Semi-Riemannian constant mean curvature surfaces via its quadric representation

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Houston J. Math. 22 (1996), 1–14

(Partially supported by DGICYT grant PB94-0750 and Fundación Séneca PIB95/34)

1. Introduction

In a series of early papers, [**2**] and [**4**], we have classified semi-Riemannian surfaces M_s^2 into semi-Riemannian space forms $\bar{M}_1^3(k)$ satisfying the Laplacian equation $\Delta x = Ax + B$ in the isometric immersion x. The key point to achieve the classification is to show that condition implies the isoparametricity of the surface. For instance, that family in $\mathbb{L}^3$ consists of the products $\mathbb{L} \times \mathbb{S}^1(r)$, $\mathbb{R} \times \mathbb{H}^1(-r)$ and $\mathbb{R} \times \mathbb{S}_1^1(r)$, as well as $\mathbb{H}^2(-r)$, $\mathbb{S}_1^2(r)$ and minimal surfaces. A richer family was obtained into De Sitter and anti De Sitter worlds where, apart from minimal surfaces, all surfaces were explicitly listed in [**4**]. Since it is well known that the class of minimal surfaces in those ambient spaces is quite interesting and large enough to deserve a deep study, we propose to distinguish minimal surfaces or, more generally, constant mean curvature (CMC) surfaces. That idea was first exploited by A. Ros [**11**], M. Barros and B.Y. Chen [**8**], and M. Barros and F. Urbano [**7**], in order to study spectral geometry of minimal submanifolds in the sphere, where they asked for the eigenvalue behaviour of the products of the Laplacian eigenfunctions.

Given a submanifold $x : M^n \longrightarrow \mathbb{R}^m$ the products of the coordinate functions allow us to define a smooth map φ from M^n into the set $SM(m)$ of $(m \times m)$-real symmetric matrices defined by $\varphi = xx^t$ (in general, not an isometric immersion). This will be called the quadric representation of M^n. In [**9**], I. Dimitric made a nice study of that map. In [**5**], M. Barros and O.J. Garay, using the quadric representation of a surface in $\mathbb{S}^3$, have obtained a new characterization of the Clifford torus, among all compact minimal and non totally geodesic surfaces in $\mathbb{S}^3$, as the only surface whose quadric representation is mass-symmetric in some hypersphere and minimal in some concentric hyperquadric. Recently, [**6**], the same authors, in order to improve their above result, stated the following problem: are there compact minimal surfaces in $\mathbb{S}^3$ with quadric representation living minimally in some hyperquadric of $SM(4)$ others than the Clifford torus and the totally geodesic 2-sphere? They show that the answer is negative.

As for semi-Riemannian surfaces, we were interesting in a more general problem: *classify CMC semi-Riemannian surfaces in the non-flat 3-dimensional semi-Riemannian space forms whose quadric representations into the set $SA(4,\nu)$ of selfadjoint matrices satisfies a certain Laplacian differential equation*. Notice that according to [**9**, Theorem 1] the flat case should be avoided. We have observed that, under Barros-Garay conditions, the quadric representation φ of a CMC semi-Riemannian surface in $\bar{M}_1^3(k)$, $k \neq 0$, satisfies the matricial Laplacian equation $\Delta\varphi = A * \varphi + B$ (see section 3 for the definition of the star product $*$). Then an interesting problem arises as follows: *could you characterize CMC semi-Riemannian surfaces into $\bar{M}_1^3(k)$ whose quadric representation satisfies that equation?*

It should be pointed out that, on one hand, we do not need the surface to be either compact or minimal. On the other hand, since the surface is now endowed with a semi-Riemannian metric, our problem naturally generalizes that of Barros-Garay. Therefore, it seems reasonable to hope for finding a richer class of CMC semi-Riemannian surfaces than in the Riemannian situation. Actually, our main result states as follows:

Let M_s^2 be a semi-Riemannian constant mean curvature surface in $\bar{M}_1^3(k)$ whose quadric representation satisfies $\Delta\varphi = A * \varphi + B$. Then M_s^2 is an open piece of one of the following surfaces:
1) $\mathbb{S}^2(r)$, $\mathbb{S}_1^2(r)$, $\mathbb{H}^2(-r)$, $\mathbb{H}_1^2(-r)$.
2) $\mathbb{H}^1(-r) \times \mathbb{S}^1(\sqrt{1+r^2})$, $\mathbb{S}_1^1(r) \times \mathbb{S}^1(\sqrt{1-r^2})$, $\mathbb{H}_1^1(-r) \times \mathbb{S}^1(\sqrt{r^2-1})$, $\mathbb{S}_1^1(r) \times \mathbb{H}^1(-\sqrt{1+r^2})$, $\mathbb{H}^1(-r) \times \mathbb{H}^1(-\sqrt{1-r^2})$.

2. Preliminaries

Let $\mathbb{R}_\nu^4$ be the pseudo-Euclidean 4-dimensional space endowed with the standard inner product of index ν given by $\langle a, b\rangle = a^t G b$, where $G = \operatorname{diag}[\delta_1, \delta_2, \delta_3, \delta_4]$, $\delta_i = \pm 1$, stands for the matrix of the metric with respect to the usual rectangular coordinates. Throughout this paper, vectors in $\mathbb{R}_\nu^4$ will be regarded as column matrices and $(\cdot)^t$ will denote the transpose matrix. As usual, let $\mathbb{S}_1^3 = \{x \in \mathbb{R}_1^4 : \langle x, x\rangle = 1\}$ and $\mathbb{H}_1^3 = \{x \in \mathbb{R}_2^4 : \langle x, x\rangle = -1\}$ be the unit pseudosphere and the unit pseudohyperbolic space, respectively, viewed as hypersurfaces of index one with constant sectional curvature $k = +1$ and $k = -1$, respectively. From now on, $\bar{M}_1^3(k)$ will denote $\mathbb{S}_1^3$ or $\mathbb{H}_1^3$ according to $k = 1$ or $k = -1$, and $\mathbb{R}_\nu^4$ the pseudo-Euclidean space where $\bar{M}_1^3(k)$ is lying.

Let $SA(4, \nu) = \{B \in \mathfrak{gl}(4, \mathbb{R}) : B^t G = GB\}$ be the set of selfadjoint endomorphisms of $\mathbb{R}_\nu^4$ equipped with the metric $g(B, C) = \frac{k}{2}\operatorname{trace}(BC)$. Let $f : \bar{M}_1^3(k) \longrightarrow SA(4, \nu)$ be the map defined by $f(x) = xx^t G$. It is easy to see that f is an isometric immersion, that is called the second standard immersion of $\bar{M}_1^3(k)$, and its second fundamental form $\bar{\sigma}$ is given by

$$\bar{\sigma}(X, Y) = (XY^t + YX^t)G - 2k\langle X, Y\rangle f(x), \tag{1}$$

for any $x \in \bar{M}_1^3(k)$ and $X, Y \in T_x\bar{M}_1^3(k)$.

Given an isometric immersion $x : M_s^2 \longrightarrow \bar{M}_1^3(k)$ of a semi-Riemannian surface M_s^2 into $\bar{M}_1^3(k)$, the map $\varphi : M_s^2 \longrightarrow SA(4, \nu)$ defined by $\varphi = f \circ x$ is also an isometric immersion that will be called the quadric representation of M_s^2. Then the mean curvature vector fields H_1 and H associate to the immersions x and φ, respectively, are related by the formula

$$H = (H_1 x^t + x H_1^t)G + \sum_{i=1}^{2} \varepsilon_i E_i E_i^t G - 2k\varphi,$$

where $\{E_1, E_2\}$ is an orthonormal frame field tangent to M_s^2 and $\varepsilon_i = \langle E_i, E_i\rangle$, $i = 1, 2$.

3. First characterization results

Let us suppose that the quadric representation $\varphi = (\varphi_{ij})$ of the isometric immersion $x : M_s^2 \longrightarrow \bar{M}_1^3(k)$ satisfies the system of differential equations

$$\Delta\varphi_{ij} = a_{ij}\varphi_{ij} + b_{ij}, \text{ for all } i, j,$$

for some real numbers a_{ij} and b_{ij}. From the definition of φ, it is easy to see that $a_{ij} = a_{ji}$ and $b_{ij} = \delta_i\delta_j b_{ji}$. Therefore the above conditions can be globally written as

$$\Delta\varphi = A * \varphi + B,$$

where $A = (a_{ij})$ is a symmetric matrix, $B = (b_{ij})$ is a selfadjoint endomorphism, and the star product $*$ associates to each pair of matrices $C = (c_{ij})$ and $D = (d_{ij})$ the matrix $C*D = (c_{ij}d_{ij})$.

The first geometric meaning of that equation is contained in the following lemma, where we prove that a surface satisfying (1) lies in a hyperquadric of $\mathbb{R}^4_\nu$.

Lemma 3.1 *Let $x : M^2_s \longrightarrow \bar{M}^3_1(k)$ be an isometric immersion whose quadric representation satisfies $\Delta\varphi = A * \varphi + B$. Then M^2_s is contained in the hyperquadric defined by $\langle Bx, x\rangle = c$, for some real constant c.*

Proof. First, observe that $\langle Bx, x\rangle = 2kg(B,\varphi)$, then it suffices to show that $Xg(B,\varphi) = 0$ for any vector field X tangent to M^2_s. Writing $\tilde\nabla$ for the Levi-Civita connection on $SA(4,\nu)$ and using the well known equation $\Delta\varphi = -2H$, we obtain

$$\begin{aligned} Xg(B,\varphi) &= g(B, \tilde\nabla_{df(X)}\varphi) = g(B, df(X)) \\ &= g(\Delta\varphi - A*\varphi, df(X)) = -g(A*\varphi, df(X)) \\ &= -g(\varphi, A * df(X)). \end{aligned}$$

Taking covariant derivative in (1) we easily get $A * df(X) = -2\tilde\nabla_{df(X)}H$ and therefore by using (2) we conclude

$$Xg(B,\varphi) = 2Xg(\varphi, H) = 0.$$

In what follows, let N be a unit vector field normal to M^2_s in $\bar{M}^3_1(k)$, with $\varepsilon = \langle N, N\rangle$, and let S be the shape operator associated to N. The following lemma gives an accurate description of the endomorphism B.

Lemma 3.2 *Let $x : M^2_s \longrightarrow \bar{M}^3_1(k)$ be a surface contained in the hyperquadric defined by $\langle Bx, x\rangle = c$, where $B \in SA(4,\nu)$ and $c \in \mathbb{R}$. Then there exists a smooth function β on M^2_s such that*

$$\begin{aligned} Bx &= \beta N + kcx, && (2) \\ BN &= \varepsilon\,\mathrm{grad}(\beta) + (\mathrm{trace}(B) - 3kc + \beta\mathrm{trace}(S))N + k\varepsilon\beta x, && (3) \\ BX &= -\beta SX + kcX + X(\beta)N, && (4) \end{aligned}$$

where $\mathrm{grad}(\beta)$ stands for the gradient of β and X is a tangent vector field.

Proof. Since Bx is normal to M^2_s in $\mathbb{R}^4_\nu$, there is a function β satisfying (2). Now equation (4) follows easily by covariant differentiation of (2). As for equation (3), we wish to compute the tangential component $(BN)^T$ of BN. To do that, we have

$$\left\langle (BN)^T, X\right\rangle = \langle BN, X\rangle = \langle N, BX\rangle = \varepsilon X(\beta),$$

and therefore $(BN)^T = \varepsilon\,\mathrm{grad}(\beta)$. The x-component is given by

$$k\,\langle BN, x\rangle = k\,\langle N, Bx\rangle = k\varepsilon\beta.$$

Finally, to compute the N-component let $\{E_1, E_2\}$ be a orthonormal frame field. Then

$$\varepsilon \langle BN, N\rangle = \mathrm{trace}(B) - \sum_{i=1}^{2} \varepsilon_i \langle BE_i, E_i\rangle - k\langle Bx, x\rangle = \mathrm{trace}(B) - 3kc + \beta \mathrm{trace}(S),$$

and the lemma follows.

It is worth pointing out that the function β contains a nice geometric information about the surface M_s^2. To show that, we need some further computations. Let us denote by $\bar\nabla$ and ∇ the Levi-Civita connections on $\mathbb{R}_\nu^4$ and M_s^2, respectively. On one hand, covariant differentiation in (3) yields

$$\begin{aligned} \bar\nabla_X(BN) &= \varepsilon\nabla_X(\mathrm{grad}(\beta)) - (\mathrm{trace}(B) - 3kc + \beta\mathrm{trace}(S))SX + k\varepsilon\beta X \\ &\quad + \langle S(\mathrm{grad}(\beta)) + \mathrm{grad}(\mathrm{trace}(S)\beta), X\rangle N. \end{aligned}$$

On the other hand, by using (4) we get

$$\bar\nabla_X(BN) = -B(SX) = \beta S^2 X - kcSX - \langle S(\mathrm{grad}(\beta)), X\rangle N.$$

These two equations lead to

$$2S(\mathrm{grad}(\beta)) + \mathrm{grad}(\beta\mathrm{trace}(S)) = 0,$$

$$\varepsilon\nabla_X(\mathrm{grad}(\beta)) = \beta S^2 X + (\mathrm{trace}(B) - 4kc + \beta\mathrm{trace}(S))SX - k\varepsilon\beta X.$$

In the following proposition we go further into the shape of the surface provided that β is a constant.

Proposition 3.3 *Let $x : M_s^2 \longrightarrow \overline{M}_1^3(k)$ be a constant mean curvature surface contained in the hyperquadric defined by $\langle Bx, x\rangle = c$, where $B \in SA(4,\nu)$ and $c \in \mathbb{R}$. Assume that the function β given in Lemma 3.2 is constant. Then:*
1) *If $\beta \neq 0$, M_s^2 is a flat isoparametric surface.*
2) *If $\beta = 0$ and $\mathrm{trace}(B) \neq 4kc$, M_s^2 is a totally geodesic surface.*

Proof. From (6) we have

$$\beta S^2 + (\mathrm{trace}(B) - 4kc + \beta\mathrm{trace}(S))S - k\varepsilon\beta = 0.$$

Now, when $\beta \neq 0$, the characteristic polynomial of the shape operator is constant and therefore M_s^2 is an isoparametric surface. Moreover, in this case $\det(S) = -k\varepsilon$ and so, from Gauss equation, M_s^2 is flat. In the second case, we have $S = 0$ and M_s^2 is totally geodesic.

4. Some examples

In this section we will explicitly exhibit certain families of surfaces in $\mathbb{S}_1^3$ and $\mathbb{H}_1^3$ satisfying equation (1) and in his turn they will support the classification we are looking for.

4.1. Totally umbilical surfaces in $\mathbb{S}^3_1$

To fix the notation we will use the metric given by $G = \mathrm{diag}[-1, 1, 1, 1]$. Non flat totally umbilical surfaces in $\mathbb{S}^3_1$ can be obtained by intersecting $\mathbb{S}^3_1$ with a coordinate hyperplane and it follows easily that all possibilities can be shown by choosing x_1 or x_4 to be constant. This is equivalent to consider the equation $\lambda x_1^2 + \mu x_4^2 = \rho^2$, where ρ is a non zero constant, $\lambda, \mu \in \{0, 1\}$ and $\lambda + \mu = 1$. The attached table lists all those surfaces, which satisfy equation (1) for matrices A and B given below.

λ	μ	Surface	A	B
1	0	$\mathbb{S}^2(r), r = \sqrt{1+\rho^2}$	$\begin{pmatrix} 0 & a & a & a \\ a & b & b & b \\ a & b & b & b \\ a & b & b & b \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$
0	1	$\mathbb{H}^2(-r), r = \sqrt{\rho^2-1}$	$\begin{pmatrix} -b & -b & -b & -a \\ -b & -b & -b & -a \\ -b & -b & -b & -a \\ -a & -a & -a & 0 \end{pmatrix}$	$\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
0	1	$\mathbb{S}^2_1(r), r = \sqrt{1-\rho^2}$	$\begin{pmatrix} b & b & b & a \\ b & b & b & a \\ b & b & b & a \\ a & a & a & 0 \end{pmatrix}$	$\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

where $a = 2/r^2$ and $b = 6/r^2$.

4.2. Totally umbilical surfaces in $\mathbb{H}^3_1$

Now we will use the metric given by $G = \mathrm{diag}[-1, -1, 1, 1]$. Reasoning as above, non flat totally umbilical surfaces in $\mathbb{H}^3_1$ satisfying the equation (1) are listed in the following table.

λ	μ	Surface	A	B
1	0	$\mathbb{S}^2_1(r), r = \sqrt{\rho^2-1}$	$\begin{pmatrix} 0 & a & a & a \\ a & b & b & b \\ a & b & b & b \\ a & b & b & b \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$
1	0	$\mathbb{H}^2(-r), r = \sqrt{1-\rho^2}$	$\begin{pmatrix} 0 & -a & -a & -a \\ -a & -b & -b & -b \\ -a & -b & -b & -b \\ -a & -b & -b & -b \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$
0	1	$\mathbb{H}^2_1(-r), r = \sqrt{1+\rho^2}$	$\begin{pmatrix} -b & -b & -b & -a \\ -b & -b & -b & -a \\ -b & -b & -b & -a \\ -a & -a & -a & 0 \end{pmatrix}$	$\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

where $a = 2/r^2$ and $b = 6/r^2$.

4.3. Standard products in $\mathbb{S}_1^3$

The standard products in $\mathbb{S}_1^3$ satisfying the condition (1) are characterized by the equation $-x_1^2 + x_2^2 = \varepsilon r^2$, where $\varepsilon = \pm 1$ stands for the sign of the surface and r is a non zero real constant. The matrices A and B are given by

$$A = \begin{pmatrix} a & a & b & b \\ a & a & b & b \\ b & b & c & c \\ b & b & c & c \end{pmatrix}, \text{ with } b = \frac{a+c}{4}, \qquad B = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

where a and c are given in the following table.

ε	Surface	a	c
1	$\mathbb{S}_1^1(r) \times \mathbb{S}^1(\sqrt{1-r^2})$	$4/r^2$	$4/(1-r^2)$
-1	$\mathbb{H}^1(-r) \times \mathbb{S}^1(\sqrt{1+r^2})$	$-4/r^2$	$4/(1+r^2)$

4.4. Standard products in $\mathbb{H}_1^3$

We have the following products $\mathbb{H}_1^1(-r)\times\mathbb{S}^1(\sqrt{r^2-1})$, $\mathbb{S}_1^1(r)\times\mathbb{H}^1(-\sqrt{1+r^2})$ and $\mathbb{H}^1(-r)\times\mathbb{H}^1(\sqrt{1-r^2})$, standardly embedded in $\mathbb{H}_1^3$ by means of the equations:

$$\begin{aligned} -x_1^2 - x_2^2 &= -r^2 \\ -x_1^2 + x_3^2 &= r^2 \\ -x_1^2 + x_3^2 &= -r^2 \end{aligned}$$

respectively. They satisfy equation (1) for the matrices A, given in the following table, and $B = -2I$, I being the identity (4×4)-matrix.

Surface	A	a	c
$\mathbb{H}_1^1(-r) \times \mathbb{S}^1(\sqrt{r^2-1})$	$\begin{pmatrix} a & a & b & b \\ a & a & b & b \\ b & b & c & c \\ b & b & c & c \end{pmatrix}$	$-4/r^2$	$4/(r^2-1)$
$\mathbb{S}_1^1(r) \times \mathbb{H}^1(-\sqrt{1+r^2})$	$\begin{pmatrix} a & b & a & b \\ b & c & b & c \\ a & b & a & b \\ b & c & b & c \end{pmatrix}$	$4/r^2$	$-4/(1+r^2)$
$\mathbb{H}^1(-r) \times \mathbb{H}^1(-\sqrt{1-r^2})$	$\begin{pmatrix} a & b & a & b \\ b & c & b & c \\ a & b & a & b \\ b & c & b & c \end{pmatrix}$	$-4/r^2$	$-4/(1-r^2)$

where $b = (a + c)/4$.

5. Main results

Before to get down to the general situation, it is worthwhile to pay attention to the following interesting case.

Consider an isometric immersion $x : M_s^2 \longrightarrow \bar{M}_1^3(k)$ whose quadric representation φ satisfies the condition

$$\Delta\varphi = \lambda\varphi + B,$$

where λ is a real constant and $B \in SA(4, \nu)$. Notice that φ satisfies condition (1) with $a_{ij} = \lambda$ for all i, j. By equation (2), and using again $\Delta\varphi = -2H$, we get

$$(\lambda - 4k)\varphi + B = -2(H_1 x^t + xH_1^t)G - 2\sum_{i=1}^{2} \varepsilon_i E_i E_i^t G.$$

By taking covariant derivative here we obtain

$$\begin{aligned}(\lambda - 4k)(Xx^t + xX^t)G &= -2\left\{(\bar{\nabla}_X H_1)x^t + H_1 X^t + XH_1^t + x(\bar{\nabla}_X H_1)^t\right\} G \\ &\quad - 2\sum_{i=1}^{2} \varepsilon_i \left\{(\bar{\nabla}_X E_i)E_i^t + E_i(\bar{\nabla}_X E_i)^t\right\} G\end{aligned}$$

and by applying this endomorphism on a tangent vector field Y, we have

$$\begin{aligned}(\lambda - 4k)\langle X, Y\rangle x &= -2\langle X, Y\rangle H_1 - 2\left\langle \bar{\nabla}_X H_1, Y\right\rangle x - 2\sum_{i=1}^{2} \varepsilon_i \langle E_i, Y\rangle \bar{\nabla}_X E_i \\ &\quad - 2\sum_{i=1}^{2} \varepsilon_i \left\langle \bar{\nabla}_X E_i, Y\right\rangle E_i.\end{aligned}$$

Equating the N-component we deduce

$$\varepsilon\langle SX, Y\rangle N = -\langle X, Y\rangle H_1,$$

and therefore $\varepsilon \mathrm{trace}(S)N = -2H_1$, which implies $H_1 = 0$ and $S = 0$. Summarizing, we have proved the following proposition.

Proposition 5.1 *Let M_s^2 be a surface in $\bar{M}_1^3(k)$ whose quadric representation satisfies $\Delta\varphi = \lambda\varphi + B$. Then M_s^2 is totally geodesic.*

To get a satisfactory solution to the problem of classifying semi-Riemannian constant mean curvature surfaces in $\bar{M}_1^3(k)$ whose quadric representation satisfies (1), there would be suitable, in view of Proposition 3.3, to know the family of isoparametric surfaces of $\bar{M}_1^3(k)$. To do that, we discuss according to the character of the shape operator. First, if S is diagonalizable, then M_s^2 is totally umbilical or, by [**1**], M_s^2 is an open piece of a semi-Riemannian product. If S is not diagonalizable with a double real eigenvalue, then M_s^2 is an open piece of a B-scroll over a null curve (see [**3**]). Finally, if S has complex eigenvalues then, from [**10**], M_s^2 is a complex circle.

Now we are ready to show our first main theorem.

Theorem 5.2 *Let $x : M_s^2 \longrightarrow \bar{M}_1^3(k)$ be a constant mean curvature isometric immersion whose quadric representation satisfies $\Delta\varphi = A * \varphi + B$. Then M_s^2 is an isoparametric surface.*

Proof. We aim to show that β is a constant. Assume that the set $\mathcal{U} = \{p \in M_s^2 : \mathrm{grad}(\beta^2)(p) \neq 0\}$ is not empty. Then (5) yields to

$$S(\mathrm{grad}(\beta)) = -\frac{\mathrm{trace}(S)}{2}\,\mathrm{grad}(\beta),$$

showing that S has no complex eigenvalues on $\mathcal{U}$. Now, if $\mathrm{trace}(S) \neq 0$ this equation implies that S is diagonalizable with constant principal curvatures given by $-\mathrm{trace}(S)/2$ and $3\mathrm{trace}(S)/2$. Thus $\mathcal{U}$ is a semi-Riemannian product. Therefore, as we have seen before, the function β is constant, which contradicts the definition of $\mathcal{U}$. On the other hand, if $\mathrm{trace}(S) = 0$ then $\mathcal{U}$ is totally geodesic or a minimal B-scroll. The former cannot hold because β should be constant, the latter neither can, because a minimal B-scroll does not satisfy (1), as one can see by a straightforward computation. Thus the function β is a constant.

In view of Proposition 3.3, we may assume $\beta = 0$ and $\mathrm{trace}(B) = 4kc$. Then by Lemma 3.2 we know that $B = kcI$ and therefore we obtain

$$0 = \Delta(\mathrm{trace}(\varphi)) = \mathrm{trace}(\Delta\varphi) = \mathrm{trace}(A * \varphi + B) = \sum_{i=1}^{4} \delta_i a_{ii} x_i^2 + 4kc.$$

This equation and the equality $\Delta(\delta_i x_i^2) = \delta_i a_{ii} x_i^2 + kc$ imply, by an easy argument, that we can find at most two different entries in the principal diagonal of A. If there are exactly two different ones, then M_s^2 is totally umbilical or a semi-Riemannian product. Thus we can assume that all entries in the principal diagonal of A are equal. Therefore one of the following statements holds:
1) All entries a_{ij} of A are equal. Then M_s^2 is, by Proposition 5.1, a totally geodesic surface.
2) There is a row containing two different entries. We distinguish between minimal and non-minimal case.
2.1) If M_s^2 is a minimal surface in $\bar{M}_1^3(k)$, then by using (2) and (1) we obtain the following coordinate equations

$$\left(\sum_{j=1}^{4} \delta_j a_{ij} x_j^2 + kc - 4\right) x_i = 0, \qquad i = 1, 2, 3, 4.$$

Now, either M_s^2 is totally geodesic or $\langle C_i x, x\rangle = 0$ for all i, where $C_i = (c-4k)I + \mathrm{diag}[a_{i1}, a_{i2}, a_{i3}, a_{i4}]$. Therefore, from Proposition 3.3, M_s^2 is a flat isoparametric surface.
2.2) Assume now that the mean curvature α of M_s^2 in $\bar{M}_1^3(k)$ does not vanish. By using again (2) and (1) we have $Hx = k\alpha N - 2x$ and $HN = \varepsilon\alpha x$, which can be written in components as

$$\begin{aligned} -2k\alpha N_i &= \langle C_i x, x\rangle\, x_i, & (7)\\ -kcN_i &= h_i(x) x_i, & (8) \end{aligned}$$

where $h_i(x) = \sum_{j=1}^{4} \delta_j a_{ij} x_j N_j + 2\varepsilon\alpha$. Since M_s^2 is not totally geodesic, the above equations yield

$$c\,\langle C_i x, x\rangle = 2\alpha h_i(x), \qquad \text{for all } i. \qquad (9)$$

Observe that we can write

$$A * \varphi + kcI = -2H = -2\left((H_1 x^t + xH_1^t)G + I - \varepsilon NN^t G - 3k\varphi\right),$$

so that by equating the (i,j)-component, according to (7), we find

$$2\alpha^2(a_{ij} - 6k) = 2k\alpha^2(\langle C_i x, x\rangle + \langle C_j x, x\rangle) + \varepsilon \langle C_i x, x\rangle \langle C_j x, x\rangle, \quad i \neq j,$$

and

$$-\left(2\alpha^2(4c + 6k) + 4k\alpha^2 \langle C_i x, x\rangle + \varepsilon \langle C_i x, x\rangle^2\right) x_i^2 + 2\alpha^2 \delta_i (kc + 2) = 0, \quad i = j,$$

where we have used that $a_{ii} = -4c$. From these equations, bearing in mind that M_s^2 is not totally geodesic, we deduce that M_s^2 is totally umbilical or $\langle C_i x, x\rangle$ are constant. Thus, from Proposition 3.3, M_s^2 is a flat isoparametric surface, and the proof finishes.

In order to get a complete classification, according to the above theorem, we reduce the problem to determine which isoparametric surfaces in $\overline{M}_1^3(k)$ satisfy (1). By considering the degree d of the minimal polynomial of the shape operator S, we distinguish two cases:
1) If $d = 1$, then M_s^2 is totally umbilical and, by [**10**], M_s^2 is an open piece of a pseudosphere, a pseudohyperbolic space or a flat totally umbilical surface. But an easy computation shows that the last one does not satisfy (1).
2) If $d = 2$, then M_s^2 is, according to [**1**], [**3**] and [**10**], a semi-Riemannian product, a B-scroll over a null curve or a complex circle. However, equation (1) is only satisfied for the first one.

Now, we are ready to state our main classification theorem.

Theorem 5.3 *Let M_s^2 be a semi-Riemannian constant mean curvature surface in $\overline{M}_1^3(k)$ whose quadric representation satisfies $\Delta\varphi = A*\varphi + B$. Then M_s^2 is an open piece of one of the following surfaces:*
1) $\mathbb{S}^2(r)$, $\mathbb{S}_1^2(r)$, $\mathbb{H}^2(-r)$, $\mathbb{H}_1^2(-r)$.
2) $\mathbb{H}^1(-r)\times\mathbb{S}^1(\sqrt{1+r^2})$, $\mathbb{S}_1^1(r)\times\mathbb{S}^1(\sqrt{1-r^2})$, $\mathbb{H}_1^1(-r)\times\mathbb{S}^1(\sqrt{r^2-1})$, $\mathbb{S}_1^1(r)\times\mathbb{H}^1(-\sqrt{1+r^2})$, $\mathbb{H}^1(-r)\times\mathbb{H}^1(-\sqrt{1-r^2})$.

Remark 5.4 Obviously, the same computations work when the ambient space is a non-flat Riemannian space form $\mathbb{S}^3$ or $\mathbb{H}^3$. In both cases, the equation (1) characterizes the totally umbilical surfaces and Riemannian standard products.

This theorem allows us to distinguish minimal surfaces in $\overline{M}_1^3(k)$ via its quadric representation. More precisely, we have the following consequences.

Corollary 5.5 *Let M_s^2 be a minimal surface in $\overline{M}_1^3(k)$ whose quadric representation satisfies $\Delta\varphi = A*\varphi + B$. Then M_s^2 is totally geodesic or an open piece of one of the following products:* $\mathbb{S}_1^1(\sqrt{2}/2)\times\mathbb{S}^1(\sqrt{2}/2)$, $\mathbb{H}^1(-\sqrt{2}/2)\times\mathbb{H}^1(-\sqrt{2}/2)$.

The Clifford torus characterization found by Barros-Garay in [**6**] can be directly obtained from Theorem 5.3 as follows.

Corollary 5.6 *Let M^2 be a compact, minimal surface in $\mathbb{S}^3$ whose quadric representation is minimal in some hyperquadric of $SA(4,0)$. Then M^2 is totally geodesic or the Clifford torus.*

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