Hypersurfaces in space forms satisfying the condition $\Delta x = Ax + B$

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Abstract

In this work we study and classify pseudo-Riemannian hypersurfaces in pseudo-Riemannian space forms which satisfy the condition $\Delta x = Ax + B$, where A is an endomorphism, B is a constant vector and x stands for the isometric immersion. We prove that the family of such hypersurfaces consists of open pieces of minimal hypersurfaces, totally umbilical hypersurfaces, products of two non-flat totally umbilical submanifolds and a special class of quadratic hypersurfaces.

1. Introduction

Let x be an isometric immersion of a hypersurface M_s^n in $\mathbb{R}_t^{n+1}$ and assume there exist an endomorphism A of $\mathbb{R}_t^{n+1}$ and a constant vector B in $\mathbb{R}_t^{n+1}$ such that $\Delta x = Ax + B$. We ask for the following question: *"What is the geometric meaning involved in that algebraic condition?"* This question was first studied in the Euclidean case by Chen and Petrovic [**4**], Dillen, Pas and Verstraelen [**5**], and Hasanis and Vlachos [**7**], who obtained some interesting classification theorems. Recently, Park [**10**], following closely the ideas in [**2**] and [**1**], has considered that condition with $B = 0$ for hypersurfaces in Euclidean spherical and hyperbolic spaces.

To study that question in its full generality, it seemed natural to us to begin with Lorentzian surfaces, [**2**]. Later, in [**1**], in order to generalize the above papers we gave a classification theorem for pseudo-Euclidean hypersurfaces. Actually, we proved that the only hypersurfaces satisfying the matricial condition on the Laplacian are open pieces of minimal hypersurfaces, totally umbilical hypersurfaces and pseudo-Riemannian products of a totally umbilical and a totally geodesic submanifold.

This paper arises as a natural continuation of [**2**] and [**1**], taking now a non-flat pseudo-Riemannian space form as the ambient space. Here, we analyze the isometric immersions x of a hypersurface M_s^n of $\bar{M}_\nu^{n+1}$ satisfying $\Delta x = Ax + B$, where $\bar{M}_\nu^{n+1}$ is the pseudo-Euclidean sphere $\mathbb{S}_\nu^{n+1} \subset \mathbb{R}_\nu^{n+2}$ or the pseudo-Euclidean hyperbolic space $\mathbb{H}_\nu^{n+1} \subset \mathbb{R}_{\nu+1}^{n+2}$.

In this new situation, the codimension of the manifold M_s^n in the pseudo-Euclidean space where it is lying is two, so that one hopes to find a richer family of examples satisfying the asked condition. On the other hand, although the proofs given in [**1**] do not work here, we follow the techniques developed there.

Before to refer to the main result, we wish pointing out that a lot of hypersurfaces having non-diagonalizable shape operator are given. This property makes substantially the difference between this case and that treated in [**1**].

The main result of this paper states that the only hypersurfaces M_s^n of $\bar{M}_\nu^{n+1}$ satisfying the matricial condition on the Laplacian are open pieces of minimal hypersurfaces, totally umbilical

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hypersurfaces, pseudo-Riemannian products of two non-flat totally umbilical submanifolds and quadratic hypersurfaces defined by $\{x \in \mathbb{R}^{n+2}_t : \langle x,x\rangle = \pm 1,\ \langle Lx, x\rangle = c\}$, where L is a self-adjoint endomorphism of $\mathbb{R}^{n+2}_t$ with minimal polynomial μ_L of degree two, and c is a real constant such that $\mu_L(kc) \neq 0$.

2. Preliminaries

Let $\mathbb{R}^{n+2}_t$ be the $(n+2)$-dimensional pseudo-Euclidean space whose metric tensor is given by

$$ds^2 = -\sum_{i=1}^{t} dx^i \otimes dx^i + \sum_{j=t+1}^{n+2} dx^j \otimes dx^j,$$

where $(x_1, \ldots, x_{n+2})$ is the standard coordinate system. For each $k \neq 0$, let $\bar{M}^{n+1}_\nu(k)$ be the complete and simply connected space with constant sectional curvature $\mathrm{sign}(k)/k^2$. A model for $\bar{M}^{n+1}_\nu(k)$ is the pseudo-Euclidean sphere $\mathbb{S}^{n+1}_\nu(k)$ if $k > 0$ and the pseudo-Euclidean hyperbolic space $\mathbb{H}^{n+1}_\nu(k)$ if $k < 0$, where $\mathbb{S}^{n+1}_\nu(k) = \{x \in \mathbb{R}^{n+2}_\nu : \langle x,x\rangle = k^2\}$ and $\mathbb{H}^{n+1}_\nu(k) = \{x \in \mathbb{R}^{n+2}_{\nu+1} : \langle x,x\rangle = -k^2\}$, $\langle,\rangle$ standing for the indefinite inner product in the pseudo-Euclidean space. Throughout this paper we will assume, without loss of generality, that $k^2 = 1$.

Let M^n_s be a pseudo-Riemannian hypersurface in $\bar{M}^{n+1}_\nu$ and let ∇ ($\bar{\nabla}$ and $\tilde{\nabla}$) denotes the Levi-Civita connection on M^n_s ($\bar{M}^{n+1}_\nu$ and $\mathbb{R}^{n+2}_t$, respectively). We will also denote by N the unit normal vector field to M^n_s in $\bar{M}^{n+1}_\nu$. Let H' and H be the mean curvature vector fields of M^n_s in $\bar{M}^{n+1}_\nu$ and $\mathbb{R}^{n+2}_t$, respectively. Thus we may write $H' = \alpha N$, α being the mean curvature of M^n_s in $\bar{M}^{n+1}_\nu$, and

$$H = H' - kx = \alpha N - kx.$$

Let $x : M^n_s \longrightarrow \bar{M}^{n+1}_\nu$ be an isometric immersion satisfying the condition

$$\Delta x = Ax + B,$$

where A is an endomorphism of $\mathbb{R}^{n+2}_t$ and B a constant vector in $\mathbb{R}^{n+2}_t$. Taking covariant derivative in (2) and using the Laplace-Beltrami equation $\Delta x = -nH$ and the Weingarten formula we get $AX = nS_HX - nD_XH$, for any vector field X tangent to M^n_s, where D denotes the normal connection on M^n_s and S_ξ the Weingarten endomorphism associated to a normal vector field ξ. Then by (1) we have $D_XH = X(\alpha)N$ and $S_HX = \alpha SX + kX$, where, for short, we have written S for the Weingarten endomorphism S_N. From now on, we will call S the shape operator of M^n_s. Now from the above formulae we deduce that

$$AX = n(\alpha SX + kX) - nX(\alpha)N.$$

From (2), taking into account the Laplace-Beltrami equation and (1), we obtain the following equation

$$Ax = -n\alpha N + nkx - B.$$

By applying the Laplacian on both sides of (2) and using again that $\Delta x = -nH$ we find $AH = \Delta H$, that along with (1) leads to $\alpha AN = \Delta H + kAx$. Now, bringing here (4) and the formula for ΔH obtained in [**3**, Lemma 3]

$$\Delta H = 2S(\nabla\alpha) + n\varepsilon\alpha\nabla\alpha + (\Delta\alpha + \varepsilon\alpha|S|^2 + nk\alpha)N - nk(k + \varepsilon\alpha^2)x,$$

where $\nabla\alpha$ stands for the gradient of α, $\varepsilon = \langle N, N\rangle$ and $|S|^2 = \mathrm{trace}(S^2)$, we get the following equation

$$\begin{aligned} \alpha A N &= 2S(\nabla\alpha) + n\varepsilon\alpha\nabla\alpha + (\Delta\alpha + \varepsilon\alpha|S|^2)N \\ &\quad - nk\varepsilon\alpha^2 x - kB. \end{aligned} \tag{5}$$

3. Some examples

In this paper we wish to classify the hypersurfaces M_s^n in $\bar{M}_\nu^{n+1}$ whose isometric immersion satisfies the condition (2). In order to get such a classification we need some examples.

2.1 Minimal hypersurfaces M_s^n in $\bar{M}_\nu^{n+1}$ obviously satisfy (2). Indeed, by using (1) we have $H = -kx$ and $\Delta x = nkx$. So we can take $A = nkI_{n+2}$ and $B = 0$.

2.2 Let M_s^n be a totally umbilical hypersurface in $\bar{M}_\nu^{n+1}$. Taking into account the classification theorem for such hypersurfaces (see, for example, [**9**, Theorem 1.4]) we get, according to $\langle H, H\rangle$ is positive, negative or zero, M_s^n is an open piece of a pseudo-Euclidean sphere $\mathbb{S}_s^n(r)$, a pseudo-Euclidean hyperbolic space $\mathbb{H}_s^n(r)$ or $\mathbb{R}_s^n$. In the last case, the immersion $f : \mathbb{R}_s^n \longrightarrow \mathbb{R}_{s+1}^{n+2}$ is given by $f(u) = (q(u), u_1, \ldots, u_n, q(u))$, where $q(u) = a\,\langle u, u\rangle + \langle b, u\rangle + c$, $a \neq 0$. The pseudo-Euclidean spheres and pseudo-Euclidean hyperbolic spaces both satisfy the condition (2). Indeed, by considering φ as the standard immersion of $\mathbb{S}_s^n(r)$ or $\mathbb{H}_s^n(r)$ in a hyperplane $\mathbb{R}_{s'}^{n+1}$ of $\mathbb{R}_t^{n+2}$, we know from [**1**] that $\Delta\varphi = L\varphi$, L being an endomorphism of $\mathbb{R}_{s'}^{n+1}$. The $(n+1)\times(n+1)$ matrix L and the immersion φ become an $(n+2)\times(n+2)$ matrix A (filling with zeros) and an immersion x in $\mathbb{R}_t^{n+2}$, respectively, in a natural way and so we get (2) with $B = 0$. Therefore the most interesting case is that with $\langle H, H\rangle = 0$. Now we can choose a point p in $\mathbb{R}_{s+1}^{n+2}$ such that $\langle f - p, f - p\rangle = \pm 1$ and then $x = f - p$ is an immersion from $\mathbb{R}_s^n$ in $\bar{M}_\nu^{n+1}$ with $\Delta x = -2n(a, 0, \ldots, 0, a)$. Thus this hypersurface satisfies (2) with $A = 0$ and $B = (-2na, 0, \ldots, 0, -2na)$. Furthermore, from the equation $\Delta x = -n\alpha N + nkx$, we easily obtain that its constant mean curvature α is given by $\alpha^2 = 1$.

2.3 Let $x : M_s^m \longrightarrow \mathbb{R}_t^{m+1}$ and $y : {M'}_{s'}^{m'} \longrightarrow \mathbb{R}_{t'}^{m'+1}$ be two isometric immersions satisfying the condition (2) and let $z = x \times y$ be the natural isometric immersion from the pseudo-Riemannian product $M_s^m \times {M'}_{s'}^{m'}$ in $\mathbb{R}_{t+t'}^{m+m'+2}$. If $\Delta x = Ax + B$ and $\Delta' y = A'y + B'$ then we can consider $\tilde{A} = \mathrm{diag}[A, A']$ and $\tilde{B} = (B, B')$. Thus it is easy to show that $\tilde{\Delta} z = \tilde{A} z + \tilde{B}$. Then from [**1**, Section 2], we can construct the following examples of hypersurfaces in $\bar{M}_\nu^{n+1}$ satisfying the condition (2):

(a) $\mathbb{S}_u^p(r) \times \mathbb{S}_{s-u}^{n-p}(\sqrt{1-r^2}) \subset \mathbb{S}_s^{n+1}$, with $0 < r < 1$ and $r \neq \sqrt{p/n}$, whose constant mean curvature is given by $\alpha^2 = (nr^2 - p)^2/(n^2r^2(1-r^2))$;
(b) $\mathbb{H}_u^p(-r) \times \mathbb{H}_{s-u}^{n-p}(-\sqrt{1-r^2}) \subset \mathbb{H}_{s+1}^{n+1}$, with $0 < r < 1$, $r \neq \sqrt{p/n}$ and $\alpha^2 = (nr^2 - p)^2/(n^2r^2(1-r^2))$;
(c) $\mathbb{S}_u^p(r) \times \mathbb{H}_{s-u}^{n-p}(-\sqrt{-1+r^2}) \subset \mathbb{S}_{s+1}^{n+1}$, with $r > 1$ and $\alpha^2 = (nr^2 - p)^2/(n^2r^2(r^2-1))$;
(d) $\mathbb{S}_u^p(r) \times \mathbb{H}_{s-u}^{n-p}(-\sqrt{1+r^2}) \subset \mathbb{H}_s^{n+1}$, with $r > 0$ and $\alpha^2 = (nr^2 + p)^2/(n^2r^2(1+r^2))$;
where $1 \leqslant p \leqslant n-1$ and $0 \leqslant u \leqslant s$. We will refer these examples as the *pseudo-Riemannian non-minimal standard products*.

2.4 The hypersurfaces in examples **2.2** and **2.3** have diagonalizable shape operator. However, it seems natural thinking of hypersurfaces with non-diagonalizable shape operator satisfying (2) into indefinite ambient spaces. Let L be a self-adjoint endomorphism of $\mathbb{R}_t^{n+2}$, that is, $\langle Lx, y\rangle = \langle x, Ly\rangle$ for all $x, y \in \mathbb{R}_t^{n+2}$. Let $f : \bar{M}_\nu^{n+1} \longrightarrow \mathbb{R}$ be the quadratic function defined by $f(x) =$

$\langle Lx, x\rangle$ and assume that the minimal polynomial of L is given by $\mu_L(t) = t^2 + at + b$, $a, b \in \mathbb{R}$. Then by computing the gradients, at each point $x \in \bar{M}^{n+1}_\nu$, we have $\tilde{\nabla} f(x) = 2Lx$ and $\bar{\nabla} f(x) = 2Lx - 2kf(x)x$. If $\tilde{\Delta}$ and $\bar{\Delta}$ denote the Laplacian operators on $\mathbb{R}^{n+2}_t$ and $\bar{M}^{n+1}_\nu$, respectively, a straightforward computation yields $\tilde{\Delta} f(x) = -2\,\mathrm{trace}(L)$ and $\bar{\Delta} f(x) = -2\,\mathrm{trace}(L) - 2k(n+1)f(x)$.

Consider the level set $M = f^{-1}(c)$ for a real constant c. Then at a point x in M we have

$$\left\langle \bar{\nabla} f(x), \bar{\nabla} f(x)\right\rangle = 4\left\langle L^2x, x\right\rangle - 4kf(x)^2 = -4k\mu_L(kc),$$

and so f is an isoparametric function (see [**6**]). Thus the level hypersurfaces $\{f^{-1}(c)\}_{c\in I}$, where $I \subset \{c \in \mathbb{R} : \mu_L(kc) \neq 0\}$ is connected, form an isoparametric family in the classical sense. The shape operator of M^n_s is given by $SX = -\dfrac{1}{|\bar{\nabla} f|}\bar{\nabla}_X(\bar{\nabla} f) = \dfrac{1}{|\mu_L(kc)|^{1/2}}(kcX - LX)$ and a messy computation gives

$$\mathrm{tr}(S) = \frac{nkc - \mathrm{tr}(L) - a}{|\mu_L(kc)|^{1/2}}.$$

Then the mean curvature α is given by

$$\alpha = \frac{\varepsilon}{n}\,\mathrm{tr}(S) = \delta\frac{a + \mathrm{tr}(L) - nkc}{nk|\mu_L(kc)|^{1/2}},$$

where δ stands for the sign of $\mu_L(kc)$. Therefore we get

$$H' = \frac{a + \mathrm{tr}L - knc}{kn\mu_L(kc)}(Lx - kcx),$$

from which we deduce, by using $\Delta x = -n(H' - kx)$, that $\Delta x = Ax$, where A is given by

$$A = \frac{knc - a - \mathrm{tr}L}{k\mu_L(kc)}L + \frac{c\mathrm{tr}L + (n+1)ac + knb}{\mu_L(kc)}I_{n+2}.$$

4. First characterization results

The aim of this section is to show that a hypersurface M^n_s of $\bar{M}^{n+1}_\nu$ satisfying the condition (2) has to be of constant mean curvature. To do that, let $\mathcal{W}$ be the open set of regular points of α^2, which we may assume a non-empty set. From (3) we have $\langle AX, x\rangle = 0$, for any vector field X tangent to M^n_s. Taking covariant derivative there we get

$$\langle A\sigma(X, Y), x\rangle = -\langle AX, Y\rangle\,,$$

for all tangent vectors X and Y, where σ represents the second fundamental form of M^n_s in $\mathbb{R}^{n+2}_t$, which is given by

$$\sigma(X, Y) = \varepsilon\langle SX, Y\rangle N - k\langle X, Y\rangle x.$$

Now equation (1), jointly with (3) and (2), leads to

$$\varepsilon\langle SX, Y\rangle\langle AN, x\rangle - k\langle X, Y\rangle\langle Ax, x\rangle = -n\alpha\langle SX, Y\rangle - nk\langle X, Y\rangle.$$

Bringing here the formulae for Ax and AN given in (4) and (5), respectively, a straightforward computation yields

$$\langle SX - \varepsilon\alpha X, Y\rangle\langle B, x\rangle = 0,$$

at the points of $\mathcal{W}$. This equation is the key to the following result.

Lemma 4.1 *Let* $x : M_s^n \longrightarrow \bar{M}_\nu^{n+1}$ *be a hypersurface such that* $\Delta x = Ax + B$*. If* M_s^n *has non-constant mean curvature, then* $B = 0$*.*

Proof. Let us consider the set $\mathcal{U} = \{p \in \mathcal{W} : \langle B, x\rangle (p) \neq 0\}$ and assume it is a non-empty set. Then at the points of $\mathcal{U}$, from (3) and (3), we have

$$AX = n(\varepsilon\alpha^2 + k)X - nX(\alpha)N.$$

Since $n \geqslant 2$, we can always find a vector field X such that $X(\alpha) = \langle X, \nabla\alpha\rangle = 0$. This shows, by using (4), that $n(\varepsilon\alpha^2 + k)$ is an eigenvalue of A and therefore locally constant on $\mathcal{U}$, which is a contradiction. Hence $\mathcal{U} = \emptyset$ and $\langle B, x\rangle = 0$ on $\mathcal{W}$. Taking covariant derivative here he deduce that B has no tangent component and therefore we get $B = \varepsilon \langle B, N\rangle N$ and $\langle B, N\rangle = 0$, because $\mathcal{W}$ is not empty.

Next we are going to make some computations before to state the main result of this section. From equation (3) it is easy to see that

$$\langle AX, Y\rangle = \langle X, AY\rangle,$$

for all tangent vector fields X and Y. Taking covariant derivative here and using the Gauss formula jointly with (5), we find

$$\begin{aligned} &\langle A\sigma(X,Z), Y\rangle - \langle A\sigma(Y,Z), X\rangle = \\ &\langle \sigma(X,Z), AY\rangle - \langle \sigma(Y,Z), AX\rangle. \end{aligned} \tag{6}$$

By (2) and (3), the equation (6) becomes

$$\begin{aligned} &\varepsilon\langle SX, Z\rangle\langle AN, Y\rangle - k\langle X, Z\rangle\langle Ax, Y\rangle - \\ &\varepsilon\langle SY, Z\rangle\langle AN, X\rangle + k\langle Y, Z\rangle\langle Ax, X\rangle = \\ &-nY(\alpha)\langle SX, Z\rangle + nX(\alpha)\langle SY, Z\rangle. \end{aligned} \tag{7}$$

Finally, by Lemma 4.1, (4) and (5), from (7) we obtain

$$TX(\alpha)SY = TY(\alpha)SX, \tag{8}$$

where T means the self-adjoint operator defined by $TX = n\alpha X + \varepsilon SX$. This equation becomes the crucial point to show the next result.

Proposition 4.2 *Let* $x : M_s^n \longrightarrow \bar{M}_\nu^{n+1}$ *be an isometric immersion such that* $\Delta x = Ax + B$*. Then* M_s^n *has constant mean curvature.*

Proof. From Lemma 4.1 we can assume $B = 0$ and then equation (8) holds on $\mathcal{W}$. First, suppose that $T(\nabla\alpha) \neq 0$ at the points of $\mathcal{W}$. Then there is a vector field X tangent to M_s^n such that $TX(\alpha) \neq 0$, so that by using (8) we find that rank$S$=1 at the points of $\mathcal{W}$. Therefore we can choose a local orthonormal frame $\{E_1, \ldots, E_n\}$ such that $SE_1 = n\varepsilon\alpha E_1$, $SE_i = 0$, $i = 2, \ldots, n$ and $\varepsilon_i = \langle E_i, E_i\rangle$. Also from (8) we have that $E_i(\alpha) = 0$, $i = 2, \ldots, n$ and using again (3), (4) and (5) we get

$$\begin{aligned} AE_1 &= n(k + \varepsilon n\alpha^2)E_1 - nE_1(\alpha)N, \\ AE_i &= nkE_i, \quad i = 2, \ldots, n, \\ AN &= 3n\varepsilon\varepsilon_1 E_1(\alpha)E_1 + \{\frac{\Delta\alpha}{\alpha} + \varepsilon n^2\alpha^2\}N - nk\varepsilon\alpha x, \\ Ax &= -n\alpha N + nkx. \end{aligned}$$

Therefore, span$\{E_1, N, x\}$ is an invariant subspace under A and the characteristic polynomial $p_A(t)$ of A is given by $p_A(t) = (t - nk)^{n-1} p_{A^*}(t)$, where A^* stands for $A|_{\mathrm{span}\{E_1,N,x\}}$. Then $p_{A^*}(t)$ is constant and we can find three real constants λ_1, λ_2 and λ_3 (which are nothing but the invariants associated to A^*) such that

$$\begin{aligned}
\lambda_1 &= \frac{\Delta\alpha}{\alpha} + 2n\{k + \varepsilon n\alpha^2\},\\
\lambda_2 &= n(2k + \varepsilon n\alpha^2)(\frac{\Delta\alpha}{\alpha} + \varepsilon n^2\alpha^2) + 3n^2\varepsilon\varepsilon_1 E_1(\alpha)^2 + n^2k(k + \varepsilon n\alpha^2) - kn^2\varepsilon\alpha^2,\\
\lambda_3 &= n^2k(k + \varepsilon n\alpha^2)(\frac{\Delta\alpha}{\alpha} + \varepsilon n^2\alpha^2) - n^3k\varepsilon\alpha^2(k + \varepsilon n\alpha^2) + 3n^3\varepsilon\varepsilon_1 kE_1(\alpha)^2.
\end{aligned}$$

Then we obtain

$$nk\lambda_2 = \lambda_3 + n^3(k + \varepsilon n\alpha^2) + n^2(\frac{\Delta\alpha}{\alpha} + \varepsilon n^2\alpha^2) + n^4k\alpha^4$$

and

$$\frac{\Delta\alpha}{\alpha} = \lambda_1 - 2n(k + \varepsilon n\alpha^2).$$

Last two equations allow us to write $n^4\alpha^4 = kp_{A^*}(kn)$ and so α is locally constant on $\mathcal{W}$, which is a contradiction.

Finally, assume now there is a point p in $\mathcal{W}$ such that $T(\nabla\alpha)(p) = 0$. Note that from (3) and (5) we have in $\mathcal{W}$, $\forall i = 1, \ldots, n$

$$\begin{aligned}
\langle AE_i, N\rangle &= -n\varepsilon E_i(\alpha) \\
\langle E_i, AN\rangle &= \frac{2\varepsilon}{\alpha}\langle T(\nabla\alpha), E_i\rangle - n\varepsilon E_i(\alpha).
\end{aligned} \tag{9}$$

It follows that at p,

$$\langle AE_i, N\rangle = \langle E_i, AN\rangle. \tag{10}$$

From (3), (4), (5), (1) and (10) we deduce that A is a self-adjoint endomorphism of $\mathbb{R}^{n+2}_t$ and thus equation (10) remains valid at every point in $\mathcal{W}$. In turn, from (10) $T(\nabla\alpha) = 0$ on $\mathcal{W}$ and so $\nabla\alpha$ is an eigenvector of S with associated eigenvalue $-n\varepsilon\alpha$. If $\langle\nabla\alpha, \nabla\alpha\rangle = \nabla\alpha(\alpha) = 0$, from (3) we could write $A(\nabla\alpha) = n(k - n\varepsilon\alpha^2)\nabla\alpha$, then $n(k - n\varepsilon\alpha^2)$ should be an eigenvalue of A and α must be locally constant on $\mathcal{W}$, which cannot be hold. Therefore we can choose a local orthonormal frame $\{E_1, \ldots, E_n\}$ with E_1 parallel to $\nabla\alpha$ such that

$$\begin{aligned}
AE_1 &= n(k - n\varepsilon\alpha^2)E_1 - nE_1(\alpha)N,\\
AE_i &= n(\alpha SE_i + kE_i), \qquad i = 2, \ldots, n,\\
AN &= -n\varepsilon\varepsilon_1 E_1(\alpha)E_1 + \{\frac{\Delta\alpha}{\alpha} + \varepsilon|S|^2\}N - nk\varepsilon\alpha x,\\
Ax &= -n\alpha N + nkx.
\end{aligned}$$

Since $SE_i \in \mathrm{span}\{E_2, \ldots, E_n\}$, $i = 2, \ldots, n$, we may write S^* for the endomorphism S restricted to span$\{E_2, \ldots, E_n\}$. Working on the characteristic polynomials, from the above equations we can deduce that

$$p_A(t) = (n\alpha)^{n-1}q(t)p_{S^*}(\frac{t - nk}{n\alpha}),$$

where $q(t)$ is a polynomial of degree three. Let $\{r_1, \dots, r_n\}$ be the possibly complex roots of $p_S(t)$, with $r_1 = -n\varepsilon\alpha$ and $\{r_2, \dots, r_n\}$ the roots of $p_{S^*}(t)$. Then the functions $nk + n\alpha r_j$, $j = 2, \dots, n$, are roots of $p_A(t)$ and therefore they are locally constant on $\mathcal{W}$. Thus from the formula $\sum_{j=2}^n (nk+n\alpha r_j) = n(n-1)k+n\alpha\sum_{j=2}^n r_j = n(n-1)k+n\alpha(\mathrm{tr}S-r_1) = n(n-1)k+2\varepsilon n^2\alpha$, we obtain that α is locally constant on $\mathcal{W}$, which is a contradiction.

Summarizing, we have got that $\mathcal{W}$ has to be empty, i.e., M_s^n has constant mean curvature.

5. Main results

We have just proved that the hypersurfaces M_s^n of $\bar{M}_\nu^{n+1}$ satisfying the condition (2) have constant mean curvature. In this section, we wish to give a classification theorem of such a class of hypersurfaces. To do that, we recall the following definition. A hypersurface M_s^n is said to be *isoparametric* if the characteristic polynomial $p_S(t)$ of its shape operator S is the same at all points of M_s^n. When S is diagonalizable (for example, in the definite case) that means that the principal curvatures of M_s^n, as well as its multiplicities, are constant. Our first main result states as follows.

Theorem 5.1 *Let* $x : M_s^n \longrightarrow \bar{M}_\nu^{n+1}$ *be an isometric immersion satisfying* $\Delta x = Ax + B$*. Then* M_s^n *is a minimal or an isoparametric hypersurface.*

Proof. Let M_s^n be a hypersurface of $\bar{M}_\nu^{n+1}$ satisfying (2). By Proposition 4.2 we can assume that the mean curvature α of M_s^n in $\bar{M}_\nu^{n+1}$ is a non-zero constant and so the equation (3) works here. If M_s^n is not totally umbilical in $\bar{M}_\nu^{n+1}$, then we have $\langle B, x\rangle = 0$ and, as in Lemma 4.1, $B = 0$. Now, from (3), (4) and (5) we get

$$\begin{aligned} AX &= n(\alpha SX + kX), \\ AN &= \varepsilon|S|^2 N - nk\varepsilon\alpha x, \\ Ax &= -n\alpha N + nkx. \end{aligned} \tag{1}$$

Working again on the characteristic polynomials $p_A(t)$ and $p_S(t)$, as in Section 4, we deduce that $p_S(t)$ is constant on M_s^n and the proof finishes.

Next with the aim of getting a complete classification of those hypersurfaces M_s^n of $\bar{M}_\nu^{n+1}$ satisfying (2), some easy computations are needed. From Theorem 5.1, M_s^n is an isoparametric hypersurface provided that the (constant) mean curvature α is not zero, and thus $|S|^2$ is also constant. Taking covariant derivative in the expression of AN in (1) we have $\tilde{\nabla}_X(AN) = -\varepsilon|S|^2 SX - nk\varepsilon\alpha X$ and $\tilde{\nabla}_X(AN) = A(\tilde{\nabla}_X N) = -n(\alpha S^2 X + kSX)$, from which we obtain

$$S^2 + \frac{nk - \varepsilon|S|^2}{n\alpha} S - k\varepsilon I = 0, \tag{2}$$

where I stands for the identity operator on the tangent bundle of M_s^n. We have just seen that if M_s^n is not totally umbilical then $B = 0$ and thus equations (1) and (2) allow us to write

$$A^2 - (\varepsilon|S|^2 + kn)A + n\varepsilon k(|S|^2 - n\alpha^2)I_{n+2} = 0,$$

and furthermore, from (1), A is a self-adjoint endomorphism of $\mathbb{R}_t^{n+2}$. If S is diagonalizable, from (2), M_s^n has exactly two constant principal curvatures. By using now similar arguments as in Theorem 2.5 of [**11**] and Lemma 2 of [**8**], we deduce that M_s^n is an open piece of a pseudo-Riemannian product of two non-flat totally umbilical submanifolds. If S is not diagonalizable, from (3), the minimal polynomial $\mu_A(t)$ of A is given by $\mu_A(t) = t^2 + at + b$, with $a = -(\varepsilon|S|^2 + kn)$ and $b = n\varepsilon k(|S|^2 - n\alpha^2)$. Since $\langle Ax, x\rangle = n$ is constant on M_s^n and $\mu_A(kn) = -n^2\alpha^2\varepsilon k \neq 0$, then M_s^n is an open piece of a quadratic hypersurface as in example **2.4**. Summing up, we have proved the following theorem.

Theorem 5.2 *Let $x : M_s^n \longrightarrow \bar{M}_\nu^{n+1}$ be an isometric immersion. Then $\Delta x = Ax + B$ if and only if M_s^n is an open piece of one of the following hypersurfaces in $\bar{M}_\nu^{n+1}$:*
1) a minimal hypersurface,
2) a totally umbilical hypersurface,
3) a pseudo-Riemannian non-minimal standard product,
4) a quadratic hypersurface as in example **2.4**, *with non-diagonalizable shape operator ($a^2 - 4b \leqslant 0$).*

As a consequence, we obtain the classification theorem for hypersurfaces in $\mathbb{S}^{n+1}$ and $\mathbb{H}^{n+1}$, which generalizes Theorem 1.3 in [**10**].

Corollary 5.3 *Let $x : M^n \longrightarrow \bar{M}^{n+1}$ be a non-minimal hypersurface. Then $\Delta x = Ax + B$ if and only if M^n is an open piece of one of the following hypersurfaces:*
1) a totally umbilical hypersurface,
2) a product $\bar{M}^p(r_1) \times \mathbb{S}^{n-p}(r_2)$.

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