Hopf cylinders, B-scrolls and solitons of the Betchov-Da Rios equation in the 3-dimensional anti-De Sitter space

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Abstract - We use the natural Hopf fibrations from $\mathbb{H}^3_1(-1)$ over $\mathbb{H}^2_s(-1/4)$ $(s = 0, 1)$ to give a geometric interpretation of the B-scrolls in terms of the Hopf cylinders shaped on non-null curves in $\mathbb{H}^2_s(-1/4)$. We also find those parametrizations of the Hopf cylinders which are solutions of the Betchov-Da Rios soliton equation in $\mathbb{H}^3_1(-1)$. In particular, the soliton solutions are the null geodesics of the Lorentzian Hopf cylinders.

Cylindres de Hopf, B-scrolls et solitons dans l'espace anti De Sitter de dimension trois

Résumé - Nous utilisons les fibrés de Hopf de $\mathbb{H}^3_1(-1)$ sur $\mathbb{H}^2_s(-1/4)$, $(s = 0, 1)$, à fin de donner une interprétation géométrique des cylindres de Hopf modelés sur courbes non-nulles dans $\mathbb{H}^2_s(-1/4)$. On trouve d'autre part les paramétrisations des cylindres de Hopf qui sont solutions de l'équation soliton de Betchov-Da Rios dans $\mathbb{H}^3_1(-1)$. En particulier, les solutions soliton sont les géodésiques nulles des cylindres de Hopf lorentziennes.

Version française abrégée -

Nous considérons dans $\mathbb{R}^4_2$ l'hypersurface $\mathbb{H}^3_1(-1) = \left\{x \in \mathbb{R}^4_2 : \langle x, x\rangle = -1\right\}$, qui est une variété lorentzienne à courbure sectionnelle constante -1 appelée l'espace anti De Sitter à dimension trois. D'après [**2**], une surface lorentzienne dans $\mathbb{H}^3_1(-1)$ est dite B-scroll modelée sur une courbe spatiale à torsion constante ± 1 si elle est parametrée par $f(t, z) = \cos(z)\bar{\beta}(t) + \sin(z)\bar{B}(t)$, où $\bar{B}(t)$ est la binormale de $\bar{\beta}(t)$. Pareillement, on obtient les B-scrolls modelées sur courbes temporelles. Soient $\pi_s : \mathbb{H}^3_1(-1) \to \mathbb{H}^2_s(-1/4)$ $(s = 0, 1)$ les fibrés de Hopf avec fibre $\mathbb{S}^1$ $(s = 0)$ et $\mathbb{H}^1$ $(s = 1)$, respectivement. Alors nous pouvons définir les cylindres de Hopf par $M_\beta = \pi_s^{-1}(\beta)$, où β est une courbe non-nulle dans $\mathbb{H}^2_s(-1/4)$. On remarque que M_β est lorentzienne si $s = 0$, tandis que elle est lorentzienne ou riemannienne si $s = 1$ selon que β soit temporelle ou spatiale, respectivement. Ainsi, nous prouvons la caractérisation géométrique des B-scrolls suivante.

Théorème 2 *Soit M une surface lorentzienne de $\mathbb{H}^3_1(-1)$. Alors M est le cylindre de Hopf d'une courbe non-nulle β dans $\mathbb{H}^2_s(-1/4)$ si et seulement si M est le B-scroll attaché à un relèvement horizontal $\bar{\beta}$ de β.*

D'autre part l'équation de Betchov-Da Rios (2) est une équation soliton par rapport à certaines applications définies sur un ouvert de $\mathbb{R}^2$ à valeurs dans une variété semi-Riemannienne à dimension trois $\bar{M}$ munie d'une connexion semi-Riemannienne $\bar{\nabla}$. Cette équation décrit le comportement d'un fluide incompressible et non visqueux dans $\bar{M}$. En utilisant la stratégie des cylindres de Hopf on obtient le théorème important (qui sera démontré dans [**1**]):

Théorème 4 *Soit β une courbe non-nulle parametrée par l'arc and soit M_β le cylindre de Hopf dans $\mathbb{H}^3_1(-1)$ attaché à β. Soit h un difféomorphisme de $\mathbb{R}^2$ et considérons $Y = X \circ h : \mathbb{R}^2 \to M_\beta \subset \mathbb{H}^3_1(-1)$, où X est le revêtement standard de $\mathbb{R}^2$ sur M_β. Alors Y est une solution de léquation soliton de Betchov-Da Rios dans $\mathbb{H}^3_1(-1)$ si et seulement si*

(1) β *est une courbe à courbure constante* κ*, et*
(2) $h(u,v) = (t(u,v), z(u,v))$ *vérifie les conditions suivantes*

$$t(u,v) = au - ag\rho v + c_1,$$
$$z(u,v) = agu - a\rho v + c_2,$$

où $(1-g^2)a^2 = \varepsilon$ *(*ε *étant le caractère causal des* u*-courbes),* $g \in \mathbb{R} - \{-1,+1,-\kappa/2\}$*,* $\rho = (\kappa+2g)a^2$ *est la courbure des* u*-courbes dans* $\mathbb{H}^3_1(-1)$ *et* $c_1, c_2 \in \mathbb{R}$*.*

1. B-scrolls in anti De Sitter space $\mathbb{H}^3_1(-1)$

Let $\mathbb{R}^4_2$ be the 4-dimensional lineal space $\mathbb{R}^4$ endowed with the inner product of signature (2,2) given by $\langle x,y\rangle = -x_1y_1 - x_2y_2 + x_3y_3 + x_4y_4$ for $x = (x_1,x_2,x_3,x_4)$, $y = (y_1,y_2,y_3,y_4) \in \mathbb{R}^4$. The space $\mathbb{H}^3_1(-1)$ is the hypersurface of $\mathbb{R}^4_2$ defined as $\mathbb{H}^3_1(-1) = \left\{x \in \mathbb{R}^4_2 : \langle x,x\rangle = -1\right\}$. Then $\mathbb{H}^3_1(-1)$ with the restriction of $\langle,\rangle$ is a Lorentzian manifold with constant sectional curvature -1, which is called the 3-dimensional anti De Sitter space.

In order to study flat surfaces, isometrically immersed, in $\mathbb{H}^3_1(-1)$, M. Dajczer and K. Nomizu, [**2**], by extending a construction of L.K. Graves, [**3**], use the notion of B-scroll of a Frenet curve (space-like, time-like or null) in $\mathbb{H}^3_1(-1)$. A curve $\bar\beta(t)$ in $\mathbb{H}^3_1(-1)$ is said to be a unit speed curve if $\langle d\bar\beta(t)/dt, d\bar\beta(t)/dt\rangle = \varepsilon$ (ε being $+1$ or -1 according to $\bar\beta$ is space-like or time-like, respectively). For a better understanding of the next construction we will bring back the notion of cross product in the tangent space $T_p\mathbb{H}^3_1(-1)$ of any point p in $\mathbb{H}^3_1(-1) \subset \mathbb{R}^4_2$. In $T_p\mathbb{H}^3_1(-1)$ there is a natural orientation defined as follows: an ordered basis $\{X,Y,X\}$ in $T_p\mathbb{H}^3_1(-1)$ is positively oriented if det$[pXYZ] > 0$, where $[pXYZ]$ is the matrix with p, X, Y, $Z \in \mathbb{R}^4_2$ as row vectors. Now let ω be the volumen element on $\mathbb{H}^3_1(-1)$ defined by $\omega(X,Y,Z) = det[pXYZ]$. Then given X, $Y \in T_p\mathbb{H}^3_1(-1)$, the cross product $X \wedge Y$ is the unique vector in $T_p\mathbb{H}^3_1(-1)$ such that $\langle X\wedge Y, Z\rangle = \omega(X,Y,Z)$, for any $Z \in T_p\mathbb{H}^3_1(-1)$. Let us recall how a B-scroll is defined, for instance in the case of a space-like curve (other cases are similarly defined with obvious changes). Given a complete space-like unit speed curve $\bar\beta(t)$ in $\mathbb{H}^3_1(-1)$, it is called a space-like Frenet curve if it admits a Frenet frame field $\left\{\bar T = d\bar\beta/dt, \bar N, \bar B\right\}$ such that $\left\langle \bar N, \bar N\right\rangle = 1$, $\bar B = \bar T \times \bar N$ and satisfying the Frenet equations

$$\bar\nabla_{\bar T}\bar T = \bar\kappa \bar N,$$
$$\bar\nabla_{\bar T}\bar N = -\bar\kappa \bar T + \bar\tau \bar B,$$
$$\bar\nabla_{\bar T}\bar B = \bar\tau \bar N,$$

where $\bar\nabla$ is the semi-Riemannian connection on $\mathbb{H}^3_1(-1)$ and $\bar\kappa = \bar\kappa(t)$ and $\bar\tau = \bar\tau(t)$ are the curvature and the torsion of $\bar\beta$, respectively. In particular, if $\bar\tau = 1$ (or -1), the mapping $f : \mathbb{R}^2 \to \mathbb{H}^3_1(-1)$ defined by $f(t,z) = \cos(z)\bar\beta(t) + \sin(z)\bar B(t)$ is an isometric immersion from $\mathbb{R}^2_1$ into $\mathbb{H}^3_1(-1)$ which is called the B-scroll of $\bar\beta$ (see [**2**] for details).

2. Geometric interpretation of B-scrolls via Hopf cylinders

As usual we identify $\mathbb{R}^4_2$ with $\mathbb{C}^2_1$. Here $\mathbb{C}^2_1$ denotes the 2-dimensional complex lineal space $\mathbb{C}^2$ endowed with the Hermitian form $(a,b) = -a_1\bar b_1 + a_2\bar b_2$, where $a = (a_1,a_2)$, $b = (b_1,b_2) \in \mathbb{C}^2$. Then $\mathbb{H}^3_1(-1) = \left\{a \in \mathbb{C}^2_1 : (a,a) = -1\right\}$ and we consider two natural actions of $\mathbb{S}^1$ (the unit circle in $\mathbb{R}^2$) and $\mathbb{H}^1$ (the unit circle in $\mathbb{R}^2_1$), respectively, over $\mathbb{H}^3_1(-1)$, namely $(r,(a_1,a_2)) = (ra_1, ra_2)$, where $r \in \mathbb{S}^1$ or $r \in \mathbb{H}^1$. Then $\mathbb{H}^2(-1/4)$ (the hyperbolic plane with Gaussian curvature -4) and $\mathbb{H}^2_1(-1/4)$ (the pseudo-hyperbolic plane with Gaussian curvature -4) are obtained as orbit spaces.

Summarizing up, we have two natural Hopf fibrations $\pi_s : \mathbb{H}^3_1(-1) \to \mathbb{H}^2_s(-1/4)$, $s = 0, 1$, with fibers $\mathbb{S}^1$ and $\mathbb{H}^1$, respectively. Actually π_s are semi-Riemannian submersions. Therefore we will use the own terminology on this topic (see [**5**] for details), in particular overbars are used to distinguish the lifts of corresponding geometrical objects on $\mathbb{H}^2_s(-1/4)$. So if ∇ denotes the semi-Riemannian connection on $\mathbb{H}^2_s(-1/4)$, we have

$$\begin{aligned}
&\bar{\nabla}_{\bar{X}}\bar{Y} = \overline{\nabla_X Y} + (-1)^s(\langle JX, Y\rangle \circ \pi_s)V \\
&\bar{\nabla}_{\bar{X}}V = \bar{\nabla}_V \bar{X} = \overline{JX} \\
&\bar{\nabla}_V V = 0
\end{aligned}$$

where J denotes the standard complex structure of both $\mathbb{H}^2_s(-1/4)$ and V is nothing but a unit vector field tangent to the fibers (that is, a vertical unit vector field).

Let β be a complete unit speed curve, immersed in $\mathbb{H}^2_s(-1/4)$, with Frenet frame $\{T, N\}$ and curvature function κ. Consider a horizontal lift $\bar{\beta}$ of β and denote by $\left\{\bar{T}, N^*, B^*\right\}$, κ^* and τ^* its corresponding Frenet objects. Now we can combine (1) with the Frenet equations of β and $\bar{\beta}$ to prove that $N^* = \bar{N}$. In particular, it yields to the horizontal distribution along $\bar{\beta}$ and it has the same causal character as N. Also it is not difficult to see that $\tau^* \equiv 1$ (or -1) and $B^* = V$ (or $-V$), that is, the binormal B^* of $\bar{\beta}$ coincides with the unit tangent to the fibers through each point of $\bar{\beta}$. Therefore we have proved the following

Lemma 2.1 *(i) The horizontal lifts of unit speed curves in $\mathbb{H}^2(-1/4)$ are space-like Frenet curves in $\mathbb{H}^3_1(-1)$ with torsion 1 (or -1).*

(ii) The horizontal lifts of unit speed curves in $\mathbb{H}^2_1(-1/4)$ are time-like Frenet curves in $\mathbb{H}^3_1(-1)$ with torsion 1 (or -1).

By pulling back via π_s a non-null curve β in $\mathbb{H}^2_s(-1/4)$ we get the total horizontal lift of β, which is an immersed flat surface M_β in $\mathbb{H}^3_1(-1)$, that will be called the semi-Riemannian Hopf cylinder associated to β. Notice that if $s = 0$, then M_β is a Lorentzian surface, whereas if $s = 1$, M_β is Riemannian or Lorentzian according to β is space-like or time-like, respectively.

Theorem 2.2 *Let M be a Lorentzian surface immersed into $\mathbb{H}^3_1(-1)$. Then M is the semi-Riemannian Hopf cylinder associated to a unit speed curve β in $\mathbb{H}^2_s(-1/4)$ if and only if M is the B-scroll of any horizontal lift $\bar{\beta}$ of β.*

Proof. Suppose $M = M_\beta$ and $\bar{\beta}$ is a horizontal lift of β. Then $\bar{\beta}$ goes through the fibers to parametrize M_β as follows,

$$X(t,z) = \begin{cases} \cos(z)\bar{\beta}(t) + \sin(z)i\bar{\beta}(t), \text{ if } s = 0 \\ \cosh(z)\bar{\beta}(t) + \sinh(z)i\bar{\beta}(t), \text{ if } s = 1. \end{cases}$$

Now observe that $i\bar{\beta}(t)$ is the unit tangent vector field to the fibers along $\bar{\beta}$, which is nothing but the binormal B^* of $\bar{\beta}$. Therefore if M_β is Lorentzian, then it is the B-scroll of $\bar{\beta}$. A similar argument works to prove the converse.

3. Hopf cylinders and solutions of the Betchov-Da Rios soliton equation

In the last section, we obtained the following nice property of Hopf cylinders: the unit normal of $\bar{\beta}$ in $\mathbb{H}^3_1(-1)$ coincides with the unit normal of M_β into $\mathbb{H}^3_1(-1)$ along any horizontal lift $\bar{\beta}$ of β and then the binormal of $\bar{\beta}$ is tangent to the fibers along $\bar{\beta}$ for any $\bar{\beta}$. Consequently, the binormal B^* of $\bar{\beta}$ can be extended to a Killing vector field on $\mathbb{H}^3_1(-1)$ and then $X(t,z)$ defines a solution of the binormal flow. In particular, if β has non-zero constant mean curvature in $\mathbb{H}^2_s(-1/4)$, one can use κ to reparametrize the fibers to get solutions $Y(t,z) = X(t,\kappa z)$ of the so called "filament flow" (see for instance [**4**] for some details about the filament equation).

More generally, we can consider the following equation for space curves $\gamma(t,z)$ in a three-dimensional pseudo-Riemannian manifold endowed with the pseudo-Riemannian connection $\overline{\nabla}$

$$\frac{\partial\gamma}{\partial t} \wedge \overline{\nabla}_{\frac{\partial\gamma}{\partial t}} \frac{\partial\gamma}{\partial t} = \frac{\partial\gamma}{\partial z}$$

The next result gives infinitely many solutions of (2) in $\mathbb{H}^3_1(-1)$, which are obtained by means of Hopf cylinders.

Theorem 3.1 *Let M_β be a Hopf cylinder in $\mathbb{H}^3_1(-1)$ over a unit speed curve β in $\mathbb{H}^2_s(-1/4)$ of curvature function κ. For any nonzero real number c and any solution $t(u)$ of $\kappa(t(u))t_u^3 - c = 0$, we define*

$$Y(u,v) = \begin{cases} \cos(cv)\overline{\beta}(t(u)) + \sin(cv)\overline{B}(t(u)), & s = 0, \\ \cosh(cv)\overline{\beta}(t(u)) + \sinh(cv)\overline{B}(t(u)), & s = 1. \end{cases}$$

Then $Y(u,v)$ are solutions of (2).

The proof of this theorem will appear in a forthcoming paper, [**1**].

In particular, the equation (2) becomes the Betchov-Da Rios equation (also called the localized induction equation, or the filament equation when viewed as an evolution equation) when t denotes the arc-length of the t-curves. This equation is a soliton equation and describes the behaviour of an incompressible, inviscid fluid.

In the following we describe all solutions of the Betchov-Da Rios soliton equation in $\mathbb{H}^3_1(-1)$ which are living in Hopf cylinders shaped on curves in $\mathbb{H}^2(-1/4)$. A similar result works for solutions in Hopf cylinders into $\mathbb{H}^3_1(-1)$ over curves in $\mathbb{H}^2_1(-1/4)$. They will appear in [**1**].

Theorem 3.2 *Let β be an arc-length parametrized curve in $\mathbb{H}^2(-1/4)$ and M_β its Hopf cylinder in $\mathbb{H}^3_1(-1)$. For any diffeomosphism h of $\mathbb{R}^2$, we consider $Y = X \circ h : \mathbb{R}^2 \to M_\beta \subset \mathbb{H}^3_1(-1)$, where X denotes the standard covering of $\mathbb{R}^2$ onto M_β (see Theorem 2.2). Then Y is a solution of the Betchov-Da Rios soliton equation in $\mathbb{H}^3_1(-1)$ if and only if*
(1) *β has constant curvature, say κ, in $\mathbb{H}^2(-1/4)$, and*
(2) *$h(u,v) = (t(u,v), z(u,v))$ is defined as*

$$t(u,v) = au - ag\rho v + c_1,$$
$$z(u,v) = agu - a\rho v + c_2,$$

where $(1-g^2)a^2 = \varepsilon$ (ε being the causal character of the u-curves), $g \in \mathbb{R} - \{-1, +1, -\kappa/2\}$, $\rho = (\kappa + 2g)a^2$ is the curvature of the u-curves in $\mathbb{H}^3_1(-1)$ and (c_1, c_2) is any couple of constants.

Remark 3.3 The u-curves in this theorem are certainly helices in $\mathbb{H}^3_1(-1)$. They are space-like if the slope $g \in (-1, 1)$, otherwise, they are time-like. A dual result for the slope can be obtained when working on Lorentzian Hopf cylinders in $\mathbb{H}^3_1(-1)$ coming from π_1. For Riemannian Hopf cylinders in $\mathbb{H}^3_1(-1)$ coming from space-like curves in $\mathbb{H}^2_1(-1/4)$ any value of the slope g is allowed. In [**1**] we prove a converse of this result, namely: any helix δ in $\mathbb{H}^3_1(-1)$ is a solution of the filament equation in $\mathbb{H}^3_1(-1)$ living in a certain Hopf cylinder M_β in $\mathbb{H}^3_1(-1)$. Moreover the curvature κ of β in $\mathbb{H}^2_s(-1/4)$ and the slope g of δ in M_β are uniquely determined from the curvature and the torsion of the helix in $\mathbb{H}^3_1(-1)$.

Theorem 4 can also be read as follows. For any value of κ, that is, for any Hopf cylinder M_γ, one has a one parameter family $\{Y_g = X \circ h_g : \mathbb{R}^2 \to M_\gamma \subset \mathbb{H}^3_1(-1)/g \in \mathbb{R} - \{-\kappa/2, -1, 1\}\}$ of congruence solutions of the Betchov-Da Rios soliton equation lying in M_γ. By computing the limit of Y_g, as g goes to ± 1, we get a pair of null geodesics in M_γ which are singular solutions. From Remark 1, that also works for Lorentzian Hopf cylinders coming from time-like curves in $\mathbb{H}^2_1(-1/4)$. However, when g goes to $-\kappa/2$ we find no solution neither Lorentzian nor Riemannian case. Summing up we have

Corollary 3.4 *Up to congruences, there is a unique soliton solution of the Betchov-Da Rios equation in $\mathbb{H}^3_1(-1)$ lying in any Lorentzian Hopf cylinder of constant mean curvature M_γ. Moreover it is a null geodesic of M_γ.*

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