On the Gauss map of B-scrolls

Luis J. Alías, Angel Ferrández, Pascual Lucas and Miguel Angel Meroño
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Abstract

B-scrolls over null curves in the 3-dimensional Lorentz-Minkowski space $\mathbb{L}^3$ are characterized as the only ruled surfaces with null rulings whose Gauss map satisfies the condition $\Delta G = \Lambda G$, Λ being and endomorphism of $\mathbb{L}^3$. This note completes the classification of such surfaces given by S.M. Choi in Tsukuba J. Math. **19** (1995), 285–304.

1. Introduction

Let M be a connected surface in Euclidean 3-space $\mathbb{R}^3$ and let $G : M \to \mathbb{S}^2 \subset \mathbb{R}^3$ be its Gauss map. It is well known (see [**9**]) that M has constant mean curvature if and only if $\Delta G = ||dG||^2 G$, Δ being the Laplace operator on M corresponding to the induced metric on M from $\mathbb{R}^3$. As a special case one can consider Euclidean surfaces whose Gauss map is an eigenfunction of the Laplacian, i.e., $\Delta G = \lambda G$, $\lambda \in \mathbb{R}$. In [**3**], C. Baikoussis and D.E. Blair asked for ruled surfaces in $\mathbb{R}^3$ whose Gauss map satisfies $\Delta G = \Lambda G$, where Λ stands for an endomorphism of $\mathbb{R}^3$. They showed that the only ones are planes and circular cylinders. Recently, S. M. Choi in [**5**], investigates the Lorentz version of the above result and she essentially obtains the same result. Namely, the only ruled surfaces in $\mathbb{L}^3$ whose Gauss map satisfies $\Delta G = \Lambda G$ are the planes $\mathbb{R}^2$ and $\mathbb{L}^2$, as well as the cylinders $\mathbb{S}^1_1 \times \mathbb{R}^1$, $\mathbb{R}^1_1 \times \mathbb{S}^1$ and $\mathbb{H}^1 \times \mathbb{R}^1$.

It should be pointed out that all surfaces obtained above have diagonalizable shape operator. However, it is well known that a self-adjoint linear operator on a 2-dimensional Lorentz vector space has a matrix of exactly three types, two of them being non-diagonalizable. This makes a chief difference with regard to the Riemannian submanifolds that has been greatly exploited (see, for example, [**1**], [**2**] and [**7**]). To illustrate the current situation, we bring here the famous example of L. K. Graves (see [**8**]), the so called *B-scroll*. This is a surface which can be parametrized as a "ruled surface" in $\mathbb{L}^3$ with null directrix curve and null rulings, i.e., $X(s,t) = x(s) + tB(s)$, $x(s)$ being a null curve and $B(s)$ a null vector field along $x(s)$ satisfying $\langle x', B\rangle = -1$.

The main purpose of this short note is to complete Choi's classification of ruled surfaces in $\mathbb{L}^3$ whose Gauss map satisfies the condition $\Delta G = \Lambda G$. Actually, we will show that B-scrolls over null curves are the only ruled surfaces in $\mathbb{L}^3$ with null rulings satisfying the above condition.

We would like to thank to the referee for bringing to our attention the preprint [**6**], where some related topics are considered.

2. Setup

Let $x : I \subset \mathbb{R} \to \mathbb{L}^3$ be a regular curve in $\mathbb{L}^3$ and $B : I \subset \mathbb{R} \to \mathbb{L}^3$ a vector field along x. Consider the ruled surface parametrized by $X(s,t) = x(s) + tB(s)$. Let us write down, as usually,

$X_s := \partial X/\partial s = x' + tB'$ and $X_t := \partial X/\partial t = B$. Observe that, at $t = 0$, $X_s(s,0) = x'(s)$ and $X_t(s,0) = B(s)$. Then $X(s,t)$ is a regular surface in $\mathbb{L}^3$ provided that the plane Π =span $\{x', B\}$ is non degenerate in $\mathbb{L}^3$. In fact, the matrix of the metric of $X(s,t)$ is given by

$$\mathbf{g}(s,t) = \begin{pmatrix} \langle x', x'\rangle + 2t\langle x', B'\rangle + t^2\langle B', B'\rangle & \langle x', B\rangle + t\langle B', B\rangle \\ \langle x', B\rangle + t\langle B', B\rangle & \langle B, B\rangle \end{pmatrix},$$

so that when the plane Π is spacelike (repectively, timelike) $X(s,t)$ parametrizes a spacelike surface (repectively, timelike surface) on the domain

$$\{(s,t) \in I \times \mathbb{R} : \det \mathbf{g}(s,t) > 0 \quad (\text{respectively}, \det \mathbf{g}(s,t) < 0)\}.$$

According to the causal character of x' and B, there are four possibilities:

(1) x' and B are non-null and linearly independent.
(2) x' is null and B is non-null with $\langle x', B\rangle \neq 0$.
(3) x' is non-null and B is null with $\langle x', B\rangle \neq 0$.
(4) x' and B are null with $\langle x', B\rangle \neq 0$.

Let us first see that, with an appropiate change of the curve x, cases (2) and (3) can be locally reduced to (1) and (4), respectively. Let $X(s,t)$ be in case (2). Reparametrizing the null curve x and normalizing the rulings B if necessary, we may assume that

$$\langle B, B\rangle = \varepsilon = \pm 1, \quad \text{and} \quad \langle x', B\rangle = -1,$$

so that

$$g(s,t) = \det \mathbf{g}(s,t) = \varepsilon(2t\langle x', B'\rangle + t^2\langle B', B'\rangle) - 1 < 0.$$

We are looking for a curve $\gamma(s) = x(s) + t(s)B(s)$ in the surface with $\langle \gamma', \gamma'\rangle = \varepsilon$ and such that γ' and B are linearly independent. Writing $\gamma' = x' + t'B + tB'$, the condition $\langle \gamma', \gamma'\rangle = \varepsilon$ is equivalent to the following differential equation for $t = t(s)$

$$(t')^2 - 2\varepsilon t' + g(s,t) = 0.$$

From (1) the discriminant of (2) is positive and we can locally integrate (2) to obtain t. Besides, γ' and B are linearly independent because $\langle \gamma', \gamma'\rangle = \langle B, B\rangle = \varepsilon$ and $\langle \gamma', B\rangle = -1 + t'\varepsilon \neq \pm\varepsilon$ due to (2). This shows that $X(s,t)$ can be reparametrized as in case (1) taking γ as the directrix curve. On the other hand, if $X(s,t)$ is in case (3), reparametrizing the null curve x and normalizing the rulings B if necessary, we may assume that

$$\langle x', x'\rangle = \varepsilon = \pm 1, \quad \text{and} \quad \langle x', B\rangle = -1.$$

We are now looking for a curve $\gamma(s) = x(s) + t(s)B(s)$ in the surface with $\langle \gamma', \gamma'\rangle = 0$ and $\langle \gamma', B\rangle \neq 0$ Writing $\gamma' = x' + t'B + tB'$, the condition $\langle \gamma', \gamma'\rangle = 0$ now becomes

$$2t' = \varepsilon + 2t\langle x', B'\rangle + t^2\langle B', B'\rangle.$$

Equation (3) can be locally integrated to obtain t. Moroever, $\langle \gamma', B\rangle = \langle x', B\rangle \neq 0$. Thus, using the curve γ as the directrix, $X(s,t)$ can be reparametrized as in case (4).

Since case (1) has been discussed in [**5**], we will pay attention to the latter one which we aim to characterize in terms of the Laplacian of its Gauss map. Therefore, let M be a ruled surface in $\mathbb{L}^3$ parametrized by $X(s,t) = x(s) + tB(s)$, where the directrix $x(s)$, as well as the rulings $B(s)$, are null. Furthermore, and without loss of generality, we may assume $\langle x', B\rangle = -1$. First of all, we will do a detailed study of this kind of surfaces.

The matrix of the metric on M writes, with respect to coordinates (s,t), as follows

$$\begin{pmatrix} 2t\langle x', B'\rangle + t^2\langle B', B'\rangle & -1 \\ -1 & 0 \end{pmatrix}.$$

In terms of local coordinates $(y_1,\ldots,y_n)$, the Laplacian Δ of a manifold is defined by (see [**4**, p. 100])

$$\Delta = -\frac{1}{\sqrt{|g|}}\sum_{i,j}\frac{\partial}{\partial y_i}\left(gg^{ij}\frac{\partial}{\partial y_j}\right),$$

where $g = det(g_{ij})$ and (g_{ij}) denotes the components of the metric with respect to $(y_1,\ldots,y_n)$. Then the Laplacian on the surface M is nothing but

$$\Delta = -2\frac{\partial^2}{\partial s\partial t} - 2\{\langle x', B'\rangle + t\langle B', B'\rangle\}\frac{\partial}{\partial t} - \{2t\langle x', B'\rangle + t^2\langle B', B'\rangle\}\frac{\partial^2}{\partial t^2}.$$

Now we will recall the notion of cross product in $\mathbb{L}^3$. There is a natural orientation in $\mathbb{L}^3$ defined as follows: an ordered basis $\{X,Y,Z\}$ in $\mathbb{L}^3$ is positively oriented if det$[XYZ] > 0$, where $[XYZ]$ is the matrix with X, Y, Z as row vectors. Now let ω be the volume element on $\mathbb{L}^3$ defined by $\omega(X,Y,Z) = \det[XYZ]$. Then given $X, Y \in \mathbb{L}^3$, the cross product $X\times Y$ is the unique vector in $\mathbb{L}^3$ such that $\langle X\times Y, Z\rangle = \omega(X,Y,Z)$, for any $Z\in\mathbb{L}^3$.

Then the Gauss map can be directly obtained from $X_s \times X_t$ getting

$$G(s,t) = x'(s)\times B(s) + tB'(s)\times B(s).$$

By putting $C = x'\times B$, then $\{x', B, C\}$ is a frame field along x of $\mathbb{L}^3$. In this frame, we easily see that $B'\times B = -fB$, f being the function defined by $f = \langle x', B'\times B\rangle$. Thus

$$G(s,t) = -tf(s)B(s) + C(s).$$

Also, and for later use, we find out that

$$B' = -\langle x', B'\rangle B - fC$$

and

$$C' = -fx' - \langle x', x''\times B\rangle B.$$

As for the shape operator S we have that

$$G_t := \frac{\partial G}{\partial t} = B'\times B = -fB = -fX_t$$

and

$$G_s := \frac{\partial G}{\partial s} = -(\langle x', x''\times B\rangle + tf')X_t - fX_s.$$

So S writes down as

$$\begin{pmatrix} f & 0 \\ tf' + \langle x', x''\times B\rangle & f \end{pmatrix}.$$

A straightforward computation yields

$$\Delta G = 2\{f' + tf\langle B', B'\rangle\}B - 2f^2C.$$

We now present a very typical example.

Example. Let $x(s)$ be a null curve in $\mathbb{L}^3$ with Cartan frame $\{A, B, C\}$, i.e., $A,\ B,\ C$ are vector fields along x in $\mathbb{L}^3$ satisfying the following conditions:

$$\langle A, A\rangle = \langle B, B\rangle = 0, \quad \langle A, B\rangle = -1,$$

$$\langle A, C\rangle = \langle B, C\rangle = 0, \quad \langle C, C\rangle = 1,$$

and

$$\begin{array}{rcl} x' & = & A, \\ C' & = & -aA - \kappa(s)B, \end{array}$$

a being a constant and $\kappa(s)$ a function vanishing nowhere. Then the map

$$\begin{array}{rcl} X : \mathbb{L}^2 & \to & \mathbb{L}^3 \\ (s,t) & \to & x(s) + tB(s) \end{array}$$

defines a Lorentz surface M in $\mathbb{L}^3$ that L.K. Graves [**8**] called a *B-scroll*. It is not difficult to see that a unit normal vector field is given by

$$G(s,t) = -atB(s) + C(s),$$

and the shape operator writes down, relative to the usual frame $\left\{\dfrac{\partial X}{\partial s}, \dfrac{\partial X}{\partial t}\right\}$, as

$$S = \begin{pmatrix} a & 0 \\ k(s) & a \end{pmatrix}.$$

Thus the B-scroll has non-diagonalizable shape operator with minimal polynomial $P_S(u) = (u - a)^2$. It has constant mean curvature $\alpha = a$ and constant Gaussian curvature $K = a^2$ and satisfies $\Delta G = \lambda G$, where $\lambda = 2a^2$.

3. Main results

It seems natural to state the following problem: *is a B-scroll the only ruled surface in $\mathbb{L}^3$ with null rulings satisfying the equation $\Delta G = \Lambda G$?*

Then our major result states as follows.

Theorem 3.1 *B-scrolls over null curves are the only ruled surfaces in $\mathbb{L}^3$ with null rulings satisfying the equation $\Delta G = \Lambda G$.*

From here and Choi's result we have got the complete classification of ruled surfaces in the 3-dimensional Lorentz-Minkowski space whose Gauss map satisfies $\Delta G = \Lambda G$.

Corollary 3.2 *A ruled surface M in $\mathbb{L}^3$ satisfies the equation $\Delta G = \Lambda G$ if and only if M is one of the following surfaces:*

(1) $\mathbb{R}^2$, $\mathbb{L}^2$ *and the cylinders* $\mathbb{S}^1_1 \times \mathbb{R}^1$, $\mathbb{R}^1_1 \times \mathbb{S}^1$ *and* $\mathbb{H}^1 \times \mathbb{R}^1$*;*

(2) *a B-scroll over a null curve.*

Proof of the theorem. Suppose that the Gauss map of M satisfies the equation $\Delta G = \Lambda G$. From Choi's result we may suppose that M has null rulings, so we only have to study the case (4). We are going to show that the function $f = \langle x', B' \times B \rangle$ is constant or, equivalently, that the open set $\mathcal{U} = \{s \in I : f(s)f'(s) \neq 0\}$ is empty. Otherwise, for $s \in \mathcal{U}$, differentiating with respect to t in $\Delta G = \Lambda G$, we have

$$2f\langle B', B' \rangle B = -f\Lambda B,$$

where we have used equations (4), (7) and (9). By (5) we obtain $\langle B', B' \rangle = f^2$, so that from (10) we see that $-2f^2$ is an eigenvalue of Λ, unless $f = 0$. Then f is a constant function, which is a contradiction that finishes the proof.

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