A quadric representation of pseudo-Riemannian product immersions

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Abstract

In this paper we introduce a quadric representation φ of the product of two pseudo-Riemannian isometric immersions. We characterize the product of submanifolds whose quadric representation satisfies $\Delta H_\varphi = \lambda H_\varphi$, for a real constant λ, where H_φ is the mean curvature vector field of φ. As for hypersurfaces, we prove that the only ones satisfying that equation are minimal products as well as products of a minimal hypersurface and another one which has constant mean and constant scalar curvatures with an appropriate relation between them. In particular, the family of these surfaces consists of $\mathbb{H}^2(-1)$ and $\mathbb{S}^1(2/3) \times \mathbb{H}^1(-2)$ in $\mathbb{S}^3_1(1)$ and $\mathbb{S}^2_1(1)$, $\mathbb{H}^1_1(-2/3) \times \mathbb{S}^1(2)$, $\mathbb{S}^1_1(2) \times \mathbb{H}^1(-2/3)$ and a B-scroll over a null Frenet curve with torsion $\pm\sqrt{2}$ in $\mathbb{H}^3_1(-1)$.

1. Introduction

Let $\mathbb{R}^{m+1}_t$ be the pseudo-Euclidean space endowed with the standard inner product of index t given by $\langle a, b\rangle = a^t G b$, where $G = \text{diag}[\underbrace{-1, \dots, -1}_{t}, \underbrace{1, \dots, 1}_{m-t+1}]$ stands for the matrix of the metric with respect to the usual rectangular coordinates. Let us now denote by $\bar{M}^m_\mu(k)$ a pseudo-Riemannian manifold of dimension m, index μ and constant curvature k and $SA(m+1, t)$ the set of selfadjoint endomorphisms of $\mathbb{R}^{m+1}_t$ equipped with the metric $g(A, B) = \frac{k}{2}\text{trace}(AB)$. Let $f : \bar{M}^m_\mu(k) \to SA(m+1, t)$ be the map defined by $f(p) = pp^tG$. Then given an isometric immersion $x : M^j_c \to \bar{M}^m_\mu(k)$ the map $\varphi : M^j_c \to SA(m, \mu)$ defined by $\varphi = f \circ x$ is also an isometric immersion which will be called the quadric representation of M^j_c. Then in [**16**] we have classified pseudo-Riemannian surfaces whose quadric representation satisfies a characteristic differential equation involving the Laplacian. Since that Laplacian equation yields isoparametric surfaces, we showed that family is made up by pseudo-Riemannian standard products and totally geodesic surfaces. We were able to distinguish the products $\mathbb{H}^1(2k) \times \mathbb{H}^1(2k) \subset \mathbb{H}^3_1(k)$, $k < 0$, and $\mathbb{S}^1_1(2k) \times \mathbb{S}^1(2k) \subset \mathbb{S}^3_1(k)$, $k > 0$, as the only minimal not totally geodesic surfaces into $\mathbb{H}^3_1(k)$ and $\mathbb{S}^3_1(k)$, respectively, whose quadric representation satisfies that Laplacian equation. Then we extended the characterization of the Clifford torus given by Barros and Garay ([**4**], [**5**]).

As standard products play a chief role in that classification problem, we are going to find a quadric representation for pseudo-Riemannian product submanifolds into indefinite space forms.

Let $\mathbb{S}^m_\mu(k)$ ($k > 0$) and $\mathbb{H}^n_\nu(k)$ ($k < 0$) be the pseudo-Euclidean hypersurfaces of constant curvature k given by

$$\mathbb{S}^m_\mu(k) = \{x \in \mathbb{R}^{m+1}_\mu : \langle x, x\rangle = \frac{1}{k}\}$$

and

$$\mathbb{H}^n_\nu(k) = \{x \in \mathbb{R}^{n+1}_{\nu+1} : \langle x, x\rangle = \frac{1}{k}\},$$

respectively. We will refer them as the hyperquadrics of constant curvature k. We consider a map f from the pseudo-Riemannian product $\bar{M}^m_\mu(k) \times \bar{N}^n_\nu(k)$ of two hyperquadrics of constant curvature k into the space of real $(m+1)\times(n+1)$ matrices $\mathfrak{M}$ which is an isometric immersion. General properties of this map are obtained, for instance, f is an isometric immersion of 1-type (in the sense of B.Y. Chen) and the associated eigenvalue is $k(m+n)$ (see Section 1).

Let us recall Chen's definition of type (see [**7**]). Let M^j_c be a pseudo-Riemannian submanifold of $\mathbb{R}^{m+1}_t$ and Δ the Laplacian on M^j_c. Then M^j_c is said to be of finite type if the position vector X of M^j_c in $\mathbb{R}^{m+1}_t$ has the following form

$$X = X_0 + \sum_{i=1}^{r} X_i, \qquad \Delta X_i = \lambda_i X_i,$$

where X_0 is a constant map and λ_i is an eigenvalue of Δ. If all eigenvalues are mutually different M^j_c is said to be of r-type. If M^j_c is of r-type and one of the λ_i is zero, M^j_c will be said of null r-type. M^j_c is said to be of infinite type if it is not of finite type.

Given isometric immersions $x : M^j_c \longrightarrow \bar{M}^m_\mu(k)$ and $y : N^\ell_d \longrightarrow \bar{N}^n_\nu(k)$, we define a new isometric immersion $\varphi : M^j_c \times N^\ell_d \longrightarrow \mathfrak{M}$ by $\varphi = f(x, y)$. Throughtout this paper the immersion φ will be called the *quadric representation of the product immersion* (x, y).

In a series of early papers ([**2**], [**3**], [**14**], [**15**]) we have pointed out substantial differences between definite and indefinite Riemannian submanifolds with regard to the spectral behaviour of the mean curvature vector field. We have shown indeed many examples of submanifolds into indefinite space forms without counterparts into definite space forms. The key point concerns to the diagonalizability of the shape operator. Now, in dealing with φ, we state the following problem:

> *Could you determine the shape of M^j_c and N^ℓ_d into the corresponding hyperquadrics via the quadric representation of the product $M^j_c \times N^\ell_d$?*

In trying to solve this question, we will study the spectral behaviour of the mean curvature vector field H_φ of φ. Actually we wish to know what kind of geometric information about M^j_c and N^ℓ_d could arise from the Laplacian equation $\Delta H_\varphi = \lambda H_\varphi$. We guess this condition will play a chief role in solving that problem, because we already know the characterization of hypersurfaces satisfying that equation into indefinite space forms (see [**14**], [**15**]). As for the Chen-type of a submanifold, it is well known that a equation like $\Delta H = \lambda H$ allows to reach only up to submanifolds of 2-type with a zero eigenvalue (the so called null 2-type) (see [**2**]). However, for our quadric representation φ, the corresponding equation $\Delta H_\varphi = \lambda H_\varphi$ yields 2-type immersions (see Theorem 3.3).

Some interesting consequences can be mentioned. Let (x, y) be a pseudo-Riemannian product of hypersurfaces. Then $\Delta H_\varphi = \lambda H_\varphi$ if and only if one of them is minimal and the other one has constant mean and constant scalar curvature with an appropriate relation between them. However the case $\lambda = 0$ deserves a special attention. In fact, from the Beltrami equation $\Delta x = -nH$, we get $\Delta^2 x = -n\Delta H$, so that the mean curvature vector field is harmonic, that is, $\Delta H = 0$ if and only if $\Delta x^2 = 0$. Then B.Y. Chen called such a submanifolds biharmonic submanifolds and stated

the following conjecture [**8**]: The only biharmonic submanifolds in Euclidean spaces are the minimal ones. In early papers [**3**, **14**, **15**] we have found that, among others, the flat totally umbilic hypersurfaces are counterexamples to that conjecture into indefinite ambient spaces. However, the products of two flat totally umbilic hypersurfaces via the quadric representation are not biharmonic. In fact, we have shown that those products are the only ones satisfying the equation $\Delta H_\varphi = C$, C being a nonzero constant vector in the normal bundle. Finally, by using the quadric representation of a pseudo-Riemannian product, we are able to give a new non-existence result: There is no pseudo-Riemannian product of surfaces with biharmonic quadric representation.

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2. General properties of a product immersion

Let $\bar{M}^m_\mu(k)$ and $\bar{N}^n_\nu(k)$ be two hyperquadrics of non-zero constant curvature k standardly embedded in $\mathbb{R}^{m+1}_t$ and $\mathbb{R}^{n+1}_s$, respectively. We can define an immersion f from the pseudo-Riemannian product $\bar{M}^m_\mu(k) \times \bar{N}^n_\nu(k)$ into the space of real $(m+1) \times (n+1)$ matrices $\mathfrak{M}$ by

$$\begin{array}{rcl} f : \bar{M}^m_\mu(k) \times \bar{N}^n_\nu(k) & \longrightarrow & \mathfrak{M} \\ (p,q) & \longrightarrow & p \otimes q \end{array}$$

where $\otimes : \mathbb{R}^{m+1}_t \times \mathbb{R}^{n+1}_s \longrightarrow \mathfrak{M}$ is given by $u \otimes v = G_1 u v^t G_2$, G_1 and G_2 standing for the matrices of the standard metrics on $\mathbb{R}^{m+1}_t$ and $\mathbb{R}^{n+1}_s$, respectively. We abbreviate $\bar{M}^m_\mu(k)$ and $\bar{N}^n_\nu(k)$ as $\bar{M}^m_\mu$ and $\bar{N}^n_\nu$.

To study general properties of f, we proceed as follows. Given $(p,q) \in \bar{M}^m_\mu \times \bar{N}^n_\nu$ and $(X_p, Y_q) \in T_{(p,q)}(\bar{M}^m_\mu \times \bar{N}^n_\nu)$, there are curves $\alpha : I \subset \mathbb{R} \longrightarrow \bar{M}^m_\mu$ and $\beta : J \subset \mathbb{R} \longrightarrow \bar{N}^n_\nu$ such that $\alpha(0) = p$, $\alpha'(0) = X_p$, $\beta(0) = q$ and $\beta'(0) = Y_q$. To compute the differential of f we have

$$\begin{aligned} df_{(p,q)}(X_p, Y_q) &= \left.\frac{d}{dt}\right|_{t=0} f(\alpha(t), \beta(t)) = \left.\frac{d}{dt}\right|_{t=0} \alpha(t) \otimes \beta(t) \\ &= \left.\frac{d}{dt}\right|_{t=0} \alpha(t) \otimes \beta(0) + \alpha(0) \otimes \left.\frac{d}{dt}\right|_{t=0} \beta(t) \\ &= X_p \otimes q + p \otimes Y_q \end{aligned}$$

Therefore, for short, we write down

$$df(X,Y) = X \otimes q + p \otimes Y.$$

Let $\widetilde{\nabla}$ be the usual flat connection on $\mathfrak{M}$. Let (V,W) be a vector field on $\bar{M}^m_\mu \times \bar{N}^n_\nu$ and take a point $(p,q) \in \bar{M}^m_\mu \times \bar{N}^n_\nu$, a tangent vector (X_p, Y_q) and curves $\alpha(t)$ and $\beta(t)$ as before. Then

for the covariant derivative we have

$$\begin{aligned}
\widetilde{\nabla}_{df(X_p,Y_q)}df(V,W) &= \left.\frac{d}{dt}\right|_{t=0} df_{(\alpha(t),\beta(t))}(V_{\alpha(t)},W_{\beta(t)}) \\
&= \left.\frac{d}{dt}\right|_{t=0} (V(\alpha(t))\otimes\beta(t)+\alpha(t)\otimes W(\alpha(t))) \\
&= \left.\frac{d}{dt}\right|_{t=0} V(\alpha(t))\otimes\beta(0)+V(\alpha(0))\otimes\left.\frac{d}{dt}\right|_{t=0}\beta(t) \\
&\quad + \left.\frac{d}{dt}\right|_{t=0}\alpha(t)\otimes W(\beta(0))+\alpha(0)\otimes\left.\frac{d}{dt}\right|_{t=0}W(\beta(t)) \\
&= \overline{\nabla}^1_{X_p}V\otimes q+V_p\otimes Y_q+X_p\otimes W_q+p\otimes\overline{\nabla}^2_{Y_q}W,
\end{aligned}$$

where $\overline{\nabla}^1$ and $\overline{\nabla}^2$ are the usual flat connections on $\mathbb{R}^{m+1}_t$ and $\mathbb{R}^{n+1}_s$, respectively. By using now the Gauss equation

$$\begin{aligned}
\overline{\nabla}^1_{X_p}V &= \nabla^1_{X_p}V-k\langle X_p,V_p\rangle p, \\
\overline{\nabla}^2_{Y_q}W &= \nabla^2_{Y_q}W-k\langle Y_q,W_q\rangle q,
\end{aligned}$$

∇^1 and ∇^2 being the Levi-Civita connections on $\bar{M}^m_\mu$ and $\bar{N}^n_\nu$, respectively, we have

$$\begin{aligned}
\widetilde{\nabla}_{df(X_p,Y_q)}df(V,W) &= df(\nabla^1_XV,\nabla^2_YW)+V\otimes Y+X\otimes W \\
&\quad -k\{\langle X,V\rangle+\langle Y,W\rangle\}f,
\end{aligned} \tag{1}$$

where, as usually, $\langle,\rangle$ denotes the metric.

Let $\widetilde{g}$ be the metric in $\mathfrak{M}$ defined by $\widetilde{g}(A,B)=k\,\mathrm{tr}(G_1AG_2B^t)$, for $A,B\in\mathfrak{M}$, then f becomes an isometric immersion. Notice that $\mathfrak{M}$, endowed with $\widetilde{g}$, is isometric to a pseudo-Euclidean space of index $t(n+1-s)+s(m+1-t)$ or $(m+1)(n+1)-s(m+1)-t(n+1)+2st$, provided that $k>0$ or $k<0$, respectively. Then it is easy to see that

$$\widetilde{g}(X\otimes V,Y\otimes W)=k\langle X,Y\rangle\langle V,W\rangle. \tag{2}$$

Now, a straightforward computation from (1) allows us to obtain the second fundamental form $\widetilde{\sigma}$ of f

$$\widetilde{\sigma}((X,Y),(V,W))=V\otimes Y+X\otimes W-k\{\langle X,V\rangle+\langle Y,W\rangle\}f.$$

We are going to get the mean curvature vector field H_f of f. To do that, let $\{E_1,\dots,E_m\}$ and $\{F_1,\dots,F_n\}$ be local orthonormal frames of $\bar{M}^m_\mu$ and $\bar{N}^n_\nu$, respectively. Then $\{(E_1,0),\dots,(E_m,0),(0,F_1),\dots,(0,F_n)\}$ is a local orthonormal frame of $\bar{M}^m_\mu\times\bar{N}^n_\nu$. From (3) we find

$$\begin{aligned}
\widetilde{\sigma}((E_i,0),(E_i,0)) &= -k\varepsilon_if, \\
\widetilde{\sigma}((0,F_j),(0,F_j)) &= -k\eta_jf,
\end{aligned}$$

where $\varepsilon_i=\langle E_i,E_i\rangle$ and $\eta_j=\langle F_j,F_j\rangle$. Therefore

$$\begin{aligned}
H_f &= \frac{1}{m+n}\left(\sum_{i=1}^m\varepsilon_i\widetilde{\sigma}((E_i,0),(E_i,0))+\sum_{j=1}^n\eta_j\widetilde{\sigma}((0,F_j),(0,F_j))\right) \\
&= -kf.
\end{aligned} \tag{4}$$

From here and the Beltrami equation $\Delta f = -(m+n)H_f$ we obtain the following interesting result.

Proposition 2.1 *The isometric immersion $f : \bar{M}^m_\mu \times \bar{N}^n_\nu \longrightarrow \mathfrak{M}$ is of 1-type with associated eigenvalue $k(m+n)$, that is, $\Delta f = k(m+n)f$.*

As a consequence of pseudo-Riemannian version of Takahashi's theorem ([**7**] and [**20**]) we have the following.

Corollary 2.2 *The isometric immersion f is minimal in the hyperquadric of $\mathfrak{M}$ given by $\{A \in \mathfrak{M} : \widetilde{g}(A,A) = k^{-1}\}$.*

3. The quadric representation

Let $x : M^j_c \longrightarrow \bar{M}^m_\mu \subset \mathbb{R}^{m+1}_t$ and $y : N^\ell_d \longrightarrow \bar{N}^n_\nu \subset \mathbb{R}^{n+1}_s$ be two isometric immersions and let $\varphi : M^j_c \times N^\ell_d \longrightarrow \mathfrak{M}$ be the isometric immersion defined by $\varphi(p,q) = f(x(p), y(q))$. From now on, φ will be called the *quadric representation of the pseudo-Riemannian product immersion* (x,y).

We are going to get properties of x and y coming from those of φ. To do that, let H_x and H_y be the mean curvature vector fields of x and y, respectively. Let $\bar{H}_x$ and $\bar{H}_y$ be the mean curvature vector fields in the corresponding pseudo-Euclidean ambient spaces. Since the hyperquadrics are totally umbilic we have

$$\bar{H}_x = H_x - kx, \qquad \bar{H}_y = H_y - ky.$$

Let σ_x and σ_y be the second fundamental forms associated to x and y, respectively. Then the second fundamental form of φ can be written as

$$\sigma_\varphi = df(\sigma_x, \sigma_y) + \widetilde{\sigma}.$$

Our first goal is to characterize the product immersions (x,y) whose quadric representation φ is of 1-type. Bearing in mind the above relation among the second fundamental forms, we have

$$\begin{aligned} \mathrm{tr}(\sigma_\varphi) &= df(\mathrm{tr}(\sigma_x), 0) - kj\varphi + df(0, \mathrm{tr}(\sigma_y)) - k\ell\varphi \\ &= df(\mathrm{tr}(\sigma_x), \mathrm{tr}(\sigma_y)) - k(j+\ell)\varphi. \end{aligned}$$

Then by letting H_φ the mean curvature vector field associated to φ we obtain

$$(j+\ell)H_\varphi = df(jH_x, \ell H_y) - k(j+\ell)\varphi. \tag{1}$$

Proposition 3.1 *The quadric representation φ of a pseudo-Riemannian product immersion (x,y) is of 1-type if and only if x and y are minimal immersions. Moreover, the associated eigenvalue is given by $k(j+\ell)$.*

Proof. First, if x and y are minimal, from (1) and the Beltrami equation $\Delta\varphi = -(j+\ell)H_\varphi$, we easily deduce that $\Delta\varphi = k(j+\ell)\varphi$.

Assume now that φ satisfies the equation

$$\Delta\varphi = \lambda(\varphi - \varphi_0), \qquad \lambda \in \mathbb{R},$$

where $\varphi_0 \in \mathfrak{M}$ is a constant matrix. By using again Beltrami equation and (1) we find

$$\lambda\varphi_0 = df(jH_x, \ell H_y) + \{\lambda - k(j+\ell)\}\varphi.$$

Now, let $V \in \mathfrak{X}(M_c^j)$, take covariant derivative in (2) and use (1) to obtain

$$\begin{aligned} 0 &= \widetilde{\nabla}_{df(V,0)} df(jH_x, \ell H_y) + \{\lambda - k(j+\ell)\}\widetilde{\nabla}_{df(V,0)}\varphi \\ &= df(j\nabla^1_V H_x, 0) + \ell V \otimes H_y + \{\lambda - k(j+\ell)\}V \otimes y \\ &= j\nabla^1_V H_x \otimes y + \ell V \otimes H_y + \{\lambda - k(j+\ell)\}V \otimes y. \end{aligned} \tag{3}$$

Since this can be viewed as an endomorphism on $\mathbb{R}^{n+1}_s$, we apply it to y to get $0 = k^{-1}jG_1(\nabla^1_V H_x) + k^{-1}\{\lambda - k(j+\ell)\}G_1 V$ and then

$$\nabla^1_V H_x = \frac{k(j+\ell) - \lambda}{j} V,$$

because G_1 is invertible. Bringing this to (3) we deduce that $\ell V \otimes H_y = 0$, which implies that $H_y = 0$. A similar reasoning, by taking in (2) covariant derivative with respect to a vector field $W \in \mathfrak{X}(N_d^\ell)$, leads to $H_x = 0$ and the proof finishes.

From now on, we will pay attention to the equation

$$\Delta H_\varphi = \lambda H_\varphi, \qquad \lambda \in \mathbb{R}.$$

Let (p,q) be a point in $M_c^j \times N_d^\ell$ and choose local orthonormal frames $\{E_1, \ldots, E_j\}$ and $\{F_1, \ldots, F_\ell\}$ on M_c^j and N_d^ℓ, respectively, such that $\nabla^x_{E_\alpha} E_\alpha(p) = 0$, for all $\alpha = 1, \ldots, j$, and $\nabla^y_{F_\beta} F_\beta(q) = 0$, for all $\beta = 1, \ldots, l$, where ∇^x and ∇^y are the Levi-Civita connections on M_c^j and N_d^ℓ, respectively. From (1) we easily get

$$(j+\ell)H_\varphi = j\bar{H}_x \otimes y + \ell x \otimes \bar{H}_y.$$

Taking covariant derivative here we obtain at (p,q)

$$\begin{aligned} \widetilde{\nabla}_{df(E_\alpha,0)}\widetilde{\nabla}_{df(E_\alpha,0)}(j+\ell)H_\varphi &= j\overline{\nabla}^1_{E_\alpha}\overline{\nabla}^1_{E_\alpha}\bar{H}_x \otimes y + \ell\overline{\nabla}^1_{E_\alpha}E_\alpha \otimes \bar{H}_y \\ &= j\overline{\nabla}^1_{E_\alpha}\overline{\nabla}^1_{E_\alpha}\bar{H}_x \otimes y + \ell\overline{\sigma}_x(E_\alpha, E_\alpha) \otimes \bar{H}_y \end{aligned}$$

and

$$\widetilde{\nabla}_{df(0,F_\beta)}\widetilde{\nabla}_{df(0,F_\beta)}(j+\ell)H_\varphi = j\bar{H}_x \otimes \overline{\sigma}_y(F_\beta, F_\beta) + \ell x \otimes \overline{\nabla}^2_{F_\beta}\overline{\nabla}^2_{F_\beta}\bar{H}_y,$$

where $\overline{\sigma}_x$ and $\overline{\sigma}_y$ are the second fundamental forms of M_c^j and N_d^ℓ in $\mathbb{R}^{m+1}_t$ and $\mathbb{R}^{n+1}_s$, respectively. Therefore we have

$$(j+\ell)\Delta H_\varphi = j\Delta\bar{H}_x \otimes y + \ell x \otimes \Delta\bar{H}_y - 2j\ell\bar{H}_x \otimes \bar{H}_y. \tag{5}$$

By assuming that $\Delta H_\varphi = \lambda H_\varphi$, we obtain from (5) that

$$j\Delta\bar{H}_x \otimes y + \ell x \otimes \Delta\bar{H}_y - 2j\ell\bar{H}_x \otimes \bar{H}_y = \lambda(j\bar{H}_x \otimes y + \ell x \otimes \bar{H}_y),$$

which we apply to y to get

$$jk^{-1}G_1\Delta\bar{H}_x + \ell\langle\Delta\bar{H}_y, y\rangle G_1 x - 2j\ell\langle\bar{H}_y, y\rangle G_1\bar{H}_x = \lambda jk^{-1}G_1\bar{H}_x + \lambda\ell\langle\bar{H}_y, y\rangle G_1 x.$$

As $\langle\bar{H}_y, y\rangle = -1$ and G_1 is invertible, the above equation writes as

$$jk^{-1}\Delta\bar{H}_x + j(2\ell - \lambda k^{-1})\bar{H}_x + \ell(\lambda + \langle\Delta\bar{H}_y, y\rangle)x = 0.$$

A similar reasoning by applying to x^t leads to

$$\ell k^{-1}\Delta\bar{H}_y + \ell(2j - \lambda k^{-1})\bar{H}_y + j(\lambda + \langle\Delta\bar{H}_x, x\rangle)y = 0.$$

By multiplying now the above equation by y we find

$$k^{-1}(j\langle\Delta\bar{H}_x, x\rangle + \ell\langle\Delta\bar{H}_y, y\rangle) + \lambda k^{-1}(\ell + j) - 2\ell j = 0.$$

From (9), the following useful lemma can be easily obtained.

Lemma 3.2 *If the quadric representation φ of a pseudo-Riemannian product immersion (x, y) satisfies the equation $\Delta H_\varphi = \lambda H_\varphi$, then the functions $c_x = \langle\Delta\bar{H}_x, x\rangle$ and $c_y = \langle\Delta\bar{H}_y, y\rangle$ are both constant and related by (9).*

Now next theorem can be proved.

Theorem 3.3 *Let $\varphi : M_c^j \times N_d^\ell \longrightarrow \mathfrak{M}$ be the quadric representation of a pseudo-Riemannian product (x, y), where $x : M_c^j \longrightarrow \bar{M}_\mu^m$ and $y : N_d^\ell \longrightarrow \bar{N}_\nu^n$ are isometric immersions. Then $\Delta H_\varphi = \lambda H_\varphi$ for nonzero constant real number λ if and only if one of the following statements holds:*
(1) *Both x and y are minimal and $\lambda = k(j + \ell)$;*
(2) *x is minimal and y is of 1-type with associate eigenvalue $-kj$ or 2-type with associated eigenvalues $\lambda - kj$ and $-kj$, that is, $y = y_1 + y_2$ ($y_1, y_2 \neq 0$) such that $\Delta y_1 = -kjy_1$ and $\Delta y_2 = (\lambda - kj)y_2$;*
(3) *y is minimal and x is of 1-type with associate eigenvalue $-k\ell$ or 2-type with associated eigenvalues $\lambda - k\ell$ and $-k\ell$, that is, $x = x_1 + x_2$ ($x_1, x_2 \neq 0$) such that $\Delta x_1 = -k\ell x_1$ and $\Delta x_2 = (\lambda - k\ell)x_2$.*

Proof. If $H_x = 0$ and $H_y = 0$ then we have $\Delta H_\varphi = -k^2(j + \ell)x \otimes y$ and $H_\varphi = -kx \otimes y$. Therefore we get $\Delta H_\varphi = \lambda H_\varphi$, with $\lambda = k(j + \ell)$.

If M_c^j is minimal in $\bar{M}_\mu^m$ and $y = y_1 + y_2$ ($y_1, y_2 \neq 0$) such that $\Delta y_1 = -kjy_1$ and $\Delta y_2 = (\lambda - kj)y_2$ then

$$\Delta^2 y + (2kj - \lambda)\Delta y - kj(\lambda - kj)y = 0,$$

which is the same as (8). Then (5) implies that $\Delta H_\varphi = \lambda H_\varphi$.

A similar reasoning applies if N_d^ℓ is minimal in $\bar{N}_\nu^n$ and M_c^j is of 1-type with associate eigenvalue $-k\ell$ or 2-type with associated eigenvalues $-k\ell$ and $\lambda - k\ell$.

To prove the converse, bring (7) and (8) to $\Delta H_\varphi = \lambda H_\varphi$. Bearing in mind Lemma 3.2 and $jc_x + \ell c_y = 2kj\ell - \lambda(j + \ell)$ we obtain $H_x \otimes H_y = 0$. This equation yields $H_x = 0$ or $H_y = 0$ or both simultaneously. Now if, for instance, $H_x \neq 0$ then $\Delta\bar{H}_y = -k^2\ell y$ and $c_y = -k\ell$. Therefore, (7) can be rewritten as

$$\Delta^2 x + (2k\ell - \lambda)\Delta x - k\ell(\lambda - k\ell)x = 0.$$

Let $p(t)$ be the polynomial $p(t) = t^2 + (2k\ell - \lambda)t - k\ell(\lambda - k\ell)$, whose discriminant is $\lambda^2 \neq 0$. Then $p(\Delta)x = 0$ and using [**11**, Proposition 4.3] we have x is of finite type less than or equal to two. If x is of 1-type, then it is totally umbilical and so with associated eigenvalue $-k\ell$. If x is of 2-type, then the associated eigenvalues are the roots of $p(t)$, that is, $\lambda - k\ell$ and $-k\ell$. That completes the proof.

Now we are going to analyze when the quadric representation is biharmonic, that is, $\Delta H_\varphi = 0$. Then we also have that $H_x = 0$ or $H_y = 0$, but not simultaneously according to Theorem 3.3. Suppose N_d^ℓ is minimal in $\bar{N}_\nu^n$, then $p(\Delta)x = 0$ where $p(t) = (t + k\ell)^2$. Hence x should be, according to [**11**, Proposition 4.2], of infinite type or of 1-type with associated eigenvalue $-k\ell$. But Theorem 3.3 implies that x should be of infinite type. So the the following result has been shown.

Proposition 3.4 *Let $\varphi : M_c^j \times N_d^\ell \longrightarrow \mathfrak{M}$ be the quadric representation of a pseudo-Riemannian product (x, y), where $x : M_c^j \longrightarrow \bar{M}_\mu^m$ and $y : N_d^\ell \longrightarrow \bar{N}_\nu^n$ are isometric immersions. Then φ is biharmonic if and only if one of the following statements holds:*
(1) *x is minimal and y is of infinite type with $\Delta^2 y + 2kj\Delta y + k^2 j^2 y = 0$;*
(2) *y is minimal and x is of infinite type with $\Delta^2 x + 2k\ell\Delta x + k^2\ell^2 x = 0$.*

4. The quadric representation of a product of hypersurfaces

This section is devoted to prove the following major result.

Theorem 4.1 *Let $x : M_c^{m-1} \longrightarrow \bar{M}_\mu^m$ and $y : N_d^{n-1} \longrightarrow \bar{N}_\nu^n$ be hypersurfaces. The quadric representation φ of a pseudo-Riemannian product of (x, y) satisfies the equation $\Delta H_\varphi = \lambda H_\varphi$, $\lambda \in \mathbb{R}$, if and only if one of the following statements holds:*
(1) *Both x and y are minimal and $\lambda = k(m + n - 2)$.*
(2) *x is minimal and y has nonzero constant mean curvature α_y and constant scalar curvature τ_y such that*

$$\begin{cases} \tau_y = \dfrac{1}{m+n-2}\{(n-1)^2(m+n-3)(k+\varepsilon_y\alpha_y^2) + k(m-1)^2\} \\ \lambda = k(m+n-2) + \dfrac{(n-1)^2}{m+n-2}\varepsilon_y\alpha_y^2 \end{cases}$$

(3) *y is minimal and x has nonzero constant mean curvature α_x and constant scalar curvature τ_x such that*

$$\begin{cases} \tau_x = \dfrac{1}{m+n-2}\{(m-1)^2(m+n-3)(k+\varepsilon_x\alpha_x^2) + k(n-1)^2\} \\ \lambda = k(m+n-2) + \dfrac{(m-1)^2}{m+n-2}\varepsilon_x\alpha_x^2 \end{cases}$$

Proof. From [**7**, Lemma 3] we can easily compute the constants c_x and c_y given in Lemma 3.2 as

$$c_x = -(m-1)(k + \varepsilon_x\alpha_x^2), \qquad c_y = -(n-1)(k + \varepsilon_y\alpha_y^2),$$

where ε_x and α_x (resp. ε_y and α_y) are the sign and mean curvature of M_c^{m-1} in $\bar{M}_\mu^m$ (resp. N_d^{n-1} in $\bar{N}_\nu^n$). It follows the constancy of the mean curvatures, and one of them vanishes according to

Theorem 3.3 and Proposition 3.4. Assume now that α_x is a non vanishing constant, then from (7) we have

$$(m-1)\Delta\bar{H}_x + k(m-1)(2(n-1)-\lambda k^{-1})\bar{H}_x + k(n-1)(\lambda - k(n-1))x = 0.$$

By using again [**7**, Lemma 3] we get

$$\mathrm{tr}(S_x^2) = \lambda - k(m-1) - 2k(n-1),$$

where S_x stands for the shape operator of M_c^{m-1} in $\bar{M}_\mu^m$. Equating the x-component we obtain

$$0 = -k(m-1)^2(k+\varepsilon_x\alpha_x^2) - 2k^2(m-1)(n-1) - k^2(n-1)^2 + \lambda k(m+n-2),$$

and then

$$\lambda = k(m+n-2) + \frac{(m-1)^2}{m+n-2}\varepsilon_x\alpha_x^2.$$

Now the Gauss equation implies that

$$\begin{aligned}\tau_x &= (m-1)^2\langle\bar{H}_x,\bar{H}_x\rangle - k(m-1) - \mathrm{tr}(S_x^2)\\ &= (m-1)^2(k+\varepsilon_x\alpha_x^2) + 2k(n-1) - \lambda,\end{aligned}$$

and so τ_x is also constant. Moreover, by substituting λ in the above equation we deduce

$$(m+n-2)\tau_x = (m-1)^2(m+n-3)(k+\varepsilon_x\alpha_x^2) + k(n-1)^2.$$

The same computation works if we assume that α_y is a nonzero constant.

To prove the converse, it suffices to consider case (2) or (3). Let us assume that $y : N_d^{n-1} \to \bar{N}_\nu^n(k)$ is minimal and $x : M_c^{m-1} \to \bar{M}_\mu^m(k)$ has nonzero constant mean curvature α_x and constant scalar curvature τ_x such that

$$\begin{cases}\tau_x = \dfrac{1}{m+n-2}\{(m-1)^2(m+n-3)(k+\varepsilon_x\alpha_x^2) + k(n-1)^2\}\\ \lambda = k(m+n-2) + \dfrac{(m-1)^2}{m+n-2}\varepsilon_x\alpha_x^2\end{cases}$$

By using [**7**, Lemma 3] we deduce

$$\Delta\bar{H}_x = (\mathrm{tr}(S_x^2) + k(m-1))H_x - k(m-1)(\varepsilon_x\alpha_x^2 + k)x.$$

From the Gauss equation jointly with the formulae for λ and τ_x we get

$$\begin{aligned}&k(m-1)(\varepsilon_x\alpha_x^2 + k) = \frac{k}{m-1}((m+n-2)\lambda - k(n-1)^2 - 2k(m-1)(n-1))\\ &\mathrm{tr}(S_x^2) + k(m-1) = \lambda - 2k(n-1).\end{aligned}$$

The last three equations lead to

$$\Delta^2 x + (2k(n-1)-\lambda)\Delta x - k(n-1)(\lambda - k(n-1))x = 0,$$

and reasoning as in Theorem 3.3 we obtain $\Delta H_\varphi = \lambda H_\varphi$.

The above theorem contains the characterization of products of hypersurfaces whose quadric representation is biharmonic ($\lambda = 0$). In the following result we extend the harmonicity condition and study the equation $\Delta H_\varphi = C$, where C is a constant vector in the normal bundle. First of all, we will show a class of hypersurfaces whose products satisfy the asked equation with $C \neq 0$. The classification of totally umbilic hypersurfaces $x : M_c^{m-1} \to \bar{M}_\mu^m(k)$ is given in [**18**, Theorem 1.4], and we know that such a hypersurface is an open piece of either a pseudo-sphere $\mathbb{S}_c^{m-1}(1/r^2)$, or a pseudo-hyperbolic space $\mathbb{H}_c^{m-1}(-1/r^2)$ or $\mathbb{R}_c^{m-1}$, according to $\langle \bar{H}_x, \bar{H}_x \rangle$ is positive, negative or zero, respectively. In the last case, the isometric immersion $x : \mathbb{R}_c^{m-1} \to \bar{M}_\mu^m(k) \subset \mathbb{R}_t^{m+1}$ is given by $x = f - x_0$, x_0 being a fixed vector in $\mathbb{R}_t^{m+1}$ and $f : \mathbb{R}_c^{m-1} \longrightarrow \mathbb{R}_t^{m+1}$ the function defined by $f(u_1, \ldots, u_{m-1}) = (q(u), u_1, \ldots, u_{m-1}, q(u))$, where $q(u) = a\langle u, u\rangle + \langle u, v\rangle + c$, where a and c are constant real numbers, especially $a \neq 0$ and v is a vector in $\mathbb{R}_c^{m-1}$. We will refer this example as a *flat totally umbilic hypersurface*. It is not difficult to see that $\Delta x = -2a(m-1)(1, 0, \ldots, 0, 1)$ and so $\Delta \bar{H}_x = 0$. Therefore, if $x : M_c^{m-1} \to \bar{M}_\mu^m(k)$ and $y : N_d^{n-1} \to \bar{N}_\nu^n(k)$ are two flat totally umbilic hypersurfaces, there exist two non-zero constants a and b such that $\Delta H_\varphi = R\Lambda$, Λ being the following nonzero matrix in $\mathfrak{M}$:

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

where $R = -8ab(m-1)(n-1)/(m+n-2)$.

Theorem 4.2 *Let $x : M_c^{m-1} \longrightarrow \bar{M}_\mu^m$ and $y : N_d^{n-1} \longrightarrow \bar{N}_\nu^n$ be hypersurfaces. The quadric representation φ of a pseudo-Riemannian product of (x,y) satisfies the equation $\Delta H_\varphi = C$, C being a constant vector in the normal bundle, if and only if one of the following statements holds:*
(1) *x is minimal and y has nonzero constant mean curvature α_y and constant scalar curvature τ_y such that*

$$\begin{cases} \tau_y = k(m-1)(5-2n-m), \\ \alpha_y^2 = -\varepsilon_y k \dfrac{(m+n-2)^2}{(n-1)^2}, \quad \varepsilon_y k < 0, \end{cases}$$

and $C = 0$.
(2) *y is minimal and x has nonzero constant mean curvature α_x and constant scalar curvature τ_x such that*

$$\begin{cases} \tau_x = k(n-1)(5-2m-n), \\ \alpha_x^2 = -\varepsilon_x k \dfrac{(m+n-2)^2}{(m-1)^2}, \quad \varepsilon_x k < 0, \end{cases}$$

and $C = 0$.
(3) *Both x and y are flat totally umbilic hypersurfaces and $C = R\Lambda$, for any $R \neq 0$, where Λ is the matrix given in the above.*

Proof. The sufficiency is a consequence of Theorem 4.1 and example exhibited before this theorem.

By using (5), $\Delta H_\varphi = C$ can be rewritten as

$$(m+n-2)C = (m-1)\Delta \bar{H}_x \otimes y + (n-1)x \otimes \Delta \bar{H}_y - 2(m-1)(n-1)\bar{H}_x \otimes \bar{H}_y.$$

Then apply (2) to y and $W \in \mathfrak{X}(N_d^{n-1})$ to obtain

$$\begin{aligned}(m+n-2)Cy &= k^{-1}(m-1)G_1\Delta\bar{H}_x + (n-1)\langle\Delta\bar{H}_y, y\rangle G_1 x \\ &\quad + 2(m-1)(n-1)G_1\bar{H}_x, \qquad (3)\\ (m+n-2)CW &= (n-1)\langle\Delta\bar{H}_y, W\rangle G_1 x. \qquad (4)\end{aligned}$$

From here, as $\widetilde{g}(C, x\otimes W) = 0$, we deduce that $\Delta\bar{H}_y$ is normal to N_d^{n-1}. Therefore, from (4), we get $G_1Cy = A$, A being a constant. Then (3) writes as follows

$$(m-1)\Delta\bar{H}_x = k(m+n-2)A - k(n-1)c_y x - 2k(m-1)(n-1)\bar{H}_x, \qquad (5)$$

where c_y is the function on N_d^{n-1} given in Lemma 3.2, from which we get c_y is constant.

A similar reasoning with x^t and $V \in \mathfrak{X}(M_c^{m-1})$, leads $\Delta\bar{H}_x$ to be normal to M_c^{m-1} and then

$$(n-1)\Delta\bar{H}_y = k(m+n-2)B - k(m-1)c_x y - 2k(m-1)(n-1)\bar{H}_y,$$

where $B = G_2C^t x$ and c_x is constant.

From the above equations the following relation between c_x and c_y can be easily obtained

$$(m+n-2)\langle A, x\rangle = k^{-1}(m-1)c_x + k^{-1}(n-1)c_y - 2(m-1)(n-1) = (m+n-2)\langle B, y\rangle.$$

From here, jointly with (5) and (6), we can rewrite (2) as follows

$$C = k(A\otimes y + x\otimes B) - k^2\langle A, x\rangle x\otimes y - 2\frac{(m-1)(n-1)}{m+n-2}H_x\otimes H_y.$$

Taking the covariant derivative along $x\otimes W$ here we deduce that

$$0 = kA\otimes W - \langle A, x\rangle x\otimes W - 2\frac{(m-1)(n-1)}{m+n-2}H_x\otimes\overline{\nabla}^2_W H_y.$$

If $H_x = 0$, an easy argument from (7) and the above equation yields $C = 0$, then from Theorem 4.1 we get (1). If $H_y = 0$, then as above we obtain (2). So we can assume that $H_x \neq 0$ and $H_y \neq 0$.

Let ξ be a vector field normal to M_c^{m-1} in $\bar{M}_\mu^m$ such that $\langle\xi, H_x\rangle \neq 0$. From (8) we have $\langle\xi, H_x\rangle\overline{\nabla}^2_W H_y = k(m+n-2)/(2(m-1)(n-1))\langle A, \xi\rangle W$, and so

$$\langle\xi, H_x\rangle\Delta H_y = -k\frac{m+n-2}{2(m-1)}\langle A, \xi\rangle\bar{H}_y.$$

Now, multiplying the above equation by y we get $k(m+n-2)/(2(m-1))\langle A, \xi\rangle = \langle\xi, H_x\rangle(c_y + k(n-1))$, (9) brings us

$$\Delta\bar{H}_y = -c_y\bar{H}_y.$$

A similar reasoning yields to

$$\Delta\bar{H}_x = -c_x\bar{H}_x.$$

By combining these two equations with (5) and (6) we deduce

$$\begin{aligned}(c_x - 2k(n-1))\Delta x + k(n-1)c_y x - k(m+n-2)A &= 0,\\ (c_y - 2k(m-1))\Delta y + k(m-1)c_x y - k(m+n-2)B &= 0.\end{aligned}$$

If $c_y = 2k(m-1)$, then $c_x = 0$ and $B = 0$. Therefore we obtain that $\Delta\bar{H}_x = k(m-1)\bar{H}_x$, which is a contradiction. Assume now that $c_x \neq 2k(n-1)$ and $c_y \neq 2k(m-1)$, then x and y satisfy $\Delta x = ax + b$ and $\Delta y = cy + d$, where $a, c \in \mathbb{R}$, $b \in \mathbb{R}^{m+1}_t$ and $d \in \mathbb{R}^{n+1}_s$ are constant. From (10) and (11) we easily get $a = -c_x$ and $c = -c_y$.

From (2)

$$\begin{aligned}(m+n-2)C &= (m-1)\Delta\bar{H}_x \otimes y + (n-1)x \otimes \Delta\bar{H}_y - 2(m-1)(n-1)\bar{H}_x \otimes \bar{H}_y \\ &= -(m-1)c_x\bar{H}_x \otimes y - (n-1)c_y x \otimes \bar{H}_y - 2(m-1)(n-1)\bar{H}_x \otimes \bar{H}_y \\ &= c_x\Delta x \otimes y + c_y x \otimes \Delta y - 2\Delta x \otimes \Delta y \\ &= (ac_x + cc_y - 2ac)x \otimes y + (c_x - 2c)b \otimes y + (c_y - 2a)x \otimes d - 2b \otimes d.\end{aligned}$$

Take the covariant derivative of C

$$0 = (m+n-2)\widetilde{\nabla}_{x\otimes W}C = (ac_x + cc_y - 2ac)x \otimes W + (c_x - 2c)b \otimes W,$$

then we get

$$(c_x - 2c)b = -(a + 2c)b = 0.$$

Similarly

$$(c_y - 2a)d = -(2a + c)d = 0.$$

So

$$(m+n-2)C = -2b \otimes d.$$

Since $H_x \neq 0$ and $H_y \neq 0$, $C \neq 0$ and hence $b \neq 0$ and $d \neq 0$. From (12) and (13), $a = 0$ and $c = 0$. Therefore the mean curvature vector fields of M^{m-1}_c and N^{n-1}_d in the corresponding pseudo-Euclidean spaces are constant.

On the other hand, by using the Beltrami equation we find $\langle \Delta x, x\rangle = -(m-1)\langle \bar{H}_x, x\rangle = (m-1)$ and so $(m-1) = \langle b, x\rangle$. This shows that M^{m-1}_c is contained in a hyperplane and therefore M^{m-1}_c is totally umbilic in $\bar{M}^m_\mu(k)$. The same is valid for N^{n-1}_d. Now from [**18**, Theorem 1.4] we know that M^{m-1}_c is an open piece of a pseudo-sphere $\mathbb{S}^{m-1}_c(1/r^2)$, a pseudo-hyperbolic space $\mathbb{H}^{m-1}_c(-1/r^2)$ or a flat totally umbilic hypersurface, but only the latter has constant mean curvature vector field. The same occurs for N^{n-1}_d and so we get (3).

It is worth noticing that this theorem gives a characterization of the products of two flat totally umbilic hypersurfaces as the only ones satisfying the equation $\Delta H_\varphi = C$, where $C \neq 0$.

5. The quadric representation of a product of surfaces

We start this section by providing some examples of surfaces, in the De Sitter space $\mathbb{S}^3_1(1)$ and in the anti-De Sitter space $\mathbb{H}^3_1(-1)$, such that the quadric representation of their product with a minimal surface satisfies the equation $\Delta H_\varphi = \lambda H_\varphi$.

Example 5.1 Let N^2_d be a minimal surface in $\bar{M}^3_1(k)$. Let M^2_c be a non flat totally umbilic surface in $\bar{M}^3_1(k)$, $k^2 = 1$, such that $\varepsilon k = -1$, ε being the sign of M^2_c in $\bar{M}^3_1(k)$. Then M^2_c is an open piece of $\mathbb{H}^1_1(-1/r^2) \subset \mathbb{H}^3_1(-1)$, $\mathbb{S}^2_1(1/r^2) \subset \mathbb{H}^3_1(-1)$, $\mathbb{H}^2(-1/r^2) \subset \mathbb{S}^3_1(1)$, or $\mathbb{S}^2(1/r^2) \subset \mathbb{S}^3_1(1)$. Then $\Delta H_\varphi = \lambda H_\varphi$ if and only if M^2_c is an open piece of $\mathbb{H}^1_1(-1) \subset \mathbb{H}^3_1(-1)$, $\mathbb{S}^2(1) \subset \mathbb{S}^3_1(1)$, $\mathbb{S}^2_1(1) \subset \mathbb{H}^3_1(-1)$ or $\mathbb{H}^2(-1) \subset \mathbb{S}^3_1(1)$, where the constant λ is given by -4, 4, -2 and 2, respectively.

Example 5.2 The families of standard products in $\bar{M}_1^3(k)$ are given by
(i)$\mathbb{S}^1(1/r^2) \times \mathbb{S}_1^1(1/(1-r^2)) \subset \mathbb{S}_1^3(1)$, $1-r^2>0$,
(ii)$\mathbb{H}^1(-1/r^2) \times \mathbb{H}^1(-1/(-1+r^2)) \subset \mathbb{H}_1^3(-1)$, $-1+r^2>0$,
(iii) $\mathbb{S}^1(1/r^2) \times \mathbb{H}^1(1/(1-r^2)) \subset \mathbb{S}_1^3(1)$, $1-r^2<0$,
(iv) $\mathbb{S}_u^1(1/r^2) \times \mathbb{H}_{1-u}^1(-1/(1+r^2)) \subset \mathbb{H}_1^3(-1)$.
Unless $r^2=1/2$ in the families (i) and (ii), those surfaces are of 2-type with eigenvalues $\{1/r^2, 1/(1-r^2)\}$, $\{-1/r^2, -1/(-1+r^2)\}$, $\{1/r^2, 1/(1-r^2)\}$and $\{1/r^2, -1/(1+r^2)\}$, respectively. Let N_d^2 be a minimal surface in $\bar{M}_1^3(k)$ and M_c^2 a standard product. Then the quadric representation φ satisfies the equation $\Delta H_\varphi = \lambda H_\varphi$ if and only if M_c^2 is $\mathbb{S}^1(2) \times \mathbb{S}_1^1(2) \subset \mathbb{S}_1^3(1)$, $\mathbb{H}^1(-2) \times \mathbb{H}^1(-2) \subset \mathbb{H}_1^3(-1)$, $\mathbb{S}^1(2/3) \times \mathbb{H}^1(-2) \subset \mathbb{S}_1^3(1)$, $\mathbb{S}_1^1(2) \times \mathbb{H}^1(-2/3) \subset \mathbb{H}_1^3(-1)$ or $\mathbb{S}^1(2) \times \mathbb{H}_1^1(-2/3) \subset \mathbb{H}_1^3(-1)$, where the constant λ is given by 4, -4, $8/3$, $-8/3$ and $-8/3$, respectively.

Example 5.3 *B-scroll over a null curve.* Let $c(s)$ be a null curve in $\mathbb{H}_1^3(-1) \subset \mathbb{R}_2^4$ with an associated Cartan frame $\{A, B, C\}$, i.e., $\{A, B, C\}$ is a pseudo-orthonormal frame of vector fields along $c(s)$,

$$\begin{aligned} \langle A, A\rangle = \langle B, B\rangle = 0, \qquad & \langle A, B\rangle = -1, \\ \langle A, C\rangle = \langle B, C\rangle = 0, \qquad & \langle C, C\rangle = 1, \end{aligned}$$

satisfying $\dot{c}(s) = A(s)$ and $\dot{C}(s) = -aA(s) - \kappa(s)B(s)$, where a is a nonzero constant and $\kappa(s) \neq 0$ for all s. Then the map $x : (s,u) \longrightarrow c(s) + uB(s)$ parametrizes a Lorentzian surface M_1^2 in $\mathbb{H}_1^3(-1)$ called a B-scroll. The B-scroll has non-diagonalizable shape operator with minimal polynomial $q(t) = (t-a)^2$ and so it has constant mean curvature $\alpha = a$ and constant Gauss curvature $G = a^2$. Therefore if N_d^2 is a minimal surface in $\mathbb{H}_1^3(-1)$ and φ is que quadric representation of $M_1^2 \times N_d^2$, the equation $\Delta H_\varphi = \lambda H_\varphi$ holds if and only if $a^2 = 2$ and $\lambda = -2$.

Theorem 5.4 *Let M_c^2 and N_d^2 be two surfaces in the De Sitter space $\mathbb{S}_1^3 = \bar{M}_1^3(1)$, and $\varphi : M_c^2 \times N_d^2 \to \mathfrak{M} \cong \mathbb{R}_6^{16}$ the quadric representation of their product. Then φ satisfies the equation $\Delta H_\varphi = \lambda H_\varphi$ if and only if one of the following statements holds:*
(1) *Both M_c^2 and N_d^2 are minimal in $\mathbb{S}_1^3$, where $\lambda = 4$;*
(2) *One surface is minimal in $\mathbb{S}_1^3$ and the other one is an open piece of the totally umbilic surface $\mathbb{H}^2(-1)$, where $\lambda = 2$;*
(3) *One surface is minimal in $\mathbb{S}_1^3$ and the other one is an open piece of the standard product surface $\mathbb{S}^1(2/3) \times \mathbb{H}^1(-2)$, where $\lambda = 8/3$.*

Theorem 5.5 *Let M_c^2 and N_d^2 be two surfaces in the anti-De Sitter space $\mathbb{H}_1^3 = \bar{M}_1^3(-1)$, and $\varphi : M_c^2 \times N_d^2 \to \mathfrak{M} \cong \mathbb{R}_8^{16}$ the quadric representation of their product. Then φ satisfies the equation $\Delta H_\varphi = \lambda H_\varphi$ if and only if one of the following statements holds:*
(1) *Both M_c^2 and N_d^2 are minimal in $\mathbb{H}_1^3$, where $\lambda = -4$;*
(2) *One surface is minimal in $\mathbb{H}_1^3$ and the other one is an open piece of the totally umbilic surface $\mathbb{S}_1^2(1)$, where $\lambda = -2$;*
(3) *One surface is minimal in $\mathbb{H}_1^3$ and the other one is an open piece of the standard product surface $\mathbb{H}_1^1(-2/3) \times \mathbb{S}^1(2)$ or $\mathbb{S}_1^1(2) \times \mathbb{H}^1(-2/3)$, where $\lambda = -8/3$;*
(4) *One surface is minimal in $\mathbb{H}_1^3$ and the other one is an open piece of a B-scroll over a null-Frenet curve with torsion $\pm\sqrt{2}$, where $\lambda = -2$.*

Proof of Theorems 5.4 and 5.5. In view of Theorem 4.1, we can assume that $M_c^2 \times N_d^2$ is not minimal. Then either M_c^2 or N_d^2 has to be minimal, so we can suppose N_d^2 is minimal. Therefore Theorem 4.1 yields M_c^2 is an isoparametric surface. Hence M_c^2 is totally umbilical, a B-scroll, a pseudo-Riemannian product or a complex circle (see the Appendix for the complete description of isoparametric surfaces in Lorentzian 3-space forms). From the above examples we see that M_c^2 is an open piece of totally umbilic surfaces $\mathbb{S}_1^2(1) \subset \mathbb{H}_1^3$, $\mathbb{H}^2(-1) \subset \mathbb{S}_1^3$ or a B-scroll in $\mathbb{H}_1^3$ with $a^2 = 2$. As for product surfaces, we get M_c^2 is an open piece of one the products given in this theorem. Finally, we are going to show that the last case that M_c^2 is a complex circle can not be given. In fact, since α_x and τ_x are related by $\tau_x = -4 + 3\varepsilon_x\alpha_x^2$, which can be rewritten by using the shape operator S_x as

$$(\mathrm{tr}(S_x))^2 - 4\,\mathrm{tr}(S_x^2) + 8\varepsilon_x = 0,$$

a straightforward computation shows that a complex circle can not satisfy that equation.

As a consequence of that theorems we obtain the following.

Corollary 5.6 *There is no pseudo-Riemannian product of surfaces with biharmonic quadric representation.*

6. A few more examples

This section is devoted to show a few more examples of hypersurfaces such that the quadric representation satisfies the equation $\Delta H_\varphi = \lambda H_\varphi$.

Example 6.1 Let $x : M_c^{m-1} \longrightarrow \bar{M}_\mu^m(k) \subset \mathbb{R}_t^{m+1}$ be a hypersurface whose shape operator has a characteristic polynomial given by $q(t) = (t-a)^{m-1}$, $a \in \mathbb{R}$, and let $y : N_d^{n-1} \longrightarrow \bar{N}_\nu^n(k) \subset \mathbb{R}_s^{n+1}$ be a minimal hypersurface. Then by the Jordan normal form we get $\mathrm{tr}(S_x) = (m-1)a$ and $\mathrm{tr}(S_x^2) = (m-1)a^2$. Since the mean curvature $\alpha_x = a$ and the scalar curvature $\tau_x = (m-1)(n-1)(k + \varepsilon_x a^2)$, it follows from Theorem 4.2 that $\Delta H_\varphi = \lambda H_\varphi$ if and only if $a^2 = -\varepsilon_x k(m+n-2)/(m-1)$, $\varepsilon_x k < 0$, and in this case $\lambda = k(n-1)$.

Let M_c^{m-1} be non-flat totally umbilic in $\bar{M}_\mu^m(k)$ and N_d^{n-1} minimal in $\bar{N}_\nu^n(k)$. Since $\varepsilon_x k < 0$, we only have the following possibilities for M_c^{m-1} and $\bar{M}_\mu^m(k)$: $\mathbb{H}_\mu^{m-1}(-1/r^2) \subset \mathbb{H}_\mu^m(k)$, $\mathbb{S}_\mu^{m-1}(1/r^2) \subset \mathbb{H}_\mu^m(k)$, $\mathbb{H}_{\mu-1}^{m-1}(-1/r^2) \subset \mathbb{S}_\mu^m(k)$, $\mathbb{S}_{\mu-1}^{m-1}(1/r^2) \subset \mathbb{S}_\mu^m(k)$. In all cases the shape operator is $S_x = aI$, where a^2 is given by $(-1-kr^2)/r^2$, $(1-kr^2)/r^2$, $(1+kr^2)/r^2$, $(-1+kr^2)/r^2$, respectively. Then $\Delta H_\varphi = \lambda H_\varphi$ if and only if M_c^{m-1} is $\mathbb{H}_\mu^{m-1}(k) \subset \mathbb{H}_\mu^m(k)$, $\mathbb{S}_{\mu-1}^{m-1}(k) \subset \mathbb{S}_\mu^m(k)$, $\mathbb{S}_\mu^{m-1}(-k(n-1)/(m-1)) \subset \mathbb{H}_\mu^m(k)$, $\mathbb{H}_{\mu-1}^{m-1}(-k(n-1)/(m-1)) \subset \mathbb{S}_\mu^m(k)$. Note that the two first ones are minimal hypersurfaces, in fact, they are totally geodesic.

To find new examples, we recall the construction of some hypersurfaces we have used in early papers.

Generalized umbilic hypersurface of degree 2 ([**3**, **19**]). Let $c : I \subset \mathbb{R} \longrightarrow \mathbb{H}_1^m(k) \subset \mathbb{R}_2^{m+1}$ be a null curve with an associated pseudo-orthonormal frame $\{A, B, Z_1, \ldots, Z_{m-3}, C\}$ along $c(s)$ such that $\dot{c} = A(s)$ and $\dot{C} = -aA(s) - \kappa(s)B(s)$, where $\kappa(s) \neq 0$ and a is a nonzero constant. Then the map $x : I \times \mathbb{R} \times \mathbb{R}^{m-3} \longrightarrow \mathbb{H}_1^m(k) \subset \mathbb{R}_2^{m+1}$ defined by

$$x(s,u,z) = (1+f(z))c(s) + uB(s) + \sum_{j=1}^{m-3} z_j Z_j(s) + (\frac{1}{a} + g(z))C(s),$$

where $f(z)$ and $g(z)$ are solutions of

$$\begin{aligned} kg + af &= -\frac{k}{a} \\ kg^2 + f^2 &= k(\frac{1}{a} - |z|^2), \end{aligned}$$

parametrizes, in a neighborhood of the origin, a Lorentzian hypersurface M_1^{m-1} of $\mathbb{H}_1^m(k)$. The mean curvature α is the nonzero constant a and the minimal polynomial of its shape operator is $q(t) = (t-a)^2$.

Generalized umbilic hypersurface of degree 3 ([**3**, **19**]). Let $c : I \subset \mathbb{R} \longrightarrow \mathbb{H}_1^m(k) \subset \mathbb{R}_2^{m+1}$ be a null curve with an associated pseudo-orthonormal frame $\{A, B, Y, Z_1, \ldots, Z_{m-4}, C\}$ such that $\dot{c} = A(s)$ and $\dot{C} = -aA(s) + \kappa(s)Y(s)$, with $\kappa(s) \neq 0$ and a a nonzero constant. Then the map $x : I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{m-4} \longrightarrow \mathbb{H}_1^m(k) \subset \mathbb{R}_2^{m+1}$ defined by

$$x(s,u,y,z) = (1 + f(z))c(s) + uB(s) + yY(s) + \sum_{j=1}^{m-4} z_j Z_j(s) + (\frac{1}{a} + g(z))C(s),$$

where $f(z)$ and $g(z)$ are solutions of

$$\begin{aligned} kg + af &= -\frac{k}{a} \\ kg^2 + f^2 &= k(\frac{1}{a} - |z|^2 - y^2), \end{aligned}$$

parametrizes, in a neighborhood of the origin, a Lorentzian hypersurface M_1^{m-1} in $\mathbb{H}_1^m$. Then M_1^{m-1} has constant mean curvature $\alpha = a \neq 0$ and the minimal polynomial of its shape operator is given by $q(t) = (t-a)^3$.

Then by taking $M_c^{m-1} \subset \bar{M}_\mu^m$ as a generalized umbilic hypersurface of degree two or three and N_d^{n-1} minimal in $\bar{N}_\nu^n(k)$, the quadric representation of the product $M_c^{m-1} \times N_d^{n-1}$ satisfies $\Delta H_\varphi = \lambda H_\varphi$ if and only if $a^2 = -(\varepsilon_x k)(m+n-2)/(m-1)$.

Example 6.2 Let N_d^{n-1} be minimal in $\bar{N}_\nu^n(k)$ and $M_c^{m-1} = \mathbb{S}_u^p(1/r^2) \times \mathbb{H}_{c-u}^{m-p-1}(k/(1-kr^2))$ into $\mathbb{S}_{c+1}^m(k)$, $k > 0$ and $1 - kr^2 < 0$, with $r^2 = (m+n-p-2)/k(n-1)$. Then it is well known (see [**7**]) that M_c^{m-1} is of 2-type with associated eigenvalues $\lambda_1 = p/r^2 = kp(n-1)/(m+n-p-2)$ and $\lambda_2 = k(m-p-1)/(1-kr^2) = -k(n-1)$. It is easy to see that $\lambda_1 = \lambda - k(n-1)$, where $\lambda = k(n-1)(m+n-2)/(m+n-p-2)$. Therefore, by applying Theorem 3.3, $\Delta H_\varphi = \lambda H_\varphi$.

Now, let N_d^{n-1} be minimal in $\bar{N}_\nu^n(k)$ and $M_c^{m-1} = \mathbb{S}_u^p(1/r^2) \times \mathbb{H}_{c-u}^{m-p-1}(k/(1-kr^2)) \subset \mathbb{H}_c^m(k)$, $k < 0$ and $1 - kr^2 > 0$, with $r^2 = -p/k(n-1)$. Then as M_c^{m-1} is of 2-type with associated eigenvalues $\lambda_1 = p/r^2 = -k(n-1)$ and $\lambda_2 = k(m-p-1)/(1-kr^2)$, it is easy to see that $\lambda_2 = \lambda - k(n-1)$, where $\lambda = k(n-1)(m+n-2)/(n+p-1)$. Therefore, by applying Theorem 3.3, $\Delta H_\varphi = \lambda H_\varphi$.

Any other choices of radii r produce examples of hypersurfaces M_c^{m-1} with both constant mean and scalar curvatures such that, for any minimal hypersurface N_d^{n-1}, the quadric representation does not satisfy the condition $\Delta H_\varphi = \lambda H_\varphi$.

As for remaining products $M_c^{m-1} = \mathbb{S}_u^p(1/r^2) \times \mathbb{S}_{c-u}^{m-p-1}(k/(1-kr^2)) \subset \mathbb{S}_c^m(k)$, $k > 0$ and $1 - kr^2 > 0$, and $M_c^{m-1} = \mathbb{H}_u^p(-1/r^2) \times \mathbb{H}_{c-u}^{m-p-1}(k/(1+kr^2)) \subset \mathbb{H}_{c+1}^m(k)$, $k < 0$ and

$1 + kr^2 > 0$, they are minimal when $r^2 = p/k(m-1)$ and so the equation $\Delta H_\varphi = \lambda H_\varphi$ holds. Otherwise, they are of 2-type with associate eigenvalues $\{p/r^2, k(m-p-1)/(1-kr^2)\}$ and $\{-p/r^2, k(m-p-1)/(1+kr^2)\}$, respectively. Therefore there is no r accomplishing Theorem 3.3. Indeed, in the former case both eigenvalues are positive and, in order to apply Theorem 3.3, one of them should be negative; in the latter, just the contrary occurs.

Note that in this example the minimal hypersurface N_d^{n-1} in $\bar{N}_\nu^n(k)$ can be replaced by a minimal submanifold N_d^ℓ and everything works fine. We must only change $(n-1)$ by ℓ.

7. Appendix: Isoparametric surfaces in Lorentzian 3-space forms

Let $\bar{M}_1^3(k)$ be a 3-space form of constant curvature $k \in \mathbb{R}$. A model for $\bar{M}_1^3(k)$ is the Lorentz-Minkowski space $\mathbb{L}^3$ if $k = 0$, the De Sitter space $\mathbb{S}_1^3(k)$ if $k > 0$ and the anti De Sitter space $\mathbb{H}_1^3(k)$ if $k < 0$. Let M_s^2 be a (spacelike or Lorentzian) surface in $\bar{M}_1^3(k)$ and denote by S the Weingarten endomorphism associated to a unit normal vector field. If the minimal polynomial of the shape operator is independent of each point of M_s^2, M_s^2 is said to be isoparametric. The possibly complex roots of that polynomial are called the principal curvatures.

The selfadjoint endomorphism S on a tangent space of M_s^2 has a matrix of exactly one of the following three types:

$$\text{I. } S \sim \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \qquad \text{II. } S \sim \begin{pmatrix} \lambda & 0 \\ -1 & \lambda \end{pmatrix} \qquad \text{III. } S \sim \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix}$$

In types I and III, S is relative to an orthonormal basis while in case II the basis is pseudo-orthonormal, that is, a basis $\{X, Y\}$ such that $\langle X, X\rangle = 0 = \langle Y, Y\rangle$ and $\langle X, Y\rangle = -1$. Thus the classification of isoparametric surfaces in $\bar{M}_1^3(k)$ should be done by distinguishing three cases, according to the canonical form of the shape operator S.

Type I: *S is diagonalizable.* If $\lambda = \mu$ then M_s^2 is nothing but an open piece of a totally umbilical surface. Otherwise, following K. Nomizu, [**22**], and N. Abe–N. Koike–S. Yamaguchi, [**1**], we get that M_s^2 is an open piece of one of the following products:
(i) $\mathbb{H}^1(-1/r^2) \times \mathbb{R}$, $\mathbb{S}_1^1(1/r^2) \times \mathbb{R}$ or $\mathbb{L} \times \mathbb{S}^1(1/r^2)$ if $k = 0$.
(ii) $\mathbb{H}^1(-1/r^2) \times \mathbb{S}^1(k/(1+kr^2))$ or $\mathbb{S}_1^1(1/r^2) \times \mathbb{S}^1(k/(1-kr^2))$, $1 - kr^2 > 0$, if $k > 0$.

(iii) $\mathbb{H}^1_1(-1/r^2)\times\mathbb{S}^1(k/(1+kr^2)),\ 1+kr^2<0$, $\mathbb{H}^1(-1/r^2)\times\mathbb{S}^1_1(k/(1+kr^2)),\ 1+kr^2<0$, or $\mathbb{H}^1(-1/r^2)\times\mathbb{H}^1(k/(1+kr^2)),\ 1+kr^2>0$, if $k<0$.

Type II: *S has a double real eigenvalue*. In this case, following L. Graves, [**17**], and M. Magid, [**19**], if $k=0$, and L.J. Alías–A. Ferrández–P. Lucas, [**2**], and M. Dajczer–K. Nomizu, [**12**], if $k\neq 0$, we deduce that M^2_s is locally an open piece of a B-scroll. This surface has been described in Example 5.3.

Type III: *S has complex eigenvalues*. Then from Codazzi's equations we can easily deduce that X and Y induce parallel vector fields on M^2_1 and therefore M^2_1 is a flat Lorentzian surface with parallel second fundamental form in the pseudo-Euclidean space where $\bar{M}^3_1(k)$ is lying. Then by using [**18**, Theorem 1.15 and 1.17] we obtain M^2_1 is congruent to a complex circle in $\mathbb{H}^3_1(k)$. Let $a+bi$ be a non-zero complex number such that $a^2-b^2=1/k$. The following map $x=(x^1,x^2,x^3,x^4):\mathbb{R}^2_1\to\mathbb{H}^3_1(k)\subset\mathbb{R}^4_2$ describes a complex circle:

$$\begin{aligned}
x^1(u_1,u_2) &= b\cosh u_2\cos u_1 - a\sinh u_2\sin u_1,\\
x^2(u_1,u_2) &= a\sinh u_2\cos u_1 + b\cosh u_2\sin u_1,\\
x^3(u_1,u_2) &= a\cosh u_2\cos u_1 + b\sinh u_2\sin u_1,\\
x^4(u_1,u_2) &= a\cosh u_2\sin u_1 - b\sinh u_2\cos u_1,
\end{aligned}$$

where (u_1,u_2) is the usual coordinate system in $\mathbb{R}^2_1$ with the Lorentz metric $ds^2=(du_1)^2-(du_2)^2$ and $\mathbb{R}^4_2$ is equipped with the metric $ds^2=(dx^1)^2+(dx^2)^2-(dx^3)^2-(dx^4)^2$. The shape operator S is given by

$$S=\begin{pmatrix}\lambda & -\mu\\ \mu & \lambda\end{pmatrix},\quad \lambda=\frac{2ab}{k(a^2+b^2)},\quad \mu=\frac{-1}{a^2+b^2},$$

with respect to the usual frame $\left\{\frac{\partial x}{\partial u_1},\frac{\partial x}{\partial u_2}\right\}$. This surface is called a *comples circle* of radius $a+bi$ by Magid, [**18**].

Note added. We would like to point out that some time after this paper was written we have met a series of papers by B.Y.Chen ([**6**], [**9**], [**10**]) where a quadric representation for Riemannian product immersions is considered as a tensor product immersions (see also [**13**], [**21**]).

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