A new viewpoint on geometry of a lightlike hypersurface in a semi-Euclidean space

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Abstract

We first show that the geometries of a hypersurface seen either as a lightlike hypersurface or as a Riemannian one are closely related. Furthermore, the Bernstein theorem for lightlike hypersurfaces is proved.

1. Introduction

As it is well known, the main difference between the geometry of submanifolds in Riemannian manifolds and in semi-Riemannian manifolds is that in the latter case the induced metric tensor field by the semi-Riemannian metric on the ambient space is not necessarily non-degenerate. If the induced metric tensor field is degenerate the classical theory of Riemannian submanifolds fails since the normal bundle and the tangent bundle of the submanifold have a non-zero intersection.

The main purpose of the present paper is to show that the geometry of a lightlike (degenerate, null) hypersurface M in a semi-Euclidean space $\mathbb{R}^{m+2}_{\nu}$ can be investigated by using the geometry of M as a Riemannian hypersurface in a Euclidean space $\mathbb{R}^{m+2}_{0}$. In the first section we present the main tools in studying the geometry of a lightlike hypersurface: the screen distribution and the lightlike transversal vector bundle (see [**2**] for details). Then, in Section 2, we show that the canonical lightlike transversal vector bundle of a lightlike immersion is just the normal bundle of a Riemannian immersion. This enables us, in Section 3, to prove the Bernstein theorem for lightlike hypersurfaces of Lorentz spaces.

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2. Preliminaries

Let $(\widetilde{M}, \widetilde{g})$ be a semi-Riemannian manifold (see [**7**]), where $\widetilde{g}$ is a semi-Riemannian metric of index $0 < \nu < \dim(\widetilde{M})$. Consider a hypersurface M of $\widetilde{M}$ and denote by g the induced tensor field by $\widetilde{g}$ on M. Then we say that M is a *lightlike (degenerate, null) hypersurface* if rank $g = \dim(M) - 1$. In order to get another characterization for a lightlike hypersurface we consider

$$T_xM^{\perp} = \{V_x \in T_x\widetilde{M} \ : \ \widetilde{g}(V_x, Y_x) = 0 \ , \ \ \forall Y_x \in T_xM\}$$

and

$$TM^\perp = \bigcup_{x\in M} T_xM^\perp.$$

Then it is easy to see that M is a lightlike hypersurface if and only if $TM^\perp$ is a distribution on M.

Next we consider a *screen distribution* STM on M, which is a complementary non-degenerate vector bundle to $TM^\perp$ in TM. As $TM^\perp$ lies in the tangent bundle, the following theorem has an important role in studying the geometry of a lightlike hypersurface.

Theorem 2.1 (Bejancu [**2**]) *Let (M, STM) be a lightlike hypersurface of $(\widetilde{M}, \widetilde{g})$. Then there exists a unique vector bundle $tr(TM)$ of rank 1 over M, such that for any non-zero section ξ of $TM^\perp$ on a coordinate neighborhhod $U \subset M$, there exists a unique section N of $tr(TM)$ on U satisfying*

$$\widetilde{g}(N, \xi) = 1$$

and

$$\widetilde{g}(N, N) = \widetilde{g}(N, W) = 0, \quad \forall W \in \Gamma(STM_{|U}).$$

By this theorem we may write down the decomposition

$$T\widetilde{M}_{|M} = TM \oplus tr(TM),$$

which enables us to call $tr(TM)$ the *lightlike transversal vector bundle* of M with respect to the screen distribution STM.

Here and in the sequel we denote by $\Gamma(E)$ the $\mathcal{F}(M)$-module of smooth sections of a vector bundle E over M, $\mathcal{F}(M)$ being the algebra of smooth functions on M. Also by $\perp$ and $\oplus$ we denote the orthogonal and non-orthogonal direct sum of two vector bundles.

In order to get an expression for N, we consider the orthogonal decomposition

$$T\widetilde{M}_{|M} = STM \perp STM^\perp,$$

where $STM^\perp$ is the orthogonal complementary vector bundle to STM in $T\widetilde{M}_{|M}$. As $TM^\perp$ is a vector subbundle of $STM^\perp$ we take a complementary vector bundle $\mathcal{V}$ of $TM^\perp$ in $STM^\perp$ and a non-zero $V \in \Gamma(\mathcal{V}_{|U})$. Then $\widetilde{g}(V, \xi) \neq 0$ on U, otherwise $STM^\perp$ would be degenerate at a point of U. Finally, the vector field N from Theorem 2.1 is given by

$$N = \frac{1}{\widetilde{g}(V,\xi)} \left\{ V - \frac{\widetilde{g}(V,V)}{2\widetilde{g}(V,\xi)}\xi \right\}.$$

The whole study of geometry of the lightlike hypersurface M is based on both the screen distribution and the lightlike transversal vector bundle. In case the ambient space is a semi-Euclidean space it is constructed in [**1**] a canonical screen distribution that induces a canonical lightlike transversal vector bundle. These vector bundles will play an important role in the rest of the paper.

3. The geometry of a lightlike hypersurface in $\mathbb{R}^{m+2}_\nu$ via its geometry in $\mathbb{R}^{m+2}_0$

It is the purpose of this section to show that the canonical lightlike transversal vector bundle of a lightlike hypersurface M in $\mathbb{R}^{m+2}_\nu$ is just the usual normal bundle of M as Riemannian hypersurface of the Euclidean space $\mathbb{R}^{m+2}_0$. As a consequence, we derive the interrelations between the geometric objects induced by the lightlike immersion and the Riemannian immersion.

Let $\mathbb{R}^{m+2}_{\nu} = (\mathbb{R}^{m+2}, \widetilde{g}_{\mathrm{SE}})$ and $\mathbb{R}^{m+2}_{0} = (\mathbb{R}^{m+2}, \widetilde{g}_{\mathrm{E}})$ be the $(m+2)$-dimensional semi-Euclidean space and Euclidean space, where $\widetilde{g}_{\mathrm{SE}}$ and $\widetilde{g}_{\mathrm{E}}$ stand for the semi-Euclidean metric of index $0 < \nu < m+2$ and the Euclidean metric given by

$$\widetilde{g}_{\mathrm{SE}}(x, y) = -\sum_{i=0}^{\nu-1} x^i y^i + \sum_{a=\nu}^{m+1} x^a y^a,$$

and

$$\widetilde{g}_{\mathrm{E}}(x, y) = \sum_{A=0}^{m+1} x^A y^A.$$

Consider a hypersurface M of $\mathbb{R}^{m+2}$ given by the quation

$$F(x^o, \cdots, x^{m+1}) = 0,$$

where F is smooth on an open set $\Omega \subset \mathbb{R}^{m+2}$ and $\mathrm{rank}[F'_{x^o}, \cdots, F'_{x^{m+1}}] = 1$ on M. Then the normal bundle of M with respect to $\widetilde{g}_{\mathrm{SE}}$ is spanned by

$$\xi = \widetilde{\nabla}_{\mathrm{SE}} F = -\sum_{i=0}^{\nu-1} F'_{x^i} \frac{\partial}{\partial x^i} + \sum_{a=\nu}^{m+1} F'_{x^a} \frac{\partial}{\partial x^a} \ ,$$

where, as usual, $\nabla_{\mathrm{SE}} F$ stands for the gradient of F with respect to $\widetilde{g}_{\mathrm{SE}}$.

Thus M is a lightlike hypersurface if and only if ξ is a null vector field with respect to $\widetilde{g}_{\mathrm{SE}}$. Hence we may state the following

Theorem 3.1 *M is a lightlike hypersurface if and only if F is a solution of the partial differential equation*

$$\sum_{i=0}^{\nu-1} (F'_{x^i})^2 = \sum_{a=\nu}^{m+1} (F'_{x^a})^2.$$

In order to obtain both the canonical transversal vector bundle and the canonical screen distribution, we consider along M the vector field

$$V = \sum_{i=0}^{\nu-1} F'_{x^i} \frac{\partial}{\partial x^i},$$

and note that V is nowhere tangent to M since

$$\widetilde{g}_{\mathrm{SE}}(V, \xi) = \sum_{i=0}^{\nu-1} (F'_{x^i})^2 \neq 0$$

on M. Then applying (1) we obtain that the canonical lightlike transversal vector bundle $tr(TM)$ is spanned by

$$N = \frac{1}{2} \left(\sum_{i=0}^{\nu-1} (F'_{x^i})^2 \right)^{-1} \sum_{A=0}^{m+1} F'_{x^A} \frac{\partial}{\partial x^A} \ .$$

Next we denote by N_0 the unit normal vector field on M with respect to the Euclidean metric $\widetilde{g}_{\mathrm{E}}$. Then due to (4) and (5) we deduce

$$N = \varepsilon(x) N_0, \qquad \varepsilon(x) = \frac{1}{\sqrt{2}} \left(\sum_{i=0}^{\nu-1} (F'_{x^i})^2 \right)^{-\frac{1}{2}} .$$

This enables us to state

Proposition 3.2 *The canonical lightlike transversal vector bundle of the lightlike hypersurface M in $\mathbb{R}^{m+2}_{\nu}$ is just the normal vector bundle of M as Riemannian hypersurface of $\mathbb{R}^{m+2}_{0}$.*

Taking into account that the canonical screen distribution STM is complementary orthogonal to $span\{N, \xi\}$, (see [**1**], [**2**]) we get that in case $\nu > 1$, STM is locally spanned in a coordinate neighborhood $U \subset M$ by

$$W_i = F'_{x^0} \frac{\partial}{\partial x^i} - F'_{x^i} \frac{\partial}{\partial x^0}, \qquad i \in \{1, \cdots, \nu - 1\},$$

$$W_a = F'_{x^{m+1}} \frac{\partial}{\partial x^a} - F'_{x^a} \frac{\partial}{\partial x^{m+1}}, \qquad a \in \{\nu, \cdots, m\},$$

provided that $F'_{x^0} \neq 0$ and $F'_{x^{m+1}} \neq 0$ on U.

In general, STM is not integrable. Indeed, the canonical screen distribution on the lightlike hypersurface M in $\mathbb{R}^4_2$ defined by

$$x^0 + \frac{1}{2}(x^1 + x^2)^2 - x^3 = 0$$

is not integrable (see [**2**]). However, we shall prove that STM is the orthogonal sum of two integrable distributions. To this end we consider another coordinate neighborhood $U^* \subset M$ such that $U \cap U^* \neq \emptyset$. Then STM is spanned on U^* by vector fields $\{W_i^*, W_a^*\}$, constructed as in (7) and (8) provided one of the partial derivatives from each group $\{F'_{x^i}\}$ and $\{F'_{x^a}\}$ is non-zero on U^*. By direct computations we obtain $span\{W_i^*\} = span\{W_i\}$ and $span\{W_a^*\} = span\{W_a\}$ on $U \cap U^*$. Hence we obtain two complementary distributions STM^- and STM^+ locally spanned on U by $\{W_i\}$ and $\{W_a\}$, respectively. Also we note that STM^- and STM^+ are timelike and spacelike orthogonal vector subbundles of STM with respect to the semi-Euclidean metric $\widetilde{g}_{\mathrm{SE}}$, respectively. Moreover, we prove

Proposition 3.3 *Both distributions STM^- and STM^+ on a lightlike hypersurface M in $\mathbb{R}^{m+2}_{\nu}$ are integrable.*

Proof. By using (7) we obtain

$$F'_{x^j} W_i - F'_{x^i} W_j = F'_{x^0} \left(F'_{x^j} \frac{\partial}{\partial x^i} - F'_{x^i} \frac{\partial}{\partial x^j} \right).$$

Then by direct calculations using again (7) and (9) we deduce that

$$[W_i, W_j] = \left(F''_{x^0 x^0} F'_{x^j} (F'_{x^0})^{-1} - F''_{x^0 x^j} \right) W_i + \left(F''_{x^0 x^i} - F''_{x^0 x^0} F'_{x^i} (F'_{x^0})^{-1} \right) W_j,$$

which proves that STM^- is integrable. In a similar way it follows that STM^+ is integrable too.

In particular, if the ambient space is the Lorentz space $\mathbb{R}^{m+2}_1$ we have $STM = STM^+$ and from Proposition 3.3 we deduce that

Corollary 3.4 ([**1**]) *Let M be a lightlike hypersurface of a Lorentz space $\mathbb{R}^{m+2}_1$. Then the canonical screen distribution STM is integrable.*

Due to Proposition 3.2 we shall write down some useful relations between the geometric objects induced by both the lightlike and Riemannian immersion of M in $\mathbb{R}^{m+2}$. Let $\widetilde{\nabla}$ be the Levi-Civita connection on $\mathbb{R}^{m+2}$ with respect to both metrics $\widetilde{g}_{\rm SE}$ and $\widetilde{g}_{\rm E}$. Then taking into account (6) we obtain that the Gauss and Weingarten equations for the lightlike immersion of M in $\mathbb{R}^{m+2}_\nu$ are given by (see [**2**])

$$\widetilde{\nabla}_X Y = \nabla_X Y + B(X,Y)N = \nabla_X Y + \varepsilon B(X,Y)N_0,\ \ \forall X,Y \in \Gamma(TM)$$

and

$$\widetilde{\nabla}_X N = -A_N X + \tau(X)N = -A_N X + \varepsilon\tau(X)N_0,\ \ \forall X \in \Gamma(TM),$$

respectively. Here, as usual, we denote by ∇, B and A_N the induced linear connection, the second fundamental form and the shape operator of the lightlike immersion. The lightlike transversal 1-form τ is specific to the lightlike immersion. By using (10), (11) and (6) we get

$$B_0(X,Y) = \varepsilon B(X,Y), \qquad \forall X,Y \in \Gamma(TM)$$

and

$$A_N X = \varepsilon A_{N_0} X, \qquad \tau(X) = X(\log\varepsilon), \qquad \forall X \in \Gamma(TM),$$

where B_0 and A_{N_0} are the second fundamental form and the shape operator of the Riemannian immersion of M in $\mathbb{R}^{m+2}_0$. Finally, we note that both immersions of M in $\mathbb{R}^{m+2}_\nu$ and in $\mathbb{R}^{m+2}_0$ induce the same linear connection ∇ on M, namely, the Levi-Civita connection with respect to the Riemannian metric induced by $\widetilde{g}_{\rm E}$ on M.

4. The Bernstein theorem for lightlike hypersurfaces of a Lorentz space

Bernstein's Theorem, which states that the only entire minimal surfaces in $\mathbb{R}^3$ are planes, is one of the most striking theorems in global geometry. In a famous paper [**4**], Cheng and Yau proved that the Bernstein Theorem holds good for maximal spacelike hypersurfaces of a Lorentz space. It is well known that such a result does not hold for entire timelike surfaces in the Minkowski 3-space $\mathbb{R}^3_1$. A few years later T.K. Milnor [**6**] presented a generalization of the Hilbert-Holmgren theorem and used it to discuss the indefinite Bernstein problem and prove a conformal analog of Bernstein's Theorem for timelike surfaces in $\mathbb{R}^3_1$. In [**5**] Magid gave one version of a solution to the indefinite Bernstein problem. Actually he looked at entire timelike surfaces in $\mathbb{R}^3_1$ which have zero mean curvature and showed that such a graph over a timelike or spacelike plane is a global translation surface. It is the purpose of the present section to prove the Bernstein Theorem for lightlike hypersurfaces with vanishing lightlike mean curvature in Lorentz spaces.

Let M be a hypersurface of the Lorentz space $\mathbb{R}^{m+2}_1$ given by equation (2). Then according to Theorem 3.1 M is a lightlike hypersurface if and only if

$$(F'_{x^0})^2 = \sum_{a=1}^{m+1} (F'_{x^a})^2.$$

As $\operatorname{rank}[F'_{x^0}, \cdots, F'_{x^{m+1}}] = 1$ on M, we deduce that F'_{x^0} is nowhere zero on M. Hence we may state the following

Proposition 4.1 *A connected lightlike hypersurface M of $\mathbb{R}^{m+1}_1$ given by (2) is globally expressed by a Monge equation*

$$x^0 = f(x^1, \cdots, x^{m+1}),$$

where f is a smooth function on a domain $\Omega \subset \mathbb{R}^{m+1}$ such that

$$\sum_{a=1}^{m+1} (f'_{x^a})^2 = 1.$$

Since $\widetilde{g}_{\rm SE}$ becomes now a Lorentz metric, i.e., it has index $\nu = 1$, we shall denote it by $\widetilde{g}_{\rm L}$. In the next proposition we find the iterrelation between $\widetilde{g}_{\rm L}$ and $\widetilde{g}_{\rm E}$ on M.

Proposition 4.2 *Let (M, STM) be a lightlike hypersurface of the Lorentz space $\mathbb{R}^{m+2}_1$, where STM is the canonical screen distribution on M. Suppose $\{V_1, \cdots, V_m\}$ is an orthonormal basis of $\Gamma(STM)$ with respect to $\widetilde{g}_{\rm L}$. Then $\{V_0, V_1, \cdots, V_m\}$ is an orthonormal field of frames on M with respect to $\widetilde{g}_{\rm E}$, where we set*

$$V_0 = (\widetilde{g}_{\rm E}(\xi, \xi))^{-\frac{1}{2}}\, \xi\ .$$

Proof. By using (3), (8) and (13) we get

$$\xi = \frac{\partial}{\partial x^0} + \sum_{a=1}^{m+1} f'_{x^a} \frac{\partial}{\partial x^a},$$

and

$$W_a = f'_{x^{m+1}} \frac{\partial}{\partial x^a} - f'_{x^a} \frac{\partial}{\partial x^{m+1}}, \quad a \in \{1, \cdots, m\}.$$

Then by direct calculations we obtain

$$\widetilde{g}_{\rm E}(W_a, \xi) = \widetilde{g}_{\rm L}(W_a, \xi) = 0, \qquad \forall a \in \{1, \cdots, m\}$$

and

$$\widetilde{g}_{\rm E}(W_a, W_b) = \widetilde{g}_{\rm L}(W_a, W_b), \qquad \forall a, b \in \{1, \cdots, m\},$$

which prove the assertion of the proposition.

As $B(\xi, \xi) = 0$ (see [**2**]), we define the *lightlike mean curvature* of M by

$$\alpha_{\rm L} = \sum_{a=1}^{m} B(V_a, V_a),$$

where $\{V_1, \cdots, V_m\}$ is an orthonormal basis of $\Gamma(STM)$. It is easy to show that $\alpha_{\rm L}$ does not depend on both the screen distribution and the orthonormal basis. By $\alpha_{\rm E}$ we shall denote the mean curvature of the Riemannian immersion of M in $\mathbb{R}^{m+2}_0$. Then by using (15), (12) and Proposition 4.2 we deduce

$$\alpha_{\rm E} = B_0(V_0, V_0) + \sum_{a=1}^{m} B_0(V_a, V_a) = \frac{1}{\sqrt{2}} \sum_{a=1}^{m} B(V_a, V_a) = \frac{1}{\sqrt{2}} \alpha_{\rm L},$$

since from (6) and (13) we have $\varepsilon = \frac{1}{\sqrt{2}}$. Due to (16) we may state the following important result

Theorem 4.3 *Let M be a lightlike hypersurface of the Lorentz space $\mathbb{R}^{m+2}_1$. Then M has zero lightlike mean curvature if and only if M is a minimal hypersurface of the Euclidean space $\mathbb{R}^{m+2}_0$.*

We now examine the partial differential equation

$$\sum_{a=1}^{m+1} \frac{\partial}{\partial x^a}\left(\frac{f'_{x^a}}{\sqrt{1+||\widetilde{\nabla}_{\mathrm{E}} f||^2}}\right) = 0,$$

whose solutions give all minimal hypersurfaces (13) of the Euclidean space $\mathbb{R}^{m+2}_0$.

In 1914 Bernstein proved a theorem that states that for $m = 2$ the only entire solution of (17) is linear. For the higher dimensional version of Bernstein's Theorem we need the following result

Theorem 4.4 (Bompieri-Giusti [**3**]) *Let f be a C^∞ solution of the minimal hypersurface equation (17) in $\mathbb{R}^{m+1}$ such that m partial derivatives f'_{x^a} are uniformly bounded in $\mathbb{R}^{m+1}$. Then f must be linear.*

For spacelike hypersurfaces of $\mathbb{R}^{m+2}_1$ the Bernstein Theorem was proved by Cheng and Yau [**4**]. By combining Theorems 4.3 and 4.4, and using (14), we obtain the following result on Bernstein's Theorem for lightlike hypersurfaces

Theorem 4.5 *Let M be a lightlike hypersurface of $\mathbb{R}^{m+2}_1$ given by (13), where f is defined on $\mathbb{R}^{m+1}$. If M has zero lightlike mean curvature then f must be linear, and thus M is a lightlike hyperplane of $\mathbb{R}^{m+2}_1$.*

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