = 1$ and $k(s) \neq 0$. Then it is not difficult to see that the map $\Psi : I \times \mathbb{R} \times \mathbb{R}^{n-2} \longrightarrow \mathbb{H}_1^{n+1} \subset \mathbb{R}_2^{n+2}$ given by

$$\Psi(s,u,z) = \left(1 + \frac{|z|^2}{2}\right) x(s) + uB(s) + \sum_{j=1}^{n-2} z_j Z_j(s) + \frac{\tau_0 |z|^2}{2} C(s),$$

parametrizes, in a neighborhood of the origin, a Lorentzian hypersurface M_1^n of $\mathbb{H}_1^{n+1}$. It is easy to see that a unit normal vector field is given by

$$N(s,u,z) = -\tau_0 uB(s) - \tau_0 \sum_{j=1}^{n-2} z_j Z_j(s) + \left(1 - \frac{|z|^2}{2}\right) C(s) - \frac{\tau_0 |z|^2}{2} x(s).$$

Moreover, its mean curvature α is the constant $\tau_0 = \pm 1$ and the minimal polynomial of its shape operator is given by $P_S(t) = (t-\tau_0)^2$. Then M_1^n satisfies condition (1) with $\lambda = 0$, that is, M_1^n is a biharmonic hypersurface.

Example 5.2 *Generalized umbilical hypersurface of degree 2.*
Let $x : I \subset \mathbb{R} \longrightarrow \overline{M}_1^{n+1} \subset \mathbb{R}_t^{n+2}$ be a null curve with an associated pseudo-orthonormal frame $\{A, B, Z_1, \ldots, Z_{n-2}, C\}$ along $x(s)$ such that

$$\begin{aligned} \dot{x} &= A(s), \\ \dot{C} &= -\tau_0 A(s) - k(s) B(s), \end{aligned}$$

where $k(s) \neq 0$ and τ_0 is a nonzero constant with $c + \tau_0^2 \neq 0$. Then the map $\Psi : I \times \mathbb{R} \times \mathbb{R}^{n-2} \longrightarrow \overline{M}_1^{n+1} \subset \mathbb{R}_t^{n+2}$ defined by

$$\Psi(s,u,z) = \frac{\tau_0^2 + cf(z)}{c+\tau_0^2} x(s) + uB(s) + \sum_{j=1}^{n-2} z_j Z_j(s) + \frac{\tau_0(1-f(z))}{c+\tau_0^2} C(s),$$

where $f(z) = \sqrt{1-(c+\tau_0^2)|z|^2}$, parametrizes, in a neighborhood of the origin, a Lorentzian hypersurface M_1^n of $\overline{M}_1^{n+1}$. It is not difficult to see that a unit normal vector field is given by

$$N(s,u,z) = -\tau_0 uB(s) - \tau_0 \sum_{j=1}^{n-2} z_j Z_j(s) + \frac{c+\tau_0^2 f(z)}{c+\tau_0^2} C(s) + \frac{\tau_0 c(1-f(z))}{c+\tau_0^2} x(s).$$

The mean curvature α is the nonzero constant τ_0 and the minimal polynomial of its shape operator is $P_S(t) = (t-\tau_0)^2$. Then M_1^n satisfies condition (1) with $\lambda = n(c+\tau_0^2)$.

Example 5.3 *Generalized umbilical hypersurface of degree 3.*
Let $x : I \subset \mathbb{R} \longrightarrow \mathbb{H}_1^{n+1} \subset \mathbb{R}_2^{n+2}$ be a null curve with an associated pseudo-orthonormal frame $\{A, B, Y, Z_1, \ldots, Z_{n-3}, C\}$ such that

$$\begin{aligned} \dot{x} &= A(s), \\ \dot{C} &= -\tau_0 A(s) + k(s) Y(s), \end{aligned}$$

with $\tau_0^2 = 1$ and $k(s) \neq 0$. Then it is easy to see that the map $\Psi : I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-3} \longrightarrow \mathbb{H}_1^{n+1} \subset \mathbb{R}_2^{n+2}$ given by

$$\Psi(s,u,y,z) = \left(1 + \frac{y^2+|z|^2}{2}\right) x(s) + uB(s) + yY(s) + \sum_{j=1}^{n-3} z_j Z_j(s) + \frac{\tau_0(y^2+|z|^2)}{2} C(s),$$

parametrizes, in a neighborhood of the origin, a Lorentzian hypersurface M_1^n in $\mathbb{H}_1^{n+1}$ with a unit normal vector field given by

$$\begin{aligned} N(s,u,y,z) &= -\left(\tau_0 u + \frac{k(s)y}{F(s,y,z)}\right) B(s) - \tau_0 y Y(s) - \tau_0 \sum_{j=1}^{n-3} z_j Z_j(s) \\ &\quad + \left(1 - \frac{y^2+|z|^2}{2}\right) C(s) - \frac{\tau_0(y^2+|z|^2)}{2} x(s), \end{aligned}$$

where $F(s,y,z) = 1 + y\left\langle \dot{B}(s), Y(s)\right\rangle + \sum_{j=1}^{n-3} z_j \left\langle \dot{B}(s), Z_j(s)\right\rangle$. A messy computation shows that M_1^n has constant mean curvature $\alpha = \tau_0 = \pm 1$ and the minimal polynomial of its shape operator is $P_S(t) = (t-\tau_0)^3$. Then M_1^n satisfies the condition $\Delta H = 0$, i.e., M_1^n is a biharmonic hypersurface of $\mathbb{H}_1^{n+1}$.

Example 5.4 *Generalized umbilical hypersurface of degree 3.*
Let $x : I \subset \mathbb{R} \longrightarrow \overline{M}_1^{n+1} \subset \mathbb{R}_t^{n+2}$ be a null curve with an associated pseudo-orthonormal frame $\{A, B, Y, Z_1, \ldots, Z_{n-3}, C\}$ such that

$$\begin{aligned} \dot{x} &= A(s), \\ \dot{C} &= -\tau_0 A(s) + k(s) Y(s), \end{aligned}$$

with $k(s) \neq 0$ and τ_0 a nonzero constant such that $c + \tau_0^2 \neq 0$. Then the map $\Psi : I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-3} \longrightarrow \overline{M}_1^{n+1} \subset \mathbb{R}_t^{n+2}$ defined by

$$\Psi(s,u,y,z) = \frac{\tau_0^2 + c f(y,z)}{c+\tau_0^2} x(s) + u B(s) + y Y(s) + \sum_{j=1}^{n-3} z_j Z_j(s) + \frac{\tau_0(1-f(y,z))}{c+\tau_0^2} C(s),$$

where $f(y,z) = \sqrt{1-(c+\tau_0^2)(y^2+|z|^2)}$, parametrizes, in a neighborhood of the origin, a Lorentzian hypersurface M_1^n in $\overline{M}_1^{n+1}$. A straightforward computation shows that a unit normal vector field is given by

$$\begin{aligned} N(s,u,y,z) &= -\left(\tau_0 u + \frac{k(s)y}{F(s,y,z)}\right) B(s) - \tau_0 y Y(s) - \tau_0 \sum_{j=1}^{n-3} z_j Z_j(s) \\ &\quad + \frac{c+\tau_0^2 f(y,z)}{c+\tau_0^2} C(s) + \frac{c\tau_0(1-f(y,z))}{c+\tau_0^2} x(s), \end{aligned}$$

where $F(s,y,z) = f(y,z) + y\left\langle \dot{B}(s), Y(s)\right\rangle + \sum_{j=1}^{n-3} z_j \left\langle \dot{B}(s), Z_j(s)\right\rangle$. Then M_1^n has constant mean curvature $\alpha = \tau_0 \neq 0$ and the minimal polynomial of its shape operator is given by $P_S(t) = (t-\tau_0)^3$. Thus M_1^n satisfies the condition $\Delta H = \lambda H$ with $\lambda = n(c+\tau_0^2)$.

In the following two theorems we are going to characterize these examples as the only ones having $(t-a)^k$, $k = 2, 3$ and a being a nonzero constant, as the minimal polynomials of their shape operators. We closely follow the ideas of Magid in [**11**].

Theorem 5.5 *Let M_1^n be a Lorentzian hypersurface isometrically immersed in $\overline{M}_1^{n+1} \subset \mathbb{R}_t^{n+2}$ and let $(t-\tau_0)^2$, $\tau_0 \neq 0$, be the minimal polynomial of its shape operator. Then, in a neighborhood of any point, M_1^n is a generalized umbilical hypersurface of degree 2.*

Proof. We can take, at any point p in M_1^n, a pseudo-orthonormal frame $\{A, B, Z_1, \ldots, Z_{n-2}\}$ in a neighborhood of p such that

$$\begin{aligned} SA &= \tau_0 A + kB, \\ SB &= \tau_0 B, \\ SZ_i &= \tau_0 Z_i, \quad i = 1, \ldots, n-2, \end{aligned}$$

where $k \neq 0$, and let N denote a unit vector field normal to M_1^n in $\overline{M}_1^{n+1}$. Dealing with M_1^n as an embedded hypersurface, let $x(s)$ be an integral curve of A starting from p, and for simplicity of notation, we shall put $A(s) = A(x(s))$, $B(s) = B(x(s))$, etc., and $C(s) = N(x(s))$. Then

$$\dot{C}(s) = \frac{\tilde{D}C}{ds}(s) = -\tau_0 A(s) - k(s)B(s).$$

By using Codazzi's equation, it is easy to see that $T = \ker(S - \tau_0 I)$ is an integrable degenerate distribution. For each fixed s, let $M(s)$ be the leaf of T through $x(s)$. We are going to show that $M(s)$ is lying in a hypersphere of $\mathbb{R}_t^{n+2}$ centered at $x(s) + (1/\tau_0)C(s)$ with radius $1/\tau_0$. Let $\gamma_s(t)$ be a curve in $M(s)$ starting from $x(s)$. Then

$$\frac{\tilde{D}N}{dt}(\gamma_s(t)) = -\tau_0 \dot{\gamma}_s(t),$$

and thus

$$N(\gamma_s(t)) + \tau_0 \gamma_s(t) = \text{const.} = C(s) + \tau_0 x(s).$$

Therefore we get

$$\left\langle \gamma_s(t) - x(s) - \frac{1}{\tau_0}C(s), \gamma_s(t) - x(s) - \frac{1}{\tau_0}C(s) \right\rangle = \frac{1}{\tau_0^2}.$$

Now let us see that $M(s)$ is also contained in the hyperplane orthogonal to $B(s)$ through the center of the above hypersphere. From (1) we have $\gamma_s(t) - x(s) - (1/\tau_0)C(s)$ is collinear to $N(\gamma_s(t))$ and then

$$\left\langle B(\gamma_s(t)), \gamma_s(t) - x(s) - \frac{1}{\tau_0}C(s) \right\rangle = 0.$$

On the other hand, for all X in T

$$\tilde{\nabla}_X B = \nabla_X B,$$

and from Codazzi's equation

$$\nabla_X(SA) - S(\nabla_X A) = \nabla_A(SX) - S(\nabla_A X)$$

we get

$$k\nabla_X B = (S - \tau_0 I)[X, A] - X(k)B.$$

Since Im$(S - \tau_0 I)$=span$\{B\}$ we deduce $\nabla_X B$ is in span$\{B\}$. Now reasoning as in Theorem 4.2 we have $B(\gamma_s(t)) = g_s(t)B(s)$, where $g_s(t) > 0$, and from (2) we get

$$\left\langle B(s), \gamma_s(t) - x(s) - \frac{1}{\tau_0}C(s) \right\rangle = 0.$$

Finally, putting together those facts the theorem follows.

Theorem 5.6 *Let M_1^n be a Lorentzian hypersurface isometrically immersed in $\overline{M}_1^{n+1} \subset \mathbb{R}_t^{n+2}$ and let $(t-\tau_0)^3$, $\tau_0 \neq 0$, be the minimal polynomial of its shape operator. Then, in a neighborhood of any point, M_1^n is a generalized umbilical hypersurface of degree 3.*

Proof. We can choose, at any point p in M_1^n, a local pseudo-orthonormal frame $\{A, B, Y, Z_1, \ldots, Z_{n-3}\}$ such that

$$\begin{aligned} SA &= \tau_0 A - kY, \\ SB &= \tau_0 B, \\ SY &= kB + \tau_0 Y, \\ SZ_i &= \tau_0 Z_i, \quad i = 1, \ldots, n-3, \end{aligned}$$

where $k \neq 0$, and let N be as in the above theorem. Considering M_1^n as an embedded hypersurface, let $x(s)$ be an integral curve of A starting from p and write $A(s) = A(x(s))$, $B(s) = B(x(s))$, etc., and $C(s) = N(x(s))$. Then

$$\dot{C}(s) = \frac{\tilde{D}C}{ds}(s) = -\tau_0 A(s) + k(s)Y(s).$$

Let T denote the kernel of $(S - \tau_0 I)^2$. Then $T = \ker(S - \tau_0 I) \oplus \operatorname{span}\{Y\}$. To prove that T is an integrable distribution it suffices to show that $[X, Y] \in T$ for all $X \in \ker(S - \tau_0 I)$. From Codazzi's equation

$$\nabla_X(SA) - S(\nabla_X A) = \nabla_A(SX) - S(\nabla_A X)$$

we have

$$\langle \nabla_X Y, B \rangle = 0 = \langle \nabla_X B, Y \rangle,$$

for all X in T. Now let $X \in \ker(S - \tau_0 I)$ and use again Codazzi's equation

$$\nabla_X(SY) - S(\nabla_X Y) = \nabla_Y(SX) - S(\nabla_Y X)$$

to get $\langle [X, Y], B \rangle = 0$, and thus $[X, Y] \in T$.

For each s, let $M(s)$ be the leaf of T through $x(s)$. We are going to show that $M(s)$ is contained in a hypersphere of $\mathbb{R}_t^{n+2}$ centered at $x(s) + (1/\tau_0)C(s)$ with radius $1/\tau_0$. Let $\gamma_s(t)$ be a curve in $M(s)$ starting from $x(s)$. Then

$$\frac{\tilde{D}N}{dt}(\gamma_s(t)) = -\tau_0 \dot{\gamma}_s(t) - k(\gamma_s(t)) \langle \dot{\gamma}_s(t), Y(\gamma_s(t)) \rangle B(\gamma_s(t)),$$

that is,

$$\frac{\tilde{D}}{dt}(N(\gamma_s(t)) + \tau_0 \gamma_s(t)) = f_s(t) B(\gamma_s(t)),$$

for a certain differentiable function $f_s(t)$. A similar reasoning as in Theorem 5.5 shows that $B(\gamma_s(t)) = g_s(t)B(s)$, $g_s(t) > 0$, and therefore we have

$$\frac{\tilde{D}}{dt}(N(\gamma_s(t)) + \tau_0 \gamma_s(t)) = h_s(t) B(s),$$

and

$$N(\gamma_s(t)) + \tau_0 \gamma_s(t) = \mu_s(t) B(s) + C(s) + \tau_0 x(s),$$

for certain differentiable functions $h_s(t)$ and $\mu_s(t)$. Thus we find

$$\left\langle \gamma_s(t) - x(s) - \frac{1}{\tau_0}C(s), \gamma_s(t) - x(s) - \frac{1}{\tau_0}C(s) \right\rangle = \frac{1}{\tau_0^2} - \frac{2}{\tau_0^2}\frac{\mu_s(t)}{g_s(t)} \langle B, N\rangle (\gamma_s(t)) = \frac{1}{\tau_0^2}.$$

The above computation also shows that $M(s)$ is contained in the hyperplane orthogonal to $B(s)$ through $x(s) + (1/\tau_0)C(s)$. Therefore M_1^n is, in a neighborhood of p, a generalized umbilical hypersurface of degree 3.

Now, we are going to state the main results of this section which will be separately treated to clarify them.

In the De Sitter world we have.

Theorem 5.7 *Let M_s^n be a hypersurface in $\mathbb{S}_1^{n+1}$ whose shape operator has no complex eigenvalues. Then $\Delta H = \lambda H$ if and only if one of the following statements holds:*

(1) $\lambda > 0$ and M_s^n is either minimal in $\mathbb{S}_1^{n+1}$ or an open piece of one of the following hypersurfaces: $\mathbb{S}^n(\sqrt{n/\lambda})$, $\mathbb{S}_1^n(\sqrt{n/\lambda})$, the generalized umbilical hypersurface of degree 2 in Example 5.2 and the one of degree 3 in Example 5.4.

(2) $\lambda < 0$ and M_0^n is an open piece of $\mathbb{H}^n(\sqrt{-n/\lambda})$.

(3) $\lambda = 0$ and M_0^n is a flat totally umbilical hypersurface.

This theorem leads us to the following characterization of biharmonic hypersurfaces in the De Sitter space.

Corollary 5.8 *A hypersurface in $\mathbb{S}_1^{n+1}$ whose shape operator has no complex eigenvalues is biharmonic if and only if is a flat totally umbilical hypersurface.*

As for hypersurfaces in $\mathbb{H}_1^{n+1}$ we have the following classification theorem.

Theorem 5.9 *Let M_s^n be a hypersurface in $\mathbb{H}_1^{n+1}$ whose shape operator has no complex eigenvalues. Then $\Delta H = \lambda H$ if and only if one of the following statements holds:*

(1) $\lambda > 0$ and M_1^n is an open piece of one of the following hypersurfaces: $\mathbb{S}_1^n(\sqrt{n/\lambda})$, the generalized umbilical hypersurface, with $\tau_0^2 > 1$, of degree 2 in Example 5.2 and the one of degree 3 in Example 5.4.

(2) $\lambda < 0$ and M_s^n is either minimal in $\mathbb{H}_1^{n+1}$ or an open piece of one of the following hypersurfaces: $\mathbb{H}^n(\sqrt{-n/\lambda})$, $\mathbb{H}_1^n(\sqrt{-n/\lambda})$, the generalized umbilical hypersurface, with $\tau_0^2 < 1$, of degree 2 in Example 5.2 and the one of degree 3 in Example 5.4.

(3) $\lambda = 0$ and M_1^n is either a flat totally umbilical hypersurface or the generalized umbilical hypersurface, with $\tau_0^2 = 1$, of degree 2 in Example 5.1 or the one of degree 3 in Example 5.3.

It is worth pointing out that the family of biharmonic hypersurfaces is richer in the anti De Sitter space than in the De Sitter space. In fact, we get the following corollary.

Corollary 5.10 *A hypersurface in $\mathbb{H}_1^{n+1}$ whose shape operator has no complex eigenvalues is biharmonic if and only if is either a flat totally umbilical hypersurface or the generalized umbilical hypersurface, with $\tau_0^2 = 1$, of degree 2 in Example 5.1 or the one of degree 3 in Example 5.3.*

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