# Pseudo-spherical and pseudo-hyperbolic submanifolds via its quadric representation II 

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## 1. Introduction

This paper is a natural continuation of [6] and both of them represent a whole. Here we present our main results. Actually, we prove that a hypersurface $M_{\nu}^{n}$ in $\bar{M}^{m+1}(k)$ satisfies $\widetilde{\Delta} \tilde{H}=$ $\lambda \tilde{H}+\mu\left(\varphi-\varphi_{0}\right)$ if and only if $M_{\nu}^{n}$ is either (i) minimal with shape operator having a double real eigenvalue; or (ii) a nonflat totally umbilical hypersurface; or (iii) a pseudo-Riemannian standard product with appropriate radii. As a consequence, in dealing with surfaces into de Sitter and anti de Sitter worlds, we have got nice characterizations, among others, for minimal $B$-scrolls and complex circles.

A characterization of submanifolds whose quadric representation satisfies $\widetilde{\Delta} \tilde{H}=\lambda \tilde{H}, \lambda$ being a real constant, is also given. Besides other conditions on shape operators, we find that only two values of $\lambda$ are permitted in order to that equation holds.

Throughout this paper references such as (I.xx) allude to equation (xx) of Part I.

## 2. A 2-type equation for hypersurfaces: Mean curvature

All examples exhibited in Section 3 of Part I are of constant mean curvature, so one can ask himself why we have not exhibited any other example without that property. The assumption of having constant mean curvature is not as restrictive as one could think, as the following result shows (compare with [2, Lemma 2.1] and [1, Section 2]).

Proposition 2.1 Let $x: M_{\nu}^{n} \rightarrow \bar{M}^{m+1}(k)$ be an isometric immersion satisfying $\widetilde{\Delta} \tilde{H}=\lambda \tilde{H}+$ $\mu\left(\varphi-\varphi_{0}\right)$, where $\lambda$ and $\mu$ are real constants and $\varphi_{0}$ a constant matrix. Then $M_{\nu}^{n}$ has constant mean curvature.

Proof. By using (I.14) we get

$$
\langle A(x), x\rangle=-2 \varepsilon k m \alpha^{2}-4(m+1)+2 k \lambda-\mu,
$$

where $A=-\mu \varphi_{0}$. Taking covariant derivative relative to a tangent vector field $Z$ we obtain

$$
2 k\langle\Delta \bar{H}, Z\rangle=-4 \varepsilon k m \alpha\langle\nabla \alpha, Z\rangle,
$$

where we have used that $A$ is a selfadjoint operator. From (I.10) the above equation yields

$$
S(\nabla \alpha)=-\frac{3}{2} \varepsilon m \alpha \nabla \alpha
$$

Let $\mathcal{U}$ be the open set of $M_{\nu}^{n}$ given by $\mathcal{U}=\left\{p \in M_{\nu}^{n}: \nabla \alpha^{2} \neq 0\right\}$ and suppose that $\mathcal{U} \neq \emptyset$. We first claim that $\langle\nabla \alpha, \nabla \alpha\rangle \neq 0$ in $\mathcal{U}$. Otherwise, from (I.12) and (1), we should have

$$
A(\nabla \alpha)=\left(3 \varepsilon m \alpha^{2}+\frac{2}{m}(2 k(m+1)-\lambda)\right) \nabla \alpha
$$

and therefore $\alpha$ is constant, which is a contradiction. So we can define $X=\frac{\nabla \alpha}{\|\nabla \alpha\|}$, where $\|\nabla \alpha\|=\sqrt{|\langle\nabla \alpha, \nabla \alpha\rangle|}$. Now, by using (I.12) and (1), we get

$$
\langle A(X), X\rangle=\delta\left(3 \varepsilon m \alpha^{2}+\frac{2}{m}(2 k(m+1)-\lambda)\right),
$$

$\delta$ being the causal character of $\nabla \alpha$. Taking covariant derivate in the above equation relative to $\nabla \alpha$, we obtain $\alpha\|\nabla \alpha\|^{2}=0$, which is again a contradiction.

## 3. Main theorem

The goal of this section is to prove the following classification result.
Theorem 3.1 Let $x: M_{\nu}^{n} \longrightarrow \bar{M}^{m+1}(k) \subset \mathbb{R}_{s}^{m+2}$ be an isometric immersion and $\varphi=f \circ x$ its quadric representation. Then $\widetilde{\Delta} \tilde{H}=\lambda \tilde{H}+\mu\left(\varphi-\varphi_{0}\right)$, where $\lambda, \mu \in \mathbb{R}$ and $\varphi_{0} \in S A(m+2, s)$, if and only if one of the following statements holds:
(1) $M_{\nu}^{n}$ is minimal in $\bar{M}^{m+1}(k)$ and $S^{2}=a I, a \in \mathbb{R}$;
(2) $M_{\nu}^{n}$ is a non flat totally umbilical hypersurface;
(3) $M_{\nu}^{n}$ is an open piece of a pseudo-Riemannian standard product $\mathcal{M}^{p}\left(k r_{1}\right) \times \mathcal{N}^{m-p}\left(k r_{2}\right)$ with radii
(a) $r_{1}^{2}=\frac{p+1}{m+2}$ and $r_{2}^{2}=\frac{m-p+1}{m+2}$,
(b) $r_{1}^{2}=\frac{p+2}{m+2}$ and $r_{2}^{2}=\frac{m-p}{m+2}$,
(c) $r_{1}^{2}=\frac{p}{m+2}$ and $r_{2}^{2}=\frac{m-p+2}{m+2}$.

Proof. Taking covariant derivate in equation (I.17) with respect to a tangent vector field $Z$ we get

$$
A(Z)=-k \alpha\left(\varepsilon \operatorname{tr}\left(S^{2}\right)+(3 m+4) k-\lambda\right) S(Z)-\left(2 \varepsilon m \alpha^{2}+4 k(m+1)-2 \lambda+k \mu\right) Z .
$$

Then from (I.15) we have

$$
\begin{align*}
& 4 \varepsilon S^{2}+m \alpha\left(\varepsilon k \operatorname{tr}\left(S^{2}\right)+3 m+8-k \lambda\right) S \\
& +\left(2 \varepsilon m^{2} \alpha^{2}+2(m+1)(2 k(m+1)-\lambda)+k m \mu\right) I=0 . \tag{2}
\end{align*}
$$

Notice that taking covariant derivative in $\langle A(N), N\rangle$ we find that $\operatorname{tr}\left(S^{2}\right)$ is constant. Then we have got a polynomial of degree two with constant coefficients which vanishes on $S$, so the minimal polynomial of $S$ is at most of degree two. If $M_{\nu}^{n}$ is totally umbilical, a straightforward computation shows that it must be non flat. So we can assume that $M_{\nu}^{n}$ is not totally umbilical and the minimal polynomial is of degree two.

By taking covariant derivative in (I.16) we get

$$
A(S(Z))=-\left(\frac{4 \varepsilon}{m} \operatorname{tr}\left(S^{2}\right)+2 \varepsilon m \alpha^{2}\right) S(Z)-\alpha\left(\operatorname{tr}\left(S^{2}\right)+(3 m+4) \varepsilon k-\varepsilon \lambda\right) Z
$$

and from (I.15) we obtain

$$
\begin{aligned}
& 4 \varepsilon S^{3}+4 m \alpha S^{2}+2\left(2 \varepsilon \operatorname{tr}\left(S^{2}\right)+\varepsilon m^{2} \alpha^{2}+2 k(m+1)-\lambda\right) S \\
& +m \alpha\left(\operatorname{tr}\left(S^{2}\right)+(3 m+4) \varepsilon k-\varepsilon \lambda\right) I=0
\end{aligned}
$$

From the above equation and (2) we find

$$
\begin{align*}
0= & m \alpha\left(\varepsilon k \operatorname{tr}\left(S^{2}\right)+3 m+4-k \lambda\right) S^{2} \\
& -\left(4 \varepsilon \operatorname{tr}\left(S^{2}\right)-2 m(2 k(m+1)-\lambda)-k m \mu\right) S \\
& -m \alpha\left(\operatorname{tr}\left(S^{2}\right)+(3 m+4) \varepsilon k-\varepsilon \lambda\right) I=0 . \tag{3}
\end{align*}
$$

If the mean curvature $\alpha$ vanishes, then from (2) we deduce that

$$
S^{2}=-\frac{1}{4} \varepsilon(2(m+1)(2 k(m+1)-\lambda)+k m \mu) I
$$

Then $M_{\nu}^{n}$ is one of the hypersurfaces exhibited in Example 3.1 in Part I. Hence, we can assume $\alpha \neq 0$.

From the hypothesis that $M_{\nu}^{n}$ is not totally umbilical, we have that $\varepsilon k \operatorname{tr}\left(S^{2}\right)+3 m+4-k \lambda \neq 0$. Then we use (2) and (3) to get

$$
2 \varepsilon m^{2} \alpha^{2}+2(m+1)(2 k(m+1)-\lambda)+k m \mu+4 k=0
$$

so that (2) writes down as follows

$$
4 \varepsilon S^{2}+m \alpha\left(\varepsilon k \operatorname{tr}\left(S^{2}\right)+3 m+8-k \lambda\right) S-4 k I=0
$$

Now, by computing the trace we obtain

$$
4 \varepsilon \operatorname{tr}\left(S^{2}\right)+\varepsilon m^{2} \alpha^{2}\left(\varepsilon k \operatorname{tr}\left(S^{2}\right)+3 m+8-k \lambda\right)-4 k m=0,
$$

and therefore equation (4) reduces to

$$
S^{2}+\frac{k m-\varepsilon \operatorname{tr}\left(S^{2}\right)}{m \alpha} S-\varepsilon k I=0
$$

If $S$ is diagonalizable, then $M_{\nu}^{n}$ is a pseudo-Riemannian standard product. Otherwise, the tangent space can be expressed as a direct sum of mutually orthogonal subspaces (hence non-degenerate) and $S$-invariant, and $S$ restricted to each subspace has a matrix of form either

$$
\text { I. }\left(\begin{array}{ccccc}
a & & & & \mathbf{0} \\
1 & a & & & \\
& \ddots & \ddots & & \\
& & 1 & a & \\
\mathbf{0} & & & 1 & a
\end{array}\right) \quad \text { or } \quad \text { II. }\left(\begin{array}{rr}
a & -b \\
b & a
\end{array}\right)
$$

In case $I$, we deduce that $a^{2}=1$ and $\varepsilon k=-1$, so the system of equations (I.15)-(I.17) writes as

$$
\begin{align*}
A(Z) & =\frac{4}{m} \varepsilon(m+2)\left(a S(Z)+\frac{2}{m} Z\right)  \tag{6}\\
A(N) & =\frac{4}{m} \varepsilon(m+2)\left(-\frac{m}{2} N+a x\right)  \tag{7}\\
A(x) & =\frac{4}{m} \varepsilon(m+2)\left(-a N+\frac{2}{m} x\right) . \tag{8}
\end{align*}
$$

By using (5) and (6) we find that

$$
A^{2}-8 \varepsilon \frac{(m+2)^{2}}{m^{2}} A+16 \frac{(m+2)^{4}}{m^{4}} I=0
$$

which we apply to the vector field $N-\frac{a m}{2} x$, and use (7) and (8), to get a contradiction.
In case II, we get $a^{2}+b^{2}=1$ and $\varepsilon k=-1$, so that the system of equations (I.15)-(I.17) reduces to

$$
\begin{align*}
A(Z) & =\frac{4}{m} \varepsilon(m+2) a S(Z)-4 \varepsilon\left(\alpha^{2}-\frac{m^{2}+2 m+4}{m^{2}}\right) Z  \tag{9}\\
A(N) & =2 \varepsilon\left(-(m+4) \alpha^{2}+2\right) N+\frac{4}{m} \varepsilon(m+2) a x  \tag{10}\\
A(x) & =-\frac{4}{m} \varepsilon(m+2) a N-4 \varepsilon\left(\alpha^{2}-\frac{m^{2}+2 m+4}{m^{2}}\right) x \tag{11}
\end{align*}
$$

Now from (5) and (9), it is easy to show that the eigenvalues of $A$ are not real. However, (10) and (11) implies that $A$ has at least a real eigenvalue. This finishes the proof.

If $M_{\nu}^{n}$ is Riemannian $(\nu=0)$ then $S$ is diagonalizable and so [3, Theorem 3] and [5, Theorem 3.1] can be deduced from our Theorem 3.1.

Corollary 3.2 Let $x: M^{n} \longrightarrow \bar{M}^{m+1}(k) \subset \mathbb{R}_{s}^{m+2}$ be an isometric immersion of a Riemannian hypersurface in $\bar{M}^{m+1}(k)$ and $\varphi=f \circ x$ its quadric representation. Then $\widetilde{\Delta} \tilde{H}=\lambda \tilde{H}+\mu\left(\varphi-\varphi_{0}\right)$ if and only if one of the following statements holds:
(1) $M_{\nu}^{n}$ is a totally umbilical hypersurface,
(2) $M_{\nu}^{n}$ is an open piece of a pseudo-Riemannian standard product $\mathcal{M}^{p}\left(k r_{1}\right) \times \mathcal{N}^{m-p}\left(k r_{2}\right)$ with radii
(a) $r_{1}^{2}=\frac{p+1}{m+2}$ and $r_{2}^{2}=\frac{m-p+1}{m+2}$,
(b) $r_{1}^{2}=\frac{p+2}{m+2}$ and $r_{2}^{2}=\frac{m-p}{m+2}$,
(c) $r_{1}^{2}=\frac{p}{m+2}$ and $r_{2}^{2}=\frac{m-p+2}{m+2}$.

Theorem 3.1 can be sharpened for surfaces as follows.
Corollary 3.3 Let $x: M_{\nu}^{2} \longrightarrow \mathbb{S}_{1}^{3}(1) \subset \mathbb{R}_{1}^{4}$ be an isometric immersion and $\varphi=f \circ x$ its quadric representation. Then $\widetilde{\Delta} \tilde{H}=\lambda \tilde{H}+\mu\left(\varphi-\varphi_{0}\right)$ if and only if one of the following statements holds: (1) $M_{\nu}^{2}$ is an open piece of a minimal $B$-scroll over a null curve;
(2) $M_{\nu}^{2}$ is a non flat totally umbilical surface;
(3) $M_{\nu}^{2}$ is an open piece of one of the following products:
$\mathbb{S}_{1}^{1}(\sqrt{2} / 2) \times \mathbb{S}^{1}(\sqrt{2} / 2), \mathbb{S}_{1}^{1}(\sqrt{3} / 2) \times \mathbb{S}^{1}(1 / 2), \mathbb{S}_{1}^{1}(1 / 2) \times \mathbb{S}^{1}(\sqrt{3} / 2)$.
Corollary 3.4 Let $x: M_{\nu}^{2} \longrightarrow \mathbb{H}_{1}^{3}(-1) \subset \mathbb{R}_{2}^{4}$ be an isometric immersion and $\varphi=f \circ x$ its quadric representation. Then $\widetilde{\Delta} \tilde{H}=\lambda \tilde{H}+\mu\left(\varphi-\varphi_{0}\right)$ if and only if one of the following statements holds:
(1) $M_{\nu}^{2}$ is an open piece of a minimal $B$-scroll over a null curve;
(2) $M_{\nu}^{2}$ is an open piece of the minimal complex circle;
(3) $M_{\nu}^{2}$ is a non flat totally umbilical surface;
(4) $M_{\nu}^{2}$ is an open piece of one of the following products:
$\mathbb{H}^{1}(-\sqrt{2} / 2) \times \mathbb{H}^{1}(-\sqrt{2} / 2), \mathbb{H}^{1}(-\sqrt{3} / 2) \times \mathbb{H}^{1}(-1 / 2)$.

## 4. Submanifolds satisfying $\widetilde{\Delta} \tilde{H}=\lambda \tilde{H}$

In this section we are going to characterize pseudo-Riemannian submanifolds satisfying the equation $\widetilde{\Delta} \tilde{H}=\lambda \tilde{H}$, where $\lambda$ is a real constant.

We know that $\widetilde{\Delta} \tilde{H}=\lambda \tilde{H}$ if and only if $\widetilde{\Delta} \tilde{H}(Z)=\lambda \tilde{H}(Z), \widetilde{\Delta} \tilde{H}(\xi)=\lambda \tilde{H}(\xi)$ and $\widetilde{\Delta} \tilde{H}(x)=$ $\lambda \tilde{H}(x)$, which is equivalent, by (I.7)-(I.9), to the following system of equations

$$
\begin{align*}
& 2\left(n S_{H}+\sum_{r=1}^{m-n+1} \delta_{r} S_{r}^{2}\right)=(\lambda-2 k(n+1)) I,  \tag{12}\\
& \nabla^{\perp} H=0,  \tag{13}\\
& 2 \sum_{r=1}^{m-n+1} \delta_{r} \operatorname{tr}\left(S_{\xi}{ }^{\circ} S_{r}\right) \xi_{r}+n^{2}\langle H, \xi\rangle H=0,  \tag{14}\\
& \langle\Delta \bar{H}, \xi\rangle=(\lambda-2 k(n+2))\langle H, \xi\rangle,  \tag{15}\\
& \langle\Delta \bar{H}, x\rangle=-(\lambda-k(n+2)) . \tag{16}
\end{align*}
$$

We first observe that equation (14) can be rewritten, by using that $\operatorname{tr}\left(S_{\xi}\right)=n\langle H, \xi\rangle$, as

$$
\begin{equation*}
2 \operatorname{tr}\left(S_{\xi} S_{\eta}\right)+\operatorname{tr}\left(S_{\xi}\right) \operatorname{tr}\left(S_{\eta}\right)=0 . \tag{17}
\end{equation*}
$$

By taking traces in (12) and using (17) we deduce that

$$
\begin{align*}
(\lambda-2 k(n+1)) n & =2 n^{2}\langle H, H\rangle-\sum_{r=1}^{m-n+1} \delta_{r} \operatorname{tr}\left(S_{r}\right)^{2} \\
& =n^{2}\langle H, H\rangle . \tag{18}
\end{align*}
$$

Bearing in mind (13) and (17), the equation of $\Delta \bar{H}$ given in [4] writes as follows

$$
\begin{align*}
\Delta \bar{H} & =\sum_{r=1}^{m-n+1} \delta_{r} \operatorname{tr}\left(S_{\bar{H}^{\circ}} S_{r}\right) \xi_{r}+k \operatorname{tr}\left(S_{\bar{H}^{\circ}} S_{x}\right) x \\
& =\left(k n-\frac{n^{2}}{2}\langle H, H\rangle\right) H-k n(k+\langle H, H\rangle) x . \tag{19}
\end{align*}
$$

On the other hand, from (13), (15) and (16) we have

$$
\begin{align*}
\Delta \bar{H} & =\sum_{r=1}^{m-n+1} \delta_{r}\left\langle\Delta \bar{H}, \xi_{r}\right\rangle \xi_{r}+k\langle\Delta \bar{H}, x\rangle x \\
& =(\lambda-2 k(n+2)) H-k(\lambda-k(n+2)) x \tag{20}
\end{align*}
$$

Therefore, from (19) and (20), we find

$$
\begin{align*}
\lambda & =2 k(n+1)+n\langle H, H\rangle  \tag{21}\\
\lambda H & =\left(3 k n+4 k-\frac{n^{2}}{2}\langle H, H\rangle\right) H \tag{22}
\end{align*}
$$

If $M_{\nu}^{n}$ is minimal then $\lambda=2 k(n+1)$ and equations (12)-(16) reduce to $\sum_{r=1}^{m-n+1} \delta_{r} S_{r}^{2}=0$ and $\operatorname{tr}\left(S_{\xi} S_{\eta}\right)=0$ for $\xi, \eta \in \mathfrak{X}^{\perp}\left(M_{\nu}^{n}\right)$. Otherwise, from (21) and (22) we get $\lambda=2 k(n+2)$ and then (12) writes as $n S_{H}+\sum_{r=1}^{m-n+1} \delta_{r} S_{r}^{2}=k I$. Observe that $\operatorname{tr}\left(S_{\xi^{\circ}} S_{\eta}\right)=0$ is equivalent to $\mathcal{S}=0$, $\mathcal{S}: \mathfrak{X}^{\perp}\left(M_{\nu}^{n}\right) \rightarrow \mathfrak{X}^{\perp}\left(M_{\nu}^{n}\right)$ being the Simons operator (see [7]). Now let $T$ be the operator defined by $T(\xi, \eta)=S_{\xi^{\circ}} S_{\eta}$, where $\xi, \eta \in \mathfrak{X}^{\perp}\left(M_{\nu}^{n}\right)$. So we have proved the following result.

Theorem 4.1 Let $x: M_{\nu}^{n} \longrightarrow \bar{M}^{m+1}(k) \subset \mathbb{R}_{s}^{m+2}$ be an isometric immersion. The mean curvature vector field $\tilde{H}$ satisfies the equation $\widetilde{\Delta} \tilde{H}=\lambda \tilde{H}, \lambda \in \mathbb{R}$, if and only if one of the following statements holds:
(1) $\lambda=2 k(n+1), H=0, \operatorname{tr}(T)=0$ and $\mathcal{S}=0$;
(2) $\lambda=2 k(n+2), \nabla^{\perp} H=0, \operatorname{tr}(T)=k I-n S_{H}$ and $\mathcal{S}=-\left(n^{2} / 2\right) H H^{b}, H^{b}$ being the 1-form metrically equivalent to $H$.

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