Pseudo-spherical and pseudo-hyperbolic submanifolds via its quadric representation I

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1. Introduction

The quadric representation of a submanifold has become a very useful tool in certain classification problems of Riemannian submanifolds (see [**3**], [**4**], [**5**], [**9**] and [**16**]). In [**13**], in order to classify constant mean curvature surfaces into non-flat pseudo-Riemannian space forms, we brought the quadric representation into the realm of indefinite space forms. Recently, in [**12**] and [**14**], the quadric representation has been also defined for product of pseudo-Riemannian submanifolds and has been used to solve some open questions related with a Chen's conjecture on biharmonicity (see for instance [**8**]). The purpose of this paper is to present the quadric representation in the most general setting and then look for a classification of submanifolds such that the mean curvature vector field of its quadric representation is proper for the Laplacian.

The reason for dealing with this condition is twofold. On one hand, viewing the quadric representation as an isometric immersion, we pointed out that is a natural assumption in terms of finite type submanifolds. On the other hand, the equation $\widetilde{\Delta}\tilde{H} = \lambda\tilde{H}$ has provided a source of properties for indefinite submanifolds without counterparts for Riemannian submanifolds (see for instance [**2**], [**10**], [**11**]). Furthermore, we know that $\widetilde{\Delta}\tilde{H} = \lambda\tilde{H}$ allows to get 1-type and null 2-type submanifolds, as well as infinite-type submanifolds (see [**7**]). Here, in Part I, we provide preliminary computations and examples which we need in Part II. Sinces Parts I and II represent a whole, we also give in advance the main results of our work which are contained in Part II. Our main theorem gives a complete characterization of hypersurfaces whose quadric representation satisfies the equation $\widetilde{\Delta}\tilde{H} = \lambda\tilde{H} + \mu(\varphi - \varphi_0)$. As a consequence, in dealing with surfaces into de Sitter and anti de Sitter worlds, we have got nice characterizations, among others, for minimal B-scrolls and complex circles. A characterization of submanifolds whose quadric representation satisfies $\widetilde{\Delta}\tilde{H} = \lambda\tilde{H}$ is also given.

2. Basics

Let $\mathbb{R}^{m+2}_s$ be the pseudo-Euclidean space of dimension $m+2$ with metric $\langle , \rangle$ having a matrix, with respect to the standard coordinate system, given by $G = \mathrm{diag}[\varepsilon_1, \ldots, \varepsilon_{m+2}]$, where $\varepsilon_1 = \cdots = \varepsilon_s = -1$ and $\varepsilon_{s+1} = \cdots = \varepsilon_{m+2} = 1$. Let us denote by $SA(m+2, s)$ the space of selfadjoint operators on $\mathbb{R}^{m+2}_s$, that is, $SA(m+2, s) = \{P \in \mathfrak{gl}(m+2, \mathbb{R}) : P^t G = GP^t\}$, P^t standing for the transpose of P. Let $\mathbb{S}^{m+1}_s(r)$ and $\mathbb{H}^{m+1}_{s-1}(-r)$, $r > 0$, be the central hyperquadrics

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of $\mathbb{R}^{m+2}_s$ defined by

$$\begin{aligned}\mathbb{S}^{m+1}_s(r) &= \{x \in \mathbb{R}^{m+2}_s : \langle x, x\rangle = r^2\},\\ \mathbb{H}^{m+1}_{s-1}(-r) &= \{x \in \mathbb{R}^{m+2}_s : \langle x, x\rangle = -r^2\}.\end{aligned}$$

These hypersurfaces are of constant curvature $1/r^2$ and $-1/r^2$, respectively. Without loss of generality, assume that $r^2 = 1$. For short we will write $\bar{M}^{m+1}(k)$, $k = \pm 1$, to indicate $\mathbb{S}^{m+1}_s(1)$ or $\mathbb{H}^{m+1}_{s-1}(-1)$, according to $k = 1$ or $k = -1$, respectively.

Let us consider the map $f : \bar{M}^{m+1}(k) \to SA(m+2, s)$ defined by $f(u) = uu^tG$, where u is regarded as a 1-column matrix. It is easy to see that f is an isometric immersion provided that $SA(m+2, s)$ is endowed, as usual, with the metric $\widetilde{g}(P, Q) = \frac{k}{2}\operatorname{tr}(PQ)$. Then f is said to be *the second standard immersion* of $\bar{M}^{m+1}(k)$ into $SA(m+2, s)$. This map has been deeply studied in the Riemannian case (see for instance [**5**], [**9**] and [**16**]).

At any point $u \in \bar{M}^{m+1}(k)$, the normal space of $\bar{M}^{m+1}(k)$ in $SA(m+2, s)$, at $f(u)$, is

$$T^{\perp}_{f(u)}\bar{M}^{m+1}(k) = \{P \in SA(m+2, s) : (P - \lambda I)u = 0, \text{ for some } \lambda \in \mathbb{R}\},$$

I being the identity matrix. Thus $f(u) \in T^{\perp}_{f(u)}\bar{M}^{m+1}(k)$. The second fundamental form $\tilde{\sigma}$ of f is given by

$$\tilde{\sigma}(X, Y) = (XY^t + YX^t)G - 2k\langle X, Y\rangle f(u),$$

for X, Y in $T_u\bar{M}^{m+1}(k)$. It is well known that $\bar{M}^{m+1}(k)$ is minimally immersed by f in a pseudosphere or a pseudohyperbolic space of $SA(m+2, s)$ centered at $\frac{k}{m+2}I$ (see [**13**]).

Let $x : M^n_\nu \to \bar{M}^{m+1}(k) \subset \mathbb{R}^{m+2}_s$ be an isometric immersion and consider the new isometric immersion $\varphi = f \circ x$, that will be called *the quadric representation* of (M^n_ν, x). Let $\overline{\mathfrak{X}}(M^n_\nu)$ be the $\mathcal{C}^\infty(M)$-module of smooth vector fields on x.

Define a map $\Phi : \overline{\mathfrak{X}}(M^n_\nu) \times \overline{\mathfrak{X}}(M^n_\nu) \to \mathcal{C}^\infty(M^n_\nu; SA(m+2, s))$ by

$$\Phi(X, Y) = (XY^t + YX^t)G,$$

where $\mathcal{C}^\infty(M^n_\nu; SA(m+2, s))$ is the algebra of differentiable functions from M^n_ν into $SA(m+2, s)$.

Then it is not difficult to see that Φ is symmetric, $\mathcal{C}^\infty(M)$-bilinear and parallel, that is, $\widetilde{\nabla}_Z(\Phi(X, Y)) = \Phi(\overline{\nabla}_Z X, Y) + \Phi(X, \overline{\nabla}_Z Y)$, $\widetilde{\nabla}$ and $\overline{\nabla}$ standing for the pseudo-Riemannian connections on $SA(m+2, s)$ and $\mathbb{R}^{m+2}_s$, respectively. The endomorphism $\Phi(X, Y)$ is characterized by $\Phi(X, Y)(Z) = \langle Y, Z\rangle X + \langle X, Z\rangle Y$.

Notice that, in the Riemannian case, this formula has a simple geometric meaning: if X and Y is an orthonormal basis for a plane Π, then $\Phi(X, Y)$ is zero on $\Pi^\perp$, and on Π is a symmetry sending X to Y and Y to X.

Let $\bar{H}$ and $\tilde{H}$ denote the mean curvature vector fields associated to $x : M^n_\nu \to \bar{M}^{m+1}(k) \subset \mathbb{R}^{m+2}_s$ and φ, respectively. Then an easy computation shows that

$$\tilde{H} = \Phi(x, \bar{H}) + \frac{1}{n}\operatorname{tr}\Phi,$$

where $\operatorname{tr}(\Phi) = \sum_{i=1}^n \varepsilon_i \Phi(E_i, E_i)$, $\{E_i\}_{i=1}^n$ being a local orthonormal frame tangent to M^n_ν and $\varepsilon_i = \langle E_i, E_i\rangle$.

Given $Z \in \overline{\mathfrak{X}}(M_\nu^n)$, define a map $\Psi_Z : \overline{\mathfrak{X}}(M_\nu^n) \times \overline{\mathfrak{X}}(M_\nu^n) \to \mathcal{C}^\infty(M_\nu^n; SA(m+2,s))$ by $\Psi_Z(X,Y) = \Phi(X, \overline{\nabla}_Y Z)$.

In what follows Δ and $\widetilde{\Delta}$ will denote the Laplacians associated to $\overline{\nabla}$ and $\widetilde{\nabla}$, respectively. Then we have the following

Lemma 2.1 *With the above notations we have*

$$\widetilde{\Delta}\tilde{H} = -n\Phi(\bar{H},\bar{H}) + \Phi(x,\Delta\bar{H}) + \frac{1}{n}\widetilde{\Delta}(\operatorname{tr}\Phi) - 2\operatorname{tr}(\Psi_{\bar{H}}).$$

Proof. Let $p \in M_\nu^n$ and consider a local orthonormal frame $\{E_i\}_{i=1}^n$ such that $\nabla_{E_i}E_j(p) = 0$, ∇ being the pseudo-Riemannian connection on M_ν^n. Then

$$\widetilde{\Delta}\tilde{H}(p) = -\sum_{i=1}^n \varepsilon_i \widetilde{\nabla}_{E_i}\widetilde{\nabla}_{E_i}\tilde{H}(p).$$

A straightforward computation yields

$$\begin{aligned}\widetilde{\nabla}_{E_i}\widetilde{\nabla}_{E_i}\tilde{H}(p) &= \Phi(\overline{\nabla}_{E_i}E_i,\bar{H})(p) + \Phi(x,\overline{\nabla}_{E_i}\overline{\nabla}_{E_i}\bar{H})(p)\\ &\quad + 2\Phi(E_i,\overline{\nabla}_{E_i}\bar{H})(p) + \frac{1}{n}\widetilde{\nabla}_{E_i}\widetilde{\nabla}_{E_i}(\operatorname{tr}\Phi)(p),\end{aligned}$$

and then the lemma follows.

Now we are going to give a complete description of $\widetilde{\Delta}(\operatorname{tr}\Phi)$ by applying this endomorphism on $Z \in \mathfrak{X}(M_\nu^n)$ and $\xi \in \mathfrak{X}^\perp(M_\nu^n) \subset \mathfrak{X}(\overline{M}^{m+1}(k))$. Let $p \in M_\nu^n$ and $\{E_i\}_{i=1}^n$ be a local orthonormal frame tangent to M_ν^n such that $\nabla_{E_i}E_j(p) = 0$. Then

$$\widetilde{\Delta}(\operatorname{tr}\Phi)(p) = 2\sum_{i=1}^n \varepsilon_i \Phi(\Delta E_i, E_i)(p) - 2\sum_{i,j}\varepsilon_i\varepsilon_j\Phi(\overline{\nabla}_{E_j}E_i, \overline{\nabla}_{E_j}E_i)(p),$$

and so

$$\begin{aligned}\widetilde{\Delta}(\operatorname{tr}\Phi)(Z)(p) &= 2\sum_{i=1}^n \varepsilon_i\{\langle\Delta E_i, Z\rangle E_i + \langle Z, E_i\rangle\Delta E_i\}(p)\\ &\quad - 4\sum_{i,j}\varepsilon_i\varepsilon_j\langle\overline{\nabla}_{E_j}E_i, Z\rangle(p)\overline{\nabla}_{E_j(p)}E_i.\end{aligned}$$

By using the Gauss and Weingarten formulae we find that

$$\begin{aligned}\overline{\nabla}_{E_j}\overline{\nabla}_{E_j}W(p) &= \sum_{i=1}^n \varepsilon_i\left\{\langle\overline{\nabla}_{E_j}\overline{\nabla}_{E_j}W, E_i\rangle E_i + 2\langle\overline{\nabla}_{E_j}W, \overline{\nabla}_{E_j}E_i\rangle E_i\right.\\ &\quad + 2\langle\overline{\nabla}_{E_j}W, E_i\rangle\overline{\nabla}_{E_j}E_i + 2\langle W, \overline{\nabla}_{E_j}E_i\rangle\overline{\nabla}_{E_j}E_i\\ &\quad \left.+ \langle W, \overline{\nabla}_{E_j}\overline{\nabla}_{E_j}E_i\rangle E_i + \langle W, E_i\rangle\overline{\nabla}_{E_j}\overline{\nabla}_{E_j}E_i\right\}(p),\end{aligned}$$

and then

$$\begin{aligned}\Delta W(p) &= (\Delta W)^T(p) - 2\sum_{i,j}\varepsilon_i\varepsilon_j\langle\overline{\nabla}_{E_j}W, \overline{\nabla}_{E_j}E_i\rangle E_i(p)\\ &\quad - 2\sum_{i,j}\varepsilon_i\varepsilon_j\langle\overline{\nabla}_{E_j}W, E_i\rangle\overline{\nabla}_{E_j}E_i(p) - 2\sum_{i,j}\varepsilon_i\varepsilon_j\langle W, \overline{\nabla}_{E_j}E_i\rangle\overline{\nabla}_{E_j}E_i(p)\\ &\quad + \sum_{i=1}^n \varepsilon_i\{\langle W, \Delta E_i\rangle E_i + \langle W, E_i\rangle\Delta E_i\}(p).\end{aligned}$$

At the point p, this equation yields

$$\begin{aligned}\widetilde{\Delta}(\operatorname{tr}\Phi)(E_k) &= 4\sum_{i,j}\varepsilon_i\varepsilon_j\{\langle\overline{\nabla}_{E_j}E_k,\overline{\nabla}_{E_j}E_i\rangle E_i+\langle\overline{\nabla}_{E_j}E_k,E_i\rangle\overline{\nabla}_{E_j}E_i\}\\ &\quad+2(\Delta E_k)^{\perp}\\ &= 2(\Delta E_k)^{\perp}+4\sum_{i,j}\varepsilon_i\varepsilon_j\langle\sigma(E_j,E_k),\sigma(E_j,E_i)\rangle E_i,\end{aligned}$$

σ being the second fundamental form of M^n_ν in $\mathbb{R}^{m+2}_s$. On the other hand, it is easy to see that

$$(\Delta E_k)^{\perp}(p)=-n\nabla^{\perp}_{E_k(p)}\bar{H},$$

$\nabla^{\perp}$ standing for the normal connection of M^n_ν in $\mathbb{R}^{m+2}_s$. Write $\sigma(E_j,E_k)=\sum_{r=1}^{m-n+2}\delta_r\langle S_rE_j,E_k\rangle\xi_r$, where $\{\xi_1,\dots,\xi_{m-n+2}\}$, $\xi_{m-n+2}=x$, is a local orthonormal frame of normal vectors to M^n_ν in $\mathbb{R}^{m+2}_s$, $\delta_r=\langle\xi_r,\xi_r\rangle$ and S_r denotes the shape operator associated to ξ_r. Then

$$\begin{aligned}\sum_{i,j}\varepsilon_i\varepsilon_j\langle\sigma(E_j,E_k),\sigma(E_j,E_i)\rangle E_i(p) &= \left(\sum_{r=1}^{m-n+2}\delta_rS_r^2\right)(E_k)(p)\\ &= \left(\sum_{r=1}^{m-n+1}\delta_rS_r^2+kI\right)(E_k)(p).\end{aligned}$$

Therefore

$$\widetilde{\Delta}(\operatorname{tr}\Phi)(E_k)(p)=-2n\nabla^{\perp}_{E_k(p)}\bar{H}+4\left(\sum_{r=1}^{m-n+1}\delta_rS_r^2+kI\right)(E_k)(p),$$

so that

$$\widetilde{\Delta}(\operatorname{tr}\Phi)(Z)=-2n\nabla^{\perp}_{Z}\bar{H}+4\left(\sum_{r=1}^{m-n+1}\delta_rS_r^2+kI\right)(Z),$$

for all $Z\in\mathfrak{X}(M^n_\nu)$.

Similar computations lead to

$$\widetilde{\Delta}(\operatorname{tr}\Phi)(x)=4n\bar{H}$$

and

$$\widetilde{\Delta}(\operatorname{tr}\Phi)(\xi)=-2n\operatorname{tr}(\Psi_{\bar{H}})(\xi)-4\sum_{r=1}^{m-n+1}\delta_r\operatorname{tr}(S_\xi\circ S_r)\xi_r+4k\operatorname{tr}(S_\xi)x,$$

for any $\xi\in\mathfrak{X}^{\perp}(M^n_\nu)\subset\mathfrak{X}(\bar{M}^{m+1}(k))$. From equations (3)–(5) we have

$$\widetilde{\Delta}\tilde{H}(Z) = 4\left(S_{\bar{H}}+\frac{1}{n}\sum_{r=1}^{m-n+2}\delta_rS_r^2\right)(Z)-4\nabla^{\perp}_{Z}\bar{H}+\langle\Delta\bar{H},Z\rangle x, \tag{7}$$

$$\begin{aligned}\widetilde{\Delta}\tilde{H}(\xi) &= -4\operatorname{tr}(\Psi_{\bar{H}})(\xi)-\frac{4}{n}\sum_{r=1}^{m-n+1}\delta_r\operatorname{tr}(S_\xi\circ S_r)\xi_r-2n\langle H,\xi\rangle H\\ &\quad+\left(2kn\langle H,\xi\rangle+\langle\Delta\bar{H},\xi\rangle+\frac{4}{n}k\operatorname{tr}(S_\xi)\right)x,\end{aligned} \tag{8}$$

$$\begin{aligned}\widetilde{\Delta}\tilde{H}(x) &= k(\Delta\bar{H})^T+2(n+2)H+2(\langle\Delta\bar{H},x\rangle-k(n+2))x\\ &\quad+k(\Delta\bar{H})^{\perp},\end{aligned} \tag{9}$$

H being the mean curvature vector field of M_ν^n in $\bar{M}^{m+1}(k)$ and $(\Delta\bar{H})^\perp$ the component of $\Delta\bar{H}$ normal to M_ν^n in $\bar{M}^{m+1}(k)$. Note that equations (7)–(9) completely characterize the endomorphism $\widetilde{\Delta}\tilde{H}$.

It is also quite easy to see that $\tilde{H}(Z) = (2/n)Z$, $\tilde{H}(\xi) = \langle H, \xi\rangle x$ and $\tilde{H}(x) = k\bar{H} - x$.

Finally we state the following useful result, just obtained in the Riemannian case in [**9**].

Lemma 2.2 *Let $\{E_1, \ldots, E_{m+1}\}$ be a local pseudo-orthonormal frame tangent to $\bar{M}^{m+1}(k)$. Then at every point $x \in \bar{M}^{m+1}(k)$ we have*

$$I = kf(x) + \sum_{i=1}^{m+1} \varepsilon_i E_i E_i^t G.$$

3. A 2-type equation for hypersurfaces: Examples

Throughout this section we shall deal with hypersurfaces M_ν^m in $\bar{M}^{m+1}(k)$. Let N be a unit normal vector field to M_ν^m in $\bar{M}^{m+1}(k)$, S the shape operator associated to N and α the mean curvature function of M_ν^m in $\bar{M}^{m+1}(k)$ defined by $\alpha = \dfrac{\varepsilon \operatorname{tr}(S)}{m}$. The following formula for $\Delta\bar{H}$ can be found in [**6**, Lemma 3],

$$\Delta\bar{H} = 2S(\nabla\alpha) + \varepsilon m\alpha\nabla\alpha + (\Delta\alpha + \varepsilon\alpha\operatorname{tr}(S^2) + km\alpha)N - km(k + \varepsilon\alpha^2)x,$$

where $\varepsilon = \langle N, N\rangle$.

Assume now that the quadric representation φ satisfies the equation

$$\widetilde{\Delta}\tilde{H} = \lambda\tilde{H} + \mu(\varphi - \varphi_0)$$

for some real constants λ and μ, φ_0 being a constant matrix. It is not difficult to see that a hypersurface of finite type less than or equal to two satisfies (11). However, the converse does not hold as we have pointed out in [**1**].

For convenience, we shall call $A = -\mu\varphi_0$ and then equation (11) will be written down as $A = \widetilde{\Delta}\tilde{H} - \lambda\tilde{H} - \mu\varphi$. Then a straightforward computation from (7)–(10) yields the following system of equations

$$\begin{aligned}
A(Z) &= \frac{4\varepsilon}{m}S^2(Z) + 4\alpha S(Z) + \frac{2}{m}(2k(m+1) - \lambda)Z \\
&\quad -4Z(\alpha)N + \langle\Delta\bar{H}, Z\rangle x, \qquad (12)\\
A(N) &= -4\varepsilon\Delta\alpha - 2\varepsilon\left(\frac{2}{m}\operatorname{tr}(S^2) + m\alpha^2\right)N \\
&\quad +\varepsilon(\nabla\alpha + (\varepsilon\operatorname{tr}(S^2) + (3m+4)k - \lambda)\alpha)x, \qquad (13)\\
A(x) &= k(\Delta\bar{H})^T + k(\Delta\alpha + \alpha(\varepsilon\operatorname{tr}(S^2) + (3m+4)k - \lambda))N \\
&\quad -(2\varepsilon m\alpha^2 + 4k(m+1) - 2\lambda + k\mu)x. \qquad (14)
\end{aligned}$$

If we suppose that the mean curvature α is constant, then the above system reduces to

$$\begin{aligned} A(Z) &= \frac{4\varepsilon}{m}S^2(Z) + 4\alpha S(Z) + \frac{2}{m}(2k(m+1)-\lambda)Z, && (15)\\ A(N) &= -2\varepsilon\left(\frac{2}{m}\operatorname{tr}(S^2) + m\alpha^2\right)N \\ &\quad + (\operatorname{tr}(S^2) + (3m+4)\varepsilon k - \varepsilon\lambda)\alpha x, && (16)\\ A(x) &= k(\varepsilon\operatorname{tr}(S^2) + (3m+4)k - \lambda)\alpha N \\ &\quad -(2\varepsilon m\alpha^2 + 4k(m+1) - 2\lambda + k\mu)x. && (17) \end{aligned}$$

Now we are going to give some examples where these equations can be checked out.

Example 3.1 Let M_ν^m be a minimal hypersurface of $\bar{M}^{m+1}(k)$ such that the shape operator verifies $S^2 = aI$, $a \in \mathbb{R}$. Then one immediately checks that M_ν^m satisfies equations (15)–(17) with

$$\lambda = 2(m+1)(k+\varepsilon a), \quad \mu = 4\varepsilon k(m+2)a$$

and

$$\varphi_0 = \frac{k}{m+2}I.$$

Example 3.2 *(Generalized totally umbilical hypersurface)* Let $\bar{M}^{m+1}(k)$ be either the anti De Sitter space $\mathbb{H}_1^{m+1}(-1)$ or the De Sitter space $\mathbb{S}_1^{m+1}(1)$. Let $\gamma : I \subset \mathbb{R} \to \bar{M}^{m+1}(k) \subset \mathbb{R}_s^{m+2}$ be a null curve with a local pseudo-orthonormal frame $\{A, B, Z_1, \ldots, Z_{m-2}, C\}$ tangent to $\bar{M}^{m+1}(k)$ along γ such that

$$\langle A, A\rangle = \langle B, B\rangle = 0, \langle A, B\rangle = -1,$$

and

$$\begin{aligned} \dot{\gamma}(s) &= A(s), \\ \dot{C}(s) &= -\kappa(s)B(s), \end{aligned}$$

for a certain function $\kappa(s) \neq 0$. Then the map $x : I \times \mathbb{R} \times \mathbb{R}^{m-2} \to \bar{M}^{m+1}(k) \subset \mathbb{R}_s^{m+2}$ given by

$$x(s,u,z) = f(z)\gamma(s) + uB(s) + \sum_{j=1}^{m-2} z_j Z_j(s), \qquad f(z) = \sqrt{1-k|z|^2},$$

parametrizes a minimal Lorentzian hypersurface which is called a generalized totally umbilical hypersurface (it is called a B-scroll over γ when $m = 2$). It is easy to show that $N(s,u) = C(s)$ defines a unit normal vector field and the shape operator S associated to N verifies $S^2 = 0$. The example before says that this hypersurface satisfies that $\Delta\tilde{H} = 2k(m+1)\tilde{H}$.

Example 3.3 *(Complex circle)* Let $x : \mathbb{R}^2 \to \mathbb{H}_1^3(-1)$ be the map defined by

$$x(u,v) = (\cos u \cosh v, \sin u \cosh v, \sin u \sinh v, -\cos u \sinh v).$$

It is easy to see that x parametrizes a minimal Lorentzian surface in $\mathbb{H}_1^3(-1)$ that is called complex circle (see [**15**]).

A unit normal vector field is given by

$$N(u,v) = (\sin u \sinh v, -\cos u \sinh v, -\cos u \cosh v, -\sin u \cosh v),$$

and the shape operator S associated to N has a matrix, relative to the usual basis $\left\{\dfrac{\partial x}{\partial u}, \dfrac{\partial x}{\partial v}\right\}$, of form

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

So we have $S^2 = -I$ and, again from Example 3.1, the complex circle satisfies

$$\Delta \tilde{H} = -12\tilde{H} + 16\left(\varphi + \frac{1}{4}I\right).$$

Example 3.4 *(Non flat totally umbilical hypersurfaces)* A non flat totally umbilical hypersurface is given by cutting $\overline{M}^{m+1}(k)$ by a hyperplane in $\mathbb{R}_s^{m+2}$. Without loss of generality we can choice the hyperplane P with the latter coordinate being constant and given by $\sqrt{k-\varepsilon r^2}$, $\varepsilon = \pm 1$, $r > 0$. Then M_ν^m can be described by the set $\{(y, \sqrt{k-\varepsilon r^2}) : \langle y, y\rangle = \varepsilon r^2\}$, and therefore the quadric representation has matrix of form

$$\varphi = \begin{pmatrix} yy^t & \sqrt{k-\varepsilon r^2}y \\ \sqrt{k-\varepsilon r^2}y^t & k-\varepsilon r^2 \end{pmatrix} G.$$

Since

$$\begin{aligned} \Delta(yy^t) &= \frac{2\varepsilon(m+1)}{r^2}yy^t + C, \\ \Delta y &= \frac{\varepsilon m}{r^2}y, \end{aligned}$$

C being a constant matrix, we deduce that φ is of 2-type and so equation $\widetilde{\Delta}\tilde{H} = \lambda\tilde{H} + \mu(\varphi - \varphi_0)$ holds, for appropriate λ and μ.

Example 3.5 *(Pseudo-Riemannian standard products)*
Let $M^p(\varepsilon_1 r_1) \times N^{m-p}(\varepsilon_2 r_2)$ be a pseudo-Riemannian product with $\varepsilon_1 r_1^2 + \varepsilon_2 r_2^2 = k$, where $M^p(\varepsilon_1 r_1)$ and $N^{m-p}(\varepsilon_2 r_2)$ are pseudo-spheres or pseudo-hyperbolic spaces according to $\varepsilon_i = 1$ or $\varepsilon_i = -1$, respectively. Let x and y be the standard immersions of $M^p(\varepsilon_1 r_1)$ and $N^{m-p}(\varepsilon_2 r_2)$ into the corresponding pseudo-Euclidean spaces $\mathbb{R}_a^{p+1}$ and $\mathbb{R}_b^{m-p+1}$, respectively. Then $x \times y$ is an isometric immersion of $M^p(\varepsilon_1 r_1) \times N^{m-p}(\varepsilon_2 r_2)$ into $\overline{M}^{m+1}(k)$ whose quadric representation $\varphi = (x \times y)(x \times y)^t G$ has matrix of form

$$\varphi = \begin{pmatrix} xx^t G_1 & xy^t G_2 \\ yx^t G_1 & yy^t G_2 \end{pmatrix},$$

G_1 and G_2 standing for the metrics on $\mathbb{R}^{p+1}_a$ and $\mathbb{R}^{m-p+1}_b$, respectively. A straightforward computation yields

$$\begin{aligned}
\Delta(xx^tG_1) &= \frac{2\varepsilon_1(p+1)}{r_1^2}xx^tG_1 - 2I_{p+1},\\
\Delta(xy^tG_2) &= \left(\frac{\varepsilon_1 p}{r_1^2} + \frac{\varepsilon_2(m-p)}{r_2^2}\right)xy^tG_2,\\
\Delta(yx^tG_1) &= \left(\frac{\varepsilon_1 p}{r_1^2} + \frac{\varepsilon_2(m-p)}{r_2^2}\right)yx^tG_1,\\
\Delta(yy^tG_2) &= \frac{2\varepsilon_2(m-p+1)}{r_2^2}yy^tG_2 - 2I_{m-p+1}.
\end{aligned}$$

Therefore $\widetilde{\Delta}\tilde{H} = \lambda\tilde{H} + \mu(\varphi - \varphi_0)$ if and only if we only have two distinct eigenvalues. Hence $k = \varepsilon_1 = \varepsilon_2$ and the radii are given by

(a) $r_1^2 = \dfrac{p+1}{m+2}$ and $r_2^2 = \dfrac{m-p+1}{m+2}$,

(b) $r_1^2 = \dfrac{p+2}{m+2}$ and $r_2^2 = \dfrac{m-p}{m+2}$,

(c) $r_1^2 = \dfrac{p}{m+2}$ and $r_2^2 = \dfrac{m-p+2}{m+2}$.

Bibliography

[**1**] L. J. Alías, A. Ferrández and P. Lucas. 2-type surfaces in $\mathbb{S}^3_1$ and $\mathbb{H}^3_1$. *Tokyo J. Math.,* **17** (1994), 447–454.

[**2**] L. J. Alías, A. Ferrández and P. Lucas. Hypersurfaces in the non-flat Lorentzian space forms with a characteristic eigenvector field. *J. of Geometry,* **52** (1995), 10–24.

[**3**] M. Barros and O. J. Garay. A new characterization of the Clifford torus in $\mathbb{S}^3$ via the quadric representation. *Japanese J. Math.*, **20** (1994), 213–224.

[**4**] M. Barros and O. J. Garay. Spherical minimal surfaces with minimal quadric representation in some hyperquadric. *Tokyo J. Math.*, **17** (1994), 479–493.

[**5**] M. Barros and F. Urbano. Spectral geometry of minimal surfaces in the sphere. *Tôhoku Math. J.*, **39** (1987), 575–588.

[**6**] B. Y. Chen. Finite-type pseudo-Riemannian submanifolds. *Tamkang J. of Math.*, **17** (1986), 137–151.

[**7**] B. Y. Chen. Null 2-type surfaces in Euclidean space. In *Algebra, Analysis and Geometry*, pages 1–18. National Taiwan Univ., 1988.

[**8**] B. Y. Chen. Some open problems and conjectures on submanifolds of finite type. *Soochow J. Math.*, **17** (1991), 169–188.

[**9**] I. Dimitric. Spherical hypersurfaces with low type quadric representation. *Tokyo J. Math.*, **13** (1990), 469–492.

[**10]** A. Ferrández and P. Lucas. Classifying hypersurfaces in the Lorentz-Minkowski space with a characteristic eigenvector. *Tokyo J. Math.*, **15** (1992), 451–459.

[**11]** A. Ferrández and P. Lucas. On surfaces in the 3-dimensional Lorentz-Minkowski space. *Pacific J. Math.*, **152** (1992), 93–100.

[**12]** A. Ferrández, P. Lucas and M. A. Meroño. A quadric representation for pseudo-Riemannian product immersions. *Tsukuba J. Math.*, **20** (1996), 1–22.

[**13]** A. Ferrández, P. Lucas and M. A. Meroño. Classification of certain semi-Riemannian constant mean curvature surfaces. In F. Dillen et al., editor, *Geometry and Topology of Submanifolds, VII*, pages 132–135. World Scientific Co., 1994.

[**14]** A. Ferrández, P. Lucas and M. A. Meroño. Biharmonic products in the normal bundle. *Commentarii Mathematici Univ. Sancti Pauli*, **45** (1996), 147–158.

[**15]** M. A. Magid. Isometric immersions of Lorentz space with parallel second fundamental forms. *Tsukuba J. Math.*, **8** (1984), 31–54.

[**16]** A. Ros. Eigenvalues inequalities for minimal submanifolds and P-manifolds. *Math. Z.*, **187** (1984), 393–404.