Willmore tori and Willmore-Chen submanifolds in pseudo-Riemannian spaces

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Abstract

We exhibit a new method to find Willmore tori and Willmore-Chen submanifolds in spaces endowed with pseudo-Riemannian warped product metrics, whose fibres are homogeneous spaces. The chief points are the invariance of the involved variational problems with respect to the conformal changes of the metrics on the ambient spaces and the principle of symmetric criticality. They allow us to relate the variational problems with that of generalized elastic curves in the conformal structure of the base space. Among others applications we get a rational one-parameter family of Willmore tori in the standard anti De Sitter 3-space shaped on an associated family of closed free elastic curves in the once punctured standard 2-sphere. We also obtain rational one-parameter families of Willmore-Chen submanifolds in standard pseudo-hyperbolic spaces. As an application of a general approach to our method, we give nice examples of pseudo-Riemannian 3-spaces which are foliated with leaves being non-trivial Willmore tori. More precisely, the leaves of this foliation are Willmore tori which are conformal to non-zero constant mean curvature flat tori.

1. Introduction

Let N be a compact manifold of dimension n, which we assume without boundary, otherwise minor changes can be done to get boundary versions of our results. Let $\mathcal{N}$ be the smooth manifold of immersions ϕ of N in a pseudo-Riemannian manifold $(M, \tilde{g})$. The Willmore-Chen functional Ω is defined on $\mathcal{N}$ to be

$$\Omega(\phi) = \int_N (\langle H, H\rangle - \tau_e)^{\frac{n}{2}} dv,$$

where H and τ_e denote, respectively, the mean curvature vector field and the extrinsic scalar curvature function of ϕ, and dv is the volume element of the induced metric (via ϕ) on N (see [14] and [27]).

The variational problem associated with this functional is known in the literature as the Willmore-Chen variational problem. It was shown by B.Y. Chen, [13], that this functional is invariant under conformal changes of the metric $\tilde{g}$ of the ambient space M. Its critical points are called Willmore-Chen submanifolds. When $n = 2$, the functional essentially agrees with the well known Willmore functional and its critical points are the so called Willmore surfaces.

Obvious examples of Willmore surfaces, in spaces of constant curvature, are those with $H \equiv 0$, in particular minimal and maximal surfaces. Articles showing different methods to get examples of non-minimal Willmore surfaces in standard spheres are well known in the literature (see, for instance, [4], [10], [15] and [22]). Examples in non-standard 3-spheres are given in [2].

The first non-trivial examples of Willmore-Chen submanifolds (of course with dimension greater than two, namely they are of dimension four) were obtained in [9]. In [3], the first author gave ample families of Willmore tori (either Riemannian or Lorentzian) in pseudo-Riemannian manifolds with non constant curvature. In particular he obtained nice examples of Riemannian Willmore tori in some kind of spacetimes close to Robertson-Walker spacetimes.

However two open problems drew our attention. The first one is that no examples of Willmore surfaces are known in the anti De Sitter space. Also, no examples of nontrivial Willmore-Chen submanifolds in pseudo-Riemannian spaces (with non zero index) are known in the literature. These two problems will be solved in this paper. Both are applications of the technique we will exhibit later.

The plan of the paper can be summarized as follows. After some preliminaries given in the next section, we will obtain the result of U.Pinkall, [22], but using a direct approach. To do that we will integrate the Euler-Lagrange equations for Willmore surfaces in spaces of constant curvature, which were computed in [25].

In three dimensional Lorentzian geometry the anti De Sitter 3-space $\mathbb{H}^3_1$ behaves, in some sense, as the 3-sphere $\mathbb{S}^3$ does in Riemannian geometry (see, for instance, [1] and [7] to compare this claim from the point of view of the behaviours of general helices). In particular, two Hopf maps can be defined from $\mathbb{H}^3_1$ over $\mathbb{H}^2$ and $\mathbb{H}^2_1$, respectively (see [6] and Section 2 for notation), the first one having closed fibres. Therefore to find Willmore tori in $\mathbb{H}^3_1$ it seems natural to deal with the class of Hopf tori and look for solutions of the corresponding Euler-Lagrange equations for the Willmore functional on this class. Two facts should be noted. On one hand, concerning Hopf tori, we mean surfaces which have a certain degree of symmetry and project, via the Hopf map, into closed curves in the hyperbolic 2-plane $\mathbb{H}^2$. On the other hand, the Euler-Lagrange equations obtained by Weiner for Riemannian manifolds work also here. However, to integrate these equations, we can use a direct method or a method based on the nice symmetries of the Hopf tori to show that there exist no Willmore Hopf tori in $\mathbb{H}^3_1$.

In Section 4 we deal with the canonical variation of the standard metric in $\mathbb{H}^3_1$ (see [11] for details) to get a one-parameter family of pseudo-Riemannian submersions over the hyperbolic 2-plane. We use a similar argument to that exposed in [2] to find interesting examples of Willmore tori in non standard anti De Sitter three spaces (these endowed with metrics of constant scalar curvature) shaped on certain closed elastic curves in the standard hyperbolic 2-plane.

Section 5 is the main one of the paper. We present a new method to obtain examples of Willmore-Chen submanifolds in the pseudo-hyperbolic space $\mathbb{H}^n_r$. We first notice that the metric on $\mathbb{H}^n_r$, obtained as the pseudo-Riemannian product of the standard metric in the once punctured $(n-r)$-sphere Σ^{n-r} and the negative definite standard one in the r-sphere, is conformal to the standard metric on $\mathbb{H}^n_r$. Then we use the homogeneous structure of the r-sphere to determine the submanifolds of $\mathbb{H}^n_r$ which are $SO(r+1)$-invariant. Next by using the invariance of the Willmore-Chen variational problem under conformal changes of the ambient metric and the principle of symmetric criticality, [21], we are able to reduce the problem of finding $SO(r+1)$-symmetric Willmore-Chen submanifolds in $\mathbb{H}^n_r$ to that of closed generalized elastic curves in Σ^{n-r}. That means critical points of the functional

$$\mathcal{F}^r(\gamma) = \int_\gamma (\kappa^2)^{\frac{r+1}{2}} ds$$

(κ being the curvature function of γ) defined on the smooth manifold of closed curves in Σ^{n-r}.

This method will allow us to give a wide family of Lorentzian Willmore tori in the standard anti De Sitter 3-space, which come from closed free elastic curves in the once punctured 2-sphere (see Corollaries 5.5 and 5.6).

We also determine, for any natural number r and any non-zero rational number, a unique closed helix in Σ^3 which is a critical point of $\mathcal{F}^r$ (see Theorem 5.7). Therefore we obtain non trivial Willmore-Chen submanifolds in the pseudo-hyperbolic space $\mathbb{H}_r^{3+r}$ (see Theorem 5.9).

In the next section we generalize the above argument to a remarkable and more general context (see Theorem 6.2). Then we apply it to get some consequences in the last section. We first obtain Willmore tori in some conformal structures on spaces which topologically are products of three circles (see Corollary 7.1). Even so, these result is widely extended with the aid of an existence result for elasticae due to N. Koiso [16] (see Corollary 7.3). We also make use of our method to give examples of pseudo-Riemannian 3-spaces which admits foliations whose leaves are non-trivial Willmore tori. These foliations come from the free elasticity of all parallels of certain surfaces of revolution obtained in [8] (see Corollary 7.2).

2. Setup

Let $\mathbb{R}_t^{n+2}$ be the $(n+2)$-dimensional pseudo-Euclidean space whose metric tensor is given by

$$\langle x, x\rangle = -\sum_{i=1}^{t} dx^i \otimes dx^i + \sum_{j=t+1}^{n+2} dx^j \otimes dx^j,$$

where $(x^1, \ldots, x^{n+2})$ is the standard coordinate system. Let $\mathcal{M}^{n+1}(\rho)$ be $\mathbb{R}_s^{n+1}$ if $\rho = 0$ or the $(n+1)$-dimensional complete and simply connected space with constant sectional curvature $K = \operatorname{sign}(\rho)/\rho^2$ and index s if $\rho \neq 0$. For each $\rho \neq 0$, a model for $\mathcal{M}^{n+1}(\rho)$ is the pseudo-Euclidean sphere $\mathbb{S}_s^{n+1}(\rho)$ if $\rho > 0$ and the pseudo-Euclidean hyperbolic space $\mathbb{H}_s^{n+1}(\rho)$ if $\rho < 0$, where

$$\mathbb{S}_s^{n+1}(\rho) = \{x \in \mathbb{R}_s^{n+2} : \langle x, x\rangle = \rho^2\} \qquad (\rho > 0)$$

and

$$\mathbb{H}_s^{n+1}(\rho) = \{x \in \mathbb{R}_{s+1}^{n+2} : \langle x, x\rangle = -\rho^2\} \qquad (\rho < 0).$$

Throughout this paper, $x : \mathcal{M}^{n+1}(\rho) \to \mathbb{R}_t^{n+2}$ will denote the standard immersion of $\mathcal{M}^{n+1}(\rho)$ in $\mathbb{R}_t^{n+2}$. For the sake of brevity we will write $\mathbb{S}^{n+1}(\rho)$ by $\mathbb{S}_0^{n+1}(\rho)$ and $\mathbb{S}_s^{n+1}$ by $\mathbb{S}_s^{n+1}(1)$. A similar convention for pseudo-hyperbolic spaces will be used.

One of the most classical topics in the calculus of variations was proposed by J. Bernoulli: the problem of the elastic rod. According to D. Bernouilli's idealization, all kinds of elastica minimize total squared curvature among curves of the same length and first order boundary data. Recently, Bryant-Griffiths, [12], and Langer-Singer, [17, 18], have generalized the notion of elastica to space forms and studied them from a geometrical point of view. Let $\gamma : I \to \mathcal{M}^m(\rho)$ be a closed curve in $\mathcal{M}^m(\rho)$, then γ is said to be an λ-*elastica* (or λ-*elastic curve*) if it is an extremal point of the functional

$$\mathcal{G}_\lambda(\gamma) = \int_0^L (\langle \nabla_T T, \nabla_T T\rangle + \lambda) ds,$$

for some λ, where ds and L stand for the arclength and length of γ, respectively. The Lagrange multiplier λ has been included partly because the case of constrained arclength will be useful later.

It is called a *free elastica* if $\lambda = 0$; in this case, γ is a critical point of $\mathcal{G}(\gamma) = \mathcal{G}_0(\gamma)$ among closed curves which are allowed to grow in length. This lack of constraint of length makes existence an interesting and non trivial question in the calculus of variations.

We can assume, without loss of generality, that γ is arclength parametrized and let $\{E_1 = T, E_2 = N, E_3 = B, \ldots, E_m\}$ be a Frenet frame along γ, with curvature functions $\{\kappa_1 = \kappa, \kappa_2 = \tau, \ldots, \kappa_{m-1}\}$, satisfying the Frenet equations

$$\begin{array}{rcl} \nabla_T T & = & \varepsilon_2 \kappa N, \\ \nabla_T N & = & -\varepsilon_1 \kappa T + \varepsilon_3 \tau B, \\ \vdots & & \vdots \\ \nabla_T E_m & = & -\varepsilon_{m-1} \kappa_{m-1} E_{m-1}, \end{array}$$

where, as usual, $T = \gamma'$, $\langle E_i, E_i \rangle = \varepsilon_i$, and ∇ stands for the Levi-Civita connection on $\mathcal{M}^m(\rho)$. Then the Euler-Lagrange equation reduces to the following system of differential equations

$$2\varepsilon_2 \kappa'' + \varepsilon_1 \kappa^3 - 2\varepsilon_3 \kappa \tau^2 + \varepsilon_1 \varepsilon_2 (2c - \lambda)\kappa = 0, \tag{2}$$
$$2\kappa' \tau + \kappa \tau' = 0, \tag{3}$$
$$\kappa \tau \delta = 0, \tag{4}$$

where $\delta \in \mathrm{span}\{T, N, B\}^\perp$. If γ does not lie in a 2-dimensional totally geodesic submanifold of $\mathcal{M}^m(\rho)$, then the equation (4) implies that $\delta = 0$ and so the curve γ lies in a 3-dimensional totally geodesic submanifold of $\mathcal{M}^m(\rho)$. Hence we can assume without loss of generality that $n = 2$ or $n = 3$. On the other hand, from (3) we deduce that $\kappa^2 \tau = a$ is constant.

Another interesting topic in the calculus of variations is concerned with the total mean curvature of immersed manifolds. The first result of this subject is due to T.J. Willmore, [28], and since then a surface M in $\mathbb{R}^3$, $\phi : M \to \mathbb{R}^3$ being the immersion, is called a *Willmore surface* if it is an extremal point of the functional

$$\Omega(\phi) = \int_M \alpha^2 dA,$$

where α and dA stand for the mean curvature function of M in $\mathbb{R}^3$ and the area element of M, respectively. In [25], Weiner extends this notion to an arbitrary 3-dimensional Riemannian manifold $\tilde{M}$: a surface $M \subset \tilde{M}$ is said to be stationary (or Willmore surface) if it is an extremal point of

$$\Omega(\phi) = \int_M (\langle H, H \rangle + R') dA,$$

where R' is the sectional curvature of $\tilde{M}$ along M and H denotes the mean curvature vector field. Of special interest is the case when $\tilde{M} = \mathcal{M}^3(\rho)$ is of constant curvature K. We define the operator $\mathcal{W}$ over sections of the normal bundle of M into $\mathcal{M}^3(\rho)$ as follows

$$\mathcal{W} : T^\perp M \to T^\perp M, \qquad \mathcal{W}(\xi) = (\Delta^D + 2\langle H, H \rangle I - \widetilde{A})\xi,$$

where $\widetilde{A}$ denotes the Simon operator, [24]. A cross section ξ will be called a Willmore section if $\mathcal{W}(\xi) = 0$. Then the operator $\mathcal{W}$ naturally appears provided that one computes the first variation formula of Ω. That can be obtained in a similar way to that given by J. L. Weiner (see [25]) in the definite case. Now Willmore surfaces are nothing but extremal points of the Willmore functional and they are characterized from the fact that their mean curvature vector fields are Willmore fields.

More generally, let (M^n, g) be an n-dimensional submanifold of a Riemannian manifold $(\tilde{M}, \tilde{g})$. Then $(\langle H, H\rangle - \tau_e)g$ is invariant under any conformal change of the ambient metric $\tilde{g}$, τ_e standing for the extrinsic scalar curvature with respect $\tilde{g}$, [14]. When $\tilde{M} = \mathcal{M}^m(\rho)$ and M^n is compact then $\tau_e = \tau - K$, τ being the scalar curvature of (M^n, g), and M is said to be stationary (or a *Willmore-Chen submanifold*) if it is an extremal point of

$$\Omega(\phi) = \int_M (\langle H, H\rangle - \tau_e)^{\frac{n}{2}} dV,$$

dV standing for the volume element on M. The variational problem associated with this functional Ω is an extrinsic conformal invariant and so are the Willmore-Chen submanifolds.

3. Willmore tori in the 3-sphere

Let $\pi : \mathbb{S}^3 \to \mathbb{S}^2(1/2)$ the usual Hopf fibration, which is a Riemannian submersion relative to canonical metrics on both spheres (we will follow the notation and terminology of [11] and [20]).

For any unit speed curve $\gamma : I \subset \mathbb{R} \to \mathbb{S}^2(1/2)$, we can talk about horizontal lifts $\overline{\gamma}(s)$ of $\gamma(s)$ and obtain unit speed curves in $\mathbb{S}^3$. All these curves define the complete lift $M_\gamma = \pi^{-1}(\gamma)$ of γ. This is a flat surface which we will call the Hopf tube over γ. It is easy to see that M_γ can be parametrized by

$$\Psi(s,t) = e^{it}\overline{\gamma}(s),$$

Ψ being a mapping $I \times \mathbb{R} \to \mathbb{S}^3$ and $\overline{\gamma}$ a fixed horizontal lift of γ.

If γ is a closed curve in $\mathbb{S}^2(1/2)$ of length L enclosing an oriented area A, then its Hopf tube M_γ is a flat torus (the Hopf torus over γ) which is isometric to $\mathbb{R}^2/\Gamma$, Γ being the lattice generated by $(0, 2\pi)$ and $(L, 2A)$.

Let us consider the manifold of all immersions of a torus in $\mathbb{S}^3$. Then the Hopf tori correspond with those immersions which are invariant under the usual $\mathbb{S}^1$-action on $\mathbb{S}^3$ in order to get $\mathbb{S}^2(1/2)$ as the orbit space. The Willmore functional of the manifold of immersions is invariant for this $\mathbb{S}^1$-action. Then we can use the nice argument of U. Pinkall (see [22]), based on the principle of symmetric criticality of R. S. Palais, [21], to reduce the problem of finding symmetric Willmore tori (i.e., Willmore Hopf tori) in $\mathbb{S}^3$ to that of finding closed elasticae (with Lagrange multiplier $\lambda = 4$) in $\mathbb{S}^2(1/2)$. This is the way used by Pinkall, [22], to get infinitely many embedded Hopf tori which are Willmore tori.

We wish to point out that Pinkall's result can be also obtained by a straightforward computation. To do that we will solve the Euler-Lagrange equation for Willmore tori in the 3-sphere (see [25] for details). That equation turns out to be

$$\Delta^D H = \tilde{A}(H) - 2\alpha^2 H.$$

H being the mean curvature vector field of the torus in $\mathbb{S}^3$, $\alpha^2 = \langle H, H\rangle$ the squared of the mean curvature function and Δ^D the Laplacian relative to the normal connection. All geometric invariant appearing in (6) can be computed from (5).

The shape operator A of M_γ is given by

$$AX_s = \overline{\kappa}X_s + X_t, \qquad AX_t = X_s,$$

where $\overline{\kappa} = \kappa \circ \pi$, κ being the curvature function of γ in $\mathbb{S}^2(1/2)$ (see [6] for details).

We also have that

$$H = \frac{1}{2}(\mathrm{tr}A)\xi = \frac{1}{2}\overline{\kappa}\xi, \qquad \Delta^D H = -\frac{1}{2}\overline{\kappa}''\xi,$$

ξ being a unit normal vector field of M_γ in $\mathbb{S}^3$ and $\overline{\kappa}'' = \frac{\mathrm{d}^2\overline{\kappa}(s)}{\mathrm{d}s^2}$.

Finally, it is easy to see that $\tilde{A}(H) = |A|^2 H$, so that

$$\tilde{A}(H) = \frac{1}{2}\overline{\kappa}(\overline{\kappa}^2 + 2)\xi.$$

Now bring (8) and (9) to (6) to get

$$2\overline{\kappa}'' + \overline{\kappa}^3 + 4\overline{\kappa} = 0.$$

This is nothing but the Euler-Lagrange equations (2) and (3) for 4-closed elastic curves in $\mathbb{S}^2(1/2)$ (see also [18, eq.1.2]).

Summing up, we have shown that M_γ *is a solution of (6), and therefore a Willmore surface in* $\mathbb{S}^3$*, if and only if* γ *is a solution of (10), and so a* 4*-elastica in* $\mathbb{S}^2(1/2)$.

Remark 3.1 In some sense, the anti De Sitter and De Sitter worlds, $\mathbb{H}^3_1$ and $\mathbb{S}^3_1$, respectively, behave as the spherical and hyperbolic space forms $\mathbb{S}^3$ and $\mathbb{H}^3$, respectively. A nice example illustrating this fact arises when one studies evolution equations associated with Killing flows in space forms (either Riemannian or Lorentzian). In particular, curves which evolves under a certain flow without changing shape, only position. For instance, general helices (see [1] and [7]).

In studying the anti De Sitter 3-space $\mathbb{H}^3_1$, we have found a couple of Hopf maps $\pi_j : \mathbb{H}^3_1 \to \mathbb{H}^2_j(-1/2)$, $j = 0, 1$, according to the base space is the hyperbolic 2-plane $\mathbb{H}^2_0 \equiv \mathbb{H}^2$ or the pseudo-hyperbolic 2-plane $\mathbb{H}^2_1$ (see [6] for details). We wish to study the Willmore problem in $\mathbb{H}^3_1$ and, in particular, we are trying to get Willmore surfaces in $\mathbb{H}^3_1$ coming from curves in $\mathbb{H}^2_j$.

The computations we have made in $\mathbb{S}^3$, via the usual Hopf map, hold now and the Euler-Lagrange equation $\mathcal{W}(H) = 0$ for $M_{\gamma,j} = \pi_j^{-1}(\gamma)$, reduces to the Euler-Lagrange equation

$$(-1)^{j+1}2\kappa'' - \kappa^3 + 4\varepsilon\kappa = 0$$

for (-4)-elastic curves in $\mathbb{H}^2_j$, ε being the sign of the surface $M_{\gamma,j}$.

We wish to point out that fibres of π_0 are circles, and thus $M_\gamma \equiv M_{\gamma,0}$ is a torus provided that γ is closed, whereas fibres of π_1 are not compact. Therefore, given a closed curve γ in $\mathbb{H}^2$, then $M_\gamma = \pi_0^{-1}(\gamma)$ is a (Lorentzian) Willmore torus if and only if γ is a (-4)-elastic curve in $\mathbb{H}^2$. Unfortunately, a recent result of Dan Steinberg in his PhD dissertation, kindly communicated to us by David A. Singer, shows that there is no closed (-4)-elasticae in $\mathbb{H}^2$. As a consequence, one should conclude that there are no (Lorentzian) Willmore tori in the anti De Sitter 3-space $\mathbb{H}^3_1$.

4. Willmore tori in non-standard anti De Sitter 3-space

Let $\pi : (M, g) \to (B, h)$ be a pseudo-Riemannian submersion. We can define a very interesting deformation of the metric g by changing the relative scales of B and the fibres (see [11]).

More precisely, it is defined the canonical variation g_t, $t > 0$, of g by setting

$$\begin{aligned} g_t|_{\mathcal{V}} &= t^2\, g|_{\mathcal{V}}\,, \\ g_t|_{\mathcal{H}} &= g|_{\mathcal{H}}\,, \\ g_t(\mathcal{V},\mathcal{H}) &= 0, \end{aligned}$$

where $\mathcal{V}$ and $\mathcal{H}$ stand for vertical and horizontal distributions, respectively, associated with the submersion. Thus we obtain a one-parameter family of pseudo-Riemannian submersions $\pi_t : (M, g_t) \to (B, h)$ with the same horizontal distribution $\mathcal{H}$, for all $t > 0$. Relative to O'Neill invariants A^t and T^t of these pseudo-Riemannian submersions, we will just recall a couple of properties. First, if g has totally geodesic fibres ($T \equiv 0$), the same happens for g_t, for all $t > 0$. Furthermore,

$$A^t_Y U = t^2 A_Y U, \tag{11}$$

for any $Y \in \mathcal{H}$ and $U \in \mathcal{V}$.

Now we consider the canonical variation of the Hopf fibration $\pi = \pi_0 : \mathbb{H}^3_1 \to \mathbb{H}^2(-1/2)$ to get a one-parameter family of pseudo-Riemannian submersions $\pi_t : (\mathbb{H}^3_1, g_t) \to (\mathbb{H}^2(-1/2), g_0)$. Let γ be a unit speed curve immersed in $\mathbb{H}^2(-1/2)$. Set $\mathcal{T}_{\gamma,t} = \pi_t^{-1}(\gamma)$. Then $\mathcal{T}_{\gamma,t}$ is a Lorentzian flat surface immersed in $\mathbb{H}^3_1$, that will be called the Lorentzian Hopf tube over γ. As the fibres of π_t are $\mathbb{H}^1_1$, which topologically are circles, then $\mathcal{T}_{\gamma,t}$ is a Hopf torus in $(\mathbb{H}^3_1, g_t)$, provided that γ is a closed curve. It is obvious that the group $G = \mathbb{S}^1$ naturally acts through isometries on $(\mathbb{H}^3_1, g_t)$, for all $t > 0$, getting $(\mathbb{H}^2(-1/2), g_0)$ as the orbit space. The following result, whose proof is omitted, gives a nice characterization of the G-invariant surfaces in $(\mathbb{H}^3_1, g_t)$.

Proposition 4.1 *Let S be an immersed surface into $(\mathbb{H}^3_1, g_t)$. Then S is G-invariant if and only if S is a Lorentzian Hopf tube $\mathcal{T}_{\gamma,t} = \pi_t(\gamma)$ over a certain curve γ immersed in the hyperbolic 2-plane $(\mathbb{H}^2(-1/2), g_0)$.*

Remark 4.2 The canonical variation of a pseudo-Riemannian submersion has been used to get examples of homogeneous Einstein metrics (see [11] for a nice and complete exposition on the subject). In dimension three, Einstein metrics correspond with constant sectional curvature metrics. Therefore, the standard metric $g = g_1$ is the only Einstein metric that one can find on the anti De Sitter 3-space. However, we can use a well known formula to compute the scalar curvature of the canonical variation of a pseudo-Riemannian submersion (see [11] again), to find that $(\mathbb{H}^3_1, g_t)$, $t > 0$, is a one-parameter family of pseudo-Riemannian manifolds with constant scalar curvature, and so the nicest metrics on the anti De Sitter 3-space after the canonical one.

In the following we will use the principle of symmetric criticality in order to reduce the problem of finding Lorentzian Willmore tori in $(\mathbb{H}^3_1, g_t)$, $t > 0$, to that of finding closed λ-elasticae in $(\mathbb{H}^2(-1/2), g_0)$.

Theorem 4.3 *Let $\pi_t : (\mathbb{H}^3_1, g_t) \to (\mathbb{H}^2(-1/2), g_0)$, $t > 0$, be the canonical variation of the pseudo-Riemannian Hopf fibration. Let γ be a closed immersed curve in $(\mathbb{H}^2(-1/2), g_0)$ and $\mathcal{T}_{\gamma,t} = \pi_t^{-1}(\gamma)$ its Lorentzian Hopf torus. Then $\mathcal{T}_{\gamma,t}$ is a Willmore surface in $(\mathbb{H}^3_1, g_t)$ if and only if γ is an elastica in $(\mathbb{H}^2(-1/2), g_0)$ with Lagrange multiplier $\lambda = -4t^2$.*

Proof. Let $T = \mathbb{S}^1 \times \mathbb{S}^1$ be a compact surface of genus one, i.e., T is a topological torus. Consider the smooth manifold of immersions of T into $(\mathbb{H}^3_1, g_t)$, say $\mathcal{M} = \{\phi : T \to (\mathbb{H}^3_1, g_t) : \phi \text{ is an immersion}\}$. The Willmore functional on $\mathcal{M}$ is

$$\Omega(\phi) = \int_T (\langle H, H\rangle + R^t) dv,$$

H and R^t standing for the mean curvature vector field of T and the sectional curvature of $(\mathbb{H}^3_1, g_t)$, measured with respect to the tangent plane to (T, ϕ), respectively. It is clear that, for any $e^{i\theta} \in \mathbb{S}^1$, we have that $\Omega(\phi) = \Omega(e^{i\theta} \cdot \phi)$. Now let us denote by $\mathcal{C}$ the set of critical points of Ω in $\mathcal{M}$, i.e., $\mathcal{C}$ is the set of genus one Willmore surfaces. Let $\mathcal{M}_G$ be the submanifold of $\mathcal{M}$ made up by those immersions of T which are $(G = \mathbb{S}^1)$-invariants and let $\mathcal{C}_G$ be the set of critical points of Ω restricted to $\mathcal{M}_G$. The principle of symmetric criticality (see [21]) can be used here to find that $\mathcal{C} \cap \mathcal{M}_G = \mathcal{C}_G$. Now from Proposition 4.1 we obtain that $\mathcal{C}_G = \{\mathcal{T}_{\gamma,t} = \pi_t^{-1}(\gamma) : \gamma \text{ is an immersed closed curve in } (\mathbb{H}^2(-1/2), g_0)\}$. To compute $\Omega(\mathcal{T}_{\gamma,t})$, i.e., the Willmore functional on $\mathcal{C}_G$, we first notice that $\alpha = \frac{1}{2}\kappa$, κ being the curvature function of γ in $(\mathbb{H}^2(-1/2), g_0)$. Now we are going to compute R^t. Let $X = \gamma'$ be the unit tangent vector field along γ and $\overline{X}$ its horizontal lift (for any $t > 0$) to $(\mathbb{H}^3_1, g_t)$. Then, at any point, the tangent plane of $\mathcal{T}_{\gamma,t}$ is spanned by $\overline{X}$ and U, U being a unit (with respect to g_t) timelike vector field which is tangent to the fibres of π_t. Then the tangent plane of $\mathcal{T}_{\gamma,t}$ is a mixed (also called "vertizontal", see [26]) section of $\pi_t : (\mathbb{H}^3_1, g_t) \to (\mathbb{H}^2(-1/2), g_0)$. As g_t has geodesic fibres, we know that $R^t = -g_t(A^t_{\overline{X}}U, A^t_{\overline{X}}U)$, A^t being the O'Neill invariant for the submersion g_t, which is known to be (see [11])

$$A^t_{\overline{X}}U = \nabla^t_{\overline{X}}U,$$

∇^t being the Levi-Civita connection of g_t. Then we have

$$-1 = g_t(U, U) = -t^2 g(U, U) = -g(tU, tU),$$

so $\xi = tU$ is a unit timelike vector field with regard to g. Now, from [6, p. 3], and bearing in mind that $\nabla^t_{\overline{X}}U$ is horizontal, we get

$$\begin{aligned} g_t(\nabla^t_{\overline{X}}U, i\overline{X}) &= \frac{1}{2} g_t(U, [i\overline{X}, \overline{X}]) \\ &= \frac{1}{2} g_t(U, \nabla_{i\overline{X}}\overline{X} - \nabla_{\overline{X}} i\overline{X}) \\ &= \frac{1}{2} g_t(U, -g_t(\overline{X}, \overline{X})\xi - g_t(i\overline{X}, i\overline{X})\xi) \\ &= \frac{1}{2} g_t(U, -2\xi) \\ &= -t^2 g(U, \xi) = t. \end{aligned} \tag{12}$$

On the other hand,

$$\begin{aligned} g_t(\nabla^t_{\overline{X}}U, \overline{X}) &= \frac{1}{2}\{\overline{X} g_t(U, \overline{X}) + U g_t(\overline{X}, \overline{X}) - \overline{X} g_t(\overline{X}, U) \\ &\quad - g_t(\overline{X}, [U, \overline{X}]) + g_t(U, [\overline{X}, \overline{X}]) + g_t(\overline{X}, [\overline{X}, U])\} \\ &= 0. \end{aligned} \tag{13}$$

From (12) and (13) we deduce that

$$\nabla^t_{\overline{X}} U = ti\overline{X}.$$

Hence,

$$R^t = -g_t(ti\overline{X}, ti\overline{X}) = -t^2.$$

Let L be the length of γ. As the fibres of g_t are circles of radii t, we have

$$\Omega(\mathcal{T}_{\gamma,t}) = \int_{\pi_t^{-1}(\gamma)} (\alpha^2 + R^t) dv = \int_0^L \int_0^{2\pi t} \left(\frac{1}{4}\kappa^2 - t^2 \right) ds dr = \frac{\pi t}{4} \int_0^L (\kappa^2 - 4t^2) ds.$$

Remark 4.4 The canonical variation g_t of the standard metric $g_1 = g$ on $\mathbb{H}^3_1$ provides an easy and useful way to get infinitely many Willmore tori in $(\mathbb{H}^3_1, g_t)$. In fact, in working with the pseudo-Riemannian Hopf fibration, we were not able to produce Lorentzian Hopf Willmore tori into $\mathbb{H}^3_1$ by pulling back elasticae in $(\mathbb{H}^2(-1/2), g_0)$. This is because the Lagrange multiplier we found is $\lambda = -4$, which is not permitted (see Remark 3.1). Now, by applying Theorem 4.3 and results by Langer and Singer, [18], we give, for $t \in (0,1)$, infinitely many Willmore tori in $(\mathbb{H}^3_1, g_t)$. It is worth pointing out that, in particular, any curve γ of constant curvature κ_0 in $(\mathbb{H}^2(-1/2), g_0)$ can be realized as an elastica in $(\mathbb{H}^2(-1/2), g_0)$ with Lagrange multiplier $\lambda = \kappa_0 - 8$. Now γ is closed provided that $\kappa_0^2 > 4$, so that taking $\kappa_0^2 \in (4,8)$, $\mathcal{T}_{\gamma,t} = \pi_t^{-1}(\gamma)$ is a Lorentzian Willmore torus with constant mean curvature in $(\mathbb{H}^3_1, g_t)$, where $t = \sqrt{2 - (\kappa_0^2/4)}$.

5. Willmore-Chen submanifolds in the pseudo-hyperbolic space

In this section we are going to introduce a new method to construct critical points of the Willmore-Chen functional in the pseudo-hyperbolic space $\mathbb{H}^n_r = \mathbb{H}^n_r(-1)$. First we will write $\mathbb{H}^n_r$ as a warped product with base space the standard hyperbolic space $\mathbb{H}^{n-r}$. Then we will use the conformal invariance of the Willmore-Chen variational problem to make a conformal change of the canonical metric of $\mathbb{H}^n_r$. Next we use the principle of symmetric criticality of R. Palais, [21], to reduce the problem to a variational one for closed curves in the once punctured standard $(n-r)$-sphere.

5.1. $\mathbb{H}^n_r$ as a warped product

Given $0 < r < n$, let

$$\mathbb{H}^{n-r} = \{(x_0, x) \in \mathbb{R} \times \mathbb{R}^{n-r} : -x_0^2 + \langle x, x \rangle = -1 \text{ and } x_0 > 0\}$$

the hyperbolic $(n-r)$-space and

$$\mathbb{H}^n_r = \{(\xi, \eta) \in \mathbb{R}^{r+1} \times \mathbb{R}^{n-r} : -\langle \xi, \xi \rangle + \langle \eta, \eta \rangle = -1\}$$

the pseudo-hyperbolic n-space. They are hypersurfaces in $\mathbb{R}^{n-r+1}_1$ and $\mathbb{R}^{n+1}_{r+1}$, respectively. The induced metrics on these spaces, from those in the corresponding pseudo-Euclidean spaces, define standard metrics h_0 on $\mathbb{H}^n_r$ and g_0 on $\mathbb{H}^{n-r}$, both with constant curvature -1.

Let $\mathbb{S}^r$ be the standard unit r-sphere endowed with its canonical metric $d\sigma^2$ and consider the mapping $\Phi : \mathbb{H}^{n-r} \times \mathbb{S}^r \to \mathbb{H}^n_r$ defined by

$$\Phi((x_0, x), u) = (x_0 u, x).$$

It is not difficult to see that Φ defines a diffeomorphism whose inverse is $\Phi^{-1}(\xi, \eta) = ((|\xi|, \eta), \xi/|\xi|)$. For any curve $\beta(t) = ((x_0(t), x(t)), u(t))$ in $\mathbb{H}^{n-r} \times \mathbb{S}^r$ we have

$$\left|\mathrm{d}\Phi_{\beta(t)}(\beta'(t))\right|^2 = -x_0'(t)^2 + |x'(t)|^2 - x_0(t)^2 |u'(t)|^2.$$

Let $f : \mathbb{H}^{n-r} \to \mathbb{R}$ be the positive function given by $f(x_0, x) = x_0$ and consider the metric $g = g_0 - f^2 d\sigma^2$ on $\mathbb{H}^{n-r} \times \mathbb{S}^r$. The pseudo-Riemannian manifold $(\mathbb{H}^{n-r} \times \mathbb{S}^r, g)$ is called the *warped product* of base $(\mathbb{H}^{n-r}, g_0)$ and fibre $(\mathbb{S}^r, -d\sigma^2)$ with warping function f.

It is usually denoted by $(\mathbb{H}^{n-r}, g_0) \times_f (\mathbb{S}^r, -d\sigma^2)$ or $\mathbb{H}^{n-r} \times_f (-\mathbb{S}^r)$ when the metrics on the base and fibre are understood (see [11] and [20] for details). Now the formula (14) shows that Φ is an isometry between $\mathbb{H}^{n-r} \times_f (-\mathbb{S}^r)$ and $(\mathbb{H}^n_r, h_0)$.

Consider a new metric h on $\mathbb{H}^n_r$ defined by

$$h = \frac{1}{f^2} h_0 = \frac{1}{f^2} g_0 - d\sigma^2,$$

with the obvious meaning by removing the pulling back via Φ. Thus $(\mathbb{H}^n_r, h)$ is the pseudo-Riemannian product of $(\mathbb{H}^{n-r}, \frac{1}{f^2} g_0)$ and $(\mathbb{S}^r, -d\sigma^2)$. Finally it is not difficult to see that $(\mathbb{H}^{n-r}, \frac{1}{f^2} g_0)$ has constant sectional curvature 1, so that it can be identified, up to isometries, with the once punctured standard $(n-r)$-sphere $(\Sigma^{n-r}, d\sigma^2)$. Consequently, $(\mathbb{H}^n_r, h)$ is nothing but $(\Sigma^{n-r}, d\sigma^2) \times (\mathbb{S}^r, -d\sigma^2)$, up to isometries.

5.2. $SO(r+1)$-invariant submanifolds in $\mathbb{H}^n_r$

For any immersed curve $\gamma : [0, L] \to \mathbb{H}^{n-r}$, we define the semi-Riemannian $(r+1)$-submanifold $\Upsilon_\gamma = \Phi(\gamma \times \mathbb{S}^r)$. It is clear that Υ_γ has index r and we will refer to Υ_γ as the tube over γ. Now let $G = SO(r+1)$ be the group of isometries of $(\mathbb{S}^r, -d\sigma^2)$. Obviously, G acts transitively on $(\mathbb{S}^r, -d\sigma^2)$. So we define an action of G on $\mathbb{H}^n_r$ as follows

$$a \cdot (\xi, \eta) = \Phi(a \cdot \Phi^{-1}(\xi, \eta)) = (a(\xi), \eta),$$

for any $a \in G$.

This action is realized through isometries of $(\mathbb{H}^n_r, h_0)$. The following statement characterizes the tubes over curves in $\mathbb{H}^{n-r}$ as symmetric points of the above mentioned G-action.

Proposition 5.1 *Let M be an $(r+1)$-dimensional submanifold in $\mathbb{H}^n_r$. Then M is G-invariant if and only if M is a tube Υ_γ over a certain curve γ in $\mathbb{H}^{n-r}$.*

Proof. Let M be a G-invariant submanifold of dimension $r+1$. For any $p \in M$, write $p = (\xi, \eta) = \Phi((x_0, x), u) = (x_0 u, x)$, where $u \in \mathbb{S}^r$. Now the G-orbit through p is given by

$$[p] = \{a \cdot p : a \in G\} = \{(x_0 a(u), x) : a \in G\} = (x_0 \mathbb{S}^r, x),$$

where we use that G acts transitively on $\mathbb{S}^r$. This proves that M is foliated by r-spheres, so that we can consider the orthogonal distribution to this foliation. Since it is one dimensional, we can integrate it to get a curve $\gamma(t) = (x_0(t), x(t))$ in $\mathbb{H}^{n-r}$ with $\Phi(\gamma(t) \times \mathbb{S}^r) = \Upsilon_\gamma = M$. The converse is trivial.

5.3. Critical points of $\mathcal{F}^r(\gamma) = \int_\gamma (\kappa^2)^{\frac{r+1}{2}} ds$

Now we deal with the functional

$$\mathcal{F}^r(\gamma) = \int_\gamma (\kappa^2)^{\frac{r+1}{2}} ds$$

defined on the manifold of regular closed curves (or curves satisfying given first order boundary data) in a given pseudo-Riemannian manifold, where r stands for any natural number (even though all computations also hold if r is a real number). Notice that we write the integrand in that form to point out that it is an even function of the curvature κ. Also $\mathcal{F}^1$ agrees with $\mathcal{G}$, which is the elastic energy functional for free elasticae.

Let $\gamma : I \subset \mathbb{R} \to \mathbb{S}^m$ be a unit speed curve in the unit m-sphere with curvatures $\{\kappa, \tau, \ldots\}$ and Frenet frame $\{T = \gamma', \xi_2, \ldots, \xi_m\}$. Given a variation $\Gamma := \Gamma(s,t) : I \times (-\varepsilon, \varepsilon) \to \mathbb{S}^m$ of γ, with $\Gamma(s,0) = \gamma(s)$, we have the associated variation vector field $W(s) = \frac{\partial \Gamma}{\partial t}(s,0)$ along γ. We will use the notation and terminology of [18]. Set $V(s,t) = \frac{\partial \Gamma}{\partial s}$, $W(s,t) = \frac{\partial \Gamma}{\partial t}$, $v(s,t) = |V(s,t)|$, $T(s,t) = \frac{1}{v}V(s,t)$, $\kappa(s,t) = |\nabla_T T|^2$, ∇ being the Levi-Civita connection of $\mathbb{S}^m$. The following lemma in [18] collects some basic facts which we will use to find the Euler-Lagrange equations relative to $\mathcal{F}^r$.

Lemma 5.2 *With the above notation, the following assertions hold:*

$$\begin{aligned}
[V,W] &= 0; \\
\frac{\partial v}{\partial t} &= \langle \nabla_T W, T\rangle v; \\
[W,T] &= -\langle \nabla_T W, T\rangle T; \\
[[W,T],T] &= T(\langle \nabla_T W, T\rangle)T; \\
\frac{\partial \kappa^2}{\partial t} &= 2\langle \nabla_T^2 W, \nabla_T T\rangle - 4\langle \nabla_T W, T\rangle \kappa^2 + 2\langle R(W,T)T, \nabla_T T\rangle,
\end{aligned}$$

R being the Riemann curvature tensor of $\mathbb{S}^m$.

To compute $\frac{\partial}{\partial t}\mathcal{F}^r(\gamma) = \frac{\partial}{\partial t}\mathcal{F}^r(\Gamma(s,t))$, we use this lemma and a standard argument involving integration by parts. Then we consider $\mathcal{F}^r$ defined on a manifold which only contains either regular closed curves or curves satisfying first order boundary data on $\mathbb{S}^m$ in order to drop out obvious boundary terms which appear in the expression of that variation. As a matter of fact, $\frac{\partial}{\partial t}\big|_{t=0}\mathcal{F}^r(\Gamma(s,t)) = 0$ allows us to get the following Euler equation, which characterizes the critical points of $\mathcal{F}^r$ on the quoted manifolds of curves:

$$\begin{aligned}
&(\kappa^2)^{(r-1)/2}\nabla_T^3 T \\
&+2\frac{\mathrm{d}}{\mathrm{d}s}((\kappa^2)^{(r-1)/2})\nabla_T^2 T \\
&+\{(\kappa^2)^{(r-1)/2} + \frac{\mathrm{d}^2}{\mathrm{d}s^2}((\kappa^2)^{(r-1)/2}) + \frac{2r+1}{r+1}(\kappa^2)^{(r+1)/2}\}\nabla_T T \\
&+\frac{2r+1}{r+1}\frac{\mathrm{d}}{\mathrm{d}s}((\kappa^2)^{(r+1)/2})T = 0.
\end{aligned}$$

From here and the Frenet equations for γ, we find the following characterization of the critical points of $\mathcal{F}^r$.

Proposition 5.3 *Let γ be a regular curve in $\mathbb{S}^m$ with curvatures $\{\kappa, \tau, \delta, \ldots\}$. Then γ is a critical point of*

$$\mathcal{F}^r(\gamma) = \int_\gamma (\kappa^2)^{(r+1)/2} ds$$

if and only if the following equations hold:

$$\begin{aligned} r\kappa'' + \frac{r}{r+1}\kappa^3 - \kappa\tau^2 + \kappa + \frac{r(r-1)}{\kappa}(\kappa')^2 &= 0, \\ (\kappa^2)^r \tau &= 0, \\ \delta &= 0. \end{aligned}$$

In particular, γ lies in some $\mathbb{S}^2$ or $\mathbb{S}^3$ totally geodesic in $\mathbb{S}^m$.

From now on we will call r-*generalized elasticae* to the critical points of $\mathcal{F}^r$. In particular, free elasticae are nothing but 1-generalized elasticae.

5.4. A key result

We are going to characterize the tubes in $(\mathbb{H}^n_r, h_0)$ which are Willmore-Chen submanifolds.

Theorem 5.4 *Let γ be a fully immersed closed curve in the hyperbolic space $\mathbb{H}^{n-r}$. The tube $\Upsilon_\gamma = \Phi(\gamma \times \mathbb{S}^r)$ in $(\mathbb{H}^n_r, h_0)$ is a Willmore-Chen submanifold if and only if γ is a generalized free elastica in the once punctured unit sphere $(\Sigma^{n-r}, d\sigma^2)$. In particular, $n - r \leqslant 3$.*

Proof. Given a closed curve γ in $\mathbb{H}^{n-r}$, let $\mathcal{Y}$ be the smooth manifold of all immersions of $\gamma \times \mathbb{S}^r$ in $(\mathbb{H}^n_r, h_0)$, i.e. $\mathcal{Y} = \{\phi : \gamma \times \mathbb{S}^r \to (\mathbb{H}^n_r, h_0) : \ \phi \text{ is an immersion}\}$. The Willmore-Chen functional on $\mathcal{Y}$ writes down as

$$\Omega(\phi) = \int_{\gamma\times\mathbb{S}^r} (\langle H, H\rangle - \tau_e)^{\frac{r+1}{2}} dv,$$

H and τ_e standing for the mean curvature vector field and the extrinsic scalar curvature function of ϕ, respectively, and dv being the volume element associated with the induced metric. Denote by $\tilde{\phi}$ the immersion ϕ when it is endowed with the induced structure coming from the metric $h = \frac{1}{f^2} h_0$. Since the Willmore-Chen variational problem and, in particular, the Willmore-Chen functional are invariants under conformal changes of the ambient metric, we have

$$\Omega(\tilde{\phi}) = \Omega(\phi).$$

Let $\mathcal{C}$ be the set of critical points of Ω on $\mathcal{Y}$, i.e., $\mathcal{C}$ is the set of Willmore-Chen immersions of $\gamma \times \mathbb{S}^r$ in $(\mathbb{H}^n_r, h_0)$. Let $\mathcal{Y}_G$ be the submanifold of $\mathcal{Y}$ made up by G-invariant immersions and $\mathcal{C}_G$ the set of critical points of Ω when it is restricted to $\mathcal{Y}_G$. By using again the principle of symmetric criticality of Palais, [21], we have

$$\mathcal{C} \cap \mathcal{Y}_G = \mathcal{C}_G.$$

Since $\mathcal{Y}_G$ in nothing but the set of tubes over closed curves in the hyperbolic space $\mathbb{H}^{n-r}$, that is $\mathcal{Y}_G = \{\Phi(\gamma \times \mathbb{S}^r) : \gamma \ \subset \mathbb{H}^{n-r}\}$, we can compute the restriction of Ω to $\mathcal{Y}_G$ to get

$$\begin{aligned} \Omega(\Phi(\gamma \times \mathbb{S}^r)) &= \frac{1}{(r+1)^{r+1}} \int_{\gamma\times\mathbb{S}^r} \left(\kappa^2\right)^{\frac{r+1}{2}} dv \\ &= \frac{\mathrm{vol}(\mathbb{S}^r, -d\sigma^2)}{(r+1)^{r+1}} \int_\gamma \left(\kappa^2\right)^{\frac{r+1}{2}} ds, \end{aligned}$$

where κ stands for the curvature function of γ in the once punctured unit sphere $(\Sigma^{n-r}, d\sigma^2) = (\mathbb{H}^{n-r}, \frac{1}{f^2} g_0)$. Notice that we have used the metric h on $\mathbb{H}^n_r$ to take advantage of the pseudo-Riemannian product structure of $(\mathbb{H}^n_r, h) = (\Sigma^{n-r}, d\sigma^2) \times (\mathbb{S}^r, -d\sigma^2)$. This proves the first part of the statement.

As for the second one, just combine Proposition 5.3 with the fullness assumption.

5.5. Some examples

In order to give examples of non trivial Willmore-Chen submanifolds in the pseudo-hyperbolic space $(\mathbb{H}^n_r, h_0)$, we apply Theorem 5.4. To do that, we start with a fully immersed closed curve γ in the hyperbolic $(n-r)$-space $\mathbb{H}^{n-r}$, with $n-r \leqslant 3$, and then we view $\mathbb{H}^{n-r}$ as a once punctured $(n-r)$-sphere $(\Sigma^{n-r}, d\sigma^2)$. Therefore, the first case we will consider is $n = 3$ and $r = 1$. Then Theorem 5.4 applied here writes down as follows:

Corollary 5.5 *Let γ be an immersed closed curve in the hyperbolic 2-plane. The Lorentzian tube $\Upsilon_\gamma = \Phi(\gamma \times \mathbb{S}^1)$ is a Willmore torus in the 3-dimensional anti De Sitter space $(\mathbb{H}^3_1, h_0)$ if and only if γ is a free elastica in the once punctured unit 2-sphere $(\Sigma^2, d\sigma^2)$.*

The complete classification of closed free elasticae in the standard 2-sphere was achieved by J.L. Langer and D.A. Singer in [18]. That classification can be briefly and geometrically described as follows: Up to rigid motions in the unit 2-sphere, the family of closed free elasticae consists of a geodesic γ_0, say the equator, and an integer two parameter family $\{\gamma_{m,n} : 0 < m < n,\ m, n \in \mathbb{Z}\}$, where $\gamma_{m,n}$ means that it closes up after n periods and m trips around the equator γ_0.

As a consequence we have

Corollary 5.6 *There exist infinitely many Lorentzian Willmore tori in the 3-dimensional anti De Sitter space. This family includes $\{\Upsilon_{\gamma_{m,n}} : 0 < m < n,\ m, n \in \mathbb{Z}\}$ and Υ_{γ_0}.*

A second case we will consider is $n - r = 3$. Then we are looking for critical points of $\mathcal{F}^r(\gamma)$, i.e., solutions of two first equations in Proposition 5.3 inside the family of helices in the standard once punctured 3-sphere $(\Sigma^3, d\sigma^2)$. (For details about the geometry of helices in the standard 3-sphere we refer to readers to [5]).

Let γ be a helix in $(\Sigma^3, d\sigma^2)$ with curvature κ and torsion τ. From now on we will assume that γ is a not a geodesic; otherwise, it is a trivial solution. Then γ is an r-generalized free elastica if and only if

$$\frac{r}{r+1}\kappa^2 - \tau^2 + 1 = 0. \tag{15}$$

That means that, in the (κ, τ)-plane of helices in $(\Sigma^3, d\sigma^2)$, $\mathcal{F}^r$ has exactly a hyperbola of critical points. To determine the closed helices which are r-generalized elasticae we use the following argument series (see [9]). First, take the usual Hopf fibration $\Pi : (\Sigma^3, d\sigma^2) \to (\mathbb{S}^2, ds^2)$, where the base space is chosen to be of radius $1/2$ in order to Π be a Riemannian submersion. Let β be an arclength parametrized curve with constant curvature $\rho \in \mathbb{R}$ into $(\mathbb{S}^2, ds^2)$. Let $S_\beta = \Pi^{-1}(\beta)$ be the Hopf tube over β (see [5] or [22] for details). Then S_β becomes a flat torus with constant mean curvature in $(\Sigma^3, d\sigma^2)$. Furthermore, it admits an obvious parametrization $\Psi(s,t)$ by means of fibres (s=constant) and horizontal lifts $\overline{\beta}$ of β (t=constant). If γ is a geodesic of S_β, with slope $\ell \in \mathbb{R}$ (slope measured with respect to Ψ), then γ is a helix in $(\Sigma^3, d\sigma^2)$ whose curvature κ and torsion τ are given by

$$\kappa = \frac{\rho + 2\ell}{1 + \ell^2}$$

and

$$\tau = \frac{1 - \rho\ell - \ell^2}{1 + \ell^2}.$$

Secondly, the converse also holds. Namely, given any helix γ in $(\Sigma^3, d\sigma^2)$, with curvature κ and torsion τ, it can be viewed as a geodesic in a certain Hopf tube of $(\Sigma^3, d\sigma^2)$. Indeed, let us just consider $S_\beta = \Pi^{-1}(\beta)$, β being a circle into $(\mathbb{S}^2, ds^2)$ with constant curvature $\rho = (\kappa^2 + \tau^2 - 1)/\kappa$ and take now a geodesic in S_β with slope $\ell = (1 - \tau)/\kappa$.

Thirdly, let L and A be the length of β and the enclosed oriented area by β in $(\mathbb{S}^2, d\sigma^2)$, respectively. As U. Pinkall showed, [22], S_β is isometric to $\mathbb{R}^2/\Gamma$, Γ being the lattice generated by $(L, 2A)$ and $(0, 2\pi)$. We notice that, due to the holonomy, the horizontal lifts of β are not closed in $(\Sigma^3, d\sigma^2)$. Now the helix γ lying in S_β is closed if and only if

$$\ell = q\sqrt{\rho^2 + 4} - \frac{\rho}{2},$$

where q is a non zero rational number (otherwise, $\kappa = 0$ and γ would be a geodesic in $(\Sigma^3, d\sigma^2)$, and therefore a trivial critical point of $\mathcal{F}^r$), and ρ is the curvature of β.

Finally, let ρ and q be any real number and any non zero rational number, respectively. Then we use (18) to get the slope, and (16) and (17) to compute the curvature κ and torsion τ of a closed helix γ in $(\Sigma^3, d\sigma^2)$. Moreover, γ will be an r-generalized free elastica provided that κ and τ satisfy (15). Therefore ρ and q satisfy

$$(r\rho + 2(2r + 1)\ell - (r + 1)\rho\ell^2)(\rho + 2\ell) = 0.$$

Since γ was assumed to be non geodesic, we have that $\rho + 2\ell \neq 0$. So we bring (18) to the equation $r\rho + 2(2r + 1)\ell - (r + 1)\rho\ell^2 = 0$ to get

$$\begin{aligned} (r+1)^2(q^2 - \frac{1}{4})^2\rho^4 \\ +4(r+1)\{(r+1)q^4 - \frac{1}{2}(3r+1)q^2 + \frac{1}{16}(r+1)\}\rho^2 \\ -4q^2(2r+1) &= 0. \end{aligned} \tag{20}$$

From here we see that for any non zero rational number q we have exactly one positive solution ρ^2 of the quoted quadratic equation. That can be summed up in the following:

Theorem 5.7 *For any natural number r, there exists a one parameter family $\{\gamma_q\}_{q \in \mathbb{Q} \setminus \{0\}}$ of closed helices in $(\Sigma^3, d\sigma^2)$ which are r-generalized free elastica.*

Remark 5.8 From $r\rho + 2(2r + 1)\ell - (r + 1)\rho\ell^2 = 0$ we easily see how to get q in terms of ρ. It suffices to write the quadratic equation

$$(r + 1)\rho\sqrt{\rho^2 + 4}q^2 - ((r + 1)\rho^2 + 2(2r + 1))q + \frac{r + 1}{4}\rho\sqrt{\rho^2 + 4} = 0.$$

We already know that for any non zero rational number q, we have exactly one ρ (up to the sign). In contrast, the last formula says that each ρ determine exactly two values of q, unless $\rho^2 = (2r+1)^2/(r(r+1))$, which corresponds to $q = \pm\frac{1}{2}$. Since both values of q are rational, that means that the corresponding Hopf tori in $(\Sigma^3, d\sigma^2)$ have transverse foliations by closed free elastic helices. As a consequence we obtain

Theorem 5.9 *Let r be any natural number. For any non zero rational number q, there exists an $(r+1)$-dimensional Willmore-Chen submanifold $\Upsilon_\gamma = \Phi(\gamma \times \mathbb{S}^r)$ in the pseudo-hyperbolic space $(\mathbb{H}_r^{r+3}, h_0)$, γ being an r-generalized free elastic closed helix in the once punctured unit 3-sphere $(\Sigma^3, d\sigma^2)$ whose slope ℓ is computed as above.*

6. A general approach

We are going to extend the argument we have used in the last section to construct Willmore-Chen submanifolds in the pseudo-hyperbolic space.

Let (M_1, g_1) and (M_2, g_2) be two pseudo-Riemannian manifolds of dimension n_1 and n_2, respectively. Given a positive function f defined on M_1 (we can assume $\inf(f) > 0$ if M_1 is not compact), define the warped product $M_1 \times_f (\varepsilon M_2)$, that is, the product manifold $M_1 \times M_2$ endowed with the metric tensor $g = g_1 + \varepsilon f^2 g_2$, where $\varepsilon = \pm 1$, f being the warping function. We simply write $M = M_1 \times_f (\varepsilon M_2)$ when the involved metrics are understood. From now on (M_2, g_2) will be a homogeneous space and G its isometry group. This action can be naturally extended to M by defining

$$\begin{array}{rcl} M \times G & \to & M \\ ((m_1, m_2), a) & \to & (m_1, m_2)\cdot a = (m_1, m_2 \cdot a), \end{array}$$

for any $(m_1, m_2) \in M$ and $a \in G$. As the action of G on M is transitive, the orbit of any point $m \in M$ is nothing but $[m] = \{m_1\} \times M_2$. Then a preliminary result states as follows.

Proposition 6.1 *Let N be a submanifold of M of dimension n_2+1. Then N is G-invariant if and only if there exists a curve γ in M_1 such that $N = \gamma \times_f (\varepsilon M_2)$.*

Proof. It is easy to see that any submanifold $\gamma \times_f (\varepsilon M_2)$ is G-invariant. Conversely, assume that N is G-invariant. Then the orbit $[p] = \{m_1\} \times M_2$ of any $p = (m_1, m_2) \in N$ is a n_2-dimensional submanifold of N. This proves that N is foliated whose leaves are totally umbilic submanifolds in (M, g) all of them diffeomorphic to M_2. In other words, the leaves of this foliation are nothing but the fibres of the warped product (M, g) along N. The transverse (orthogonal) distribution, being of dimension one, can be integrated. Therefore, we can choose a curve γ in M_1 such that the submanifold N writes down as $N = \gamma \times_f (\varepsilon M_2)$, which concludes the proof.

Now the main result states as follows.

Theorem 6.2 *Let $(M, g) = (M_1, g_1) \times_f (M_2, \varepsilon g_2)$ be a warped product, where (M_2, g_2) is a compact homogeneous space of dimension n_2. Let γ be an immersed closed curve in (M_1, g_1). The submanifold $N = \gamma \times_f (\varepsilon M_2)$ is a Willmore-Chen submanifold in (M, g) if and only if γ is a $\frac{n_2+1}{2}$-generalized free elastica in $(M_1, \frac{1}{f^2} g_1)$. That means that γ is a critical point of the functional*

$$\mathcal{F}^{n_2}(\gamma) = \int_\gamma (\kappa^2)^{\frac{n_2+1}{2}} ds,$$

κ being the curvature function of γ into $(M_1, \frac{1}{f^2}g_1)$.

Proof. Since the Willmore-Chen variational problem is invariant under conformal changes of the metric of the ambient space, we are allowed to consider a new metric $\tilde{g}$ on M defined by

$$\tilde{g} = \frac{1}{f^2}g = \frac{1}{f^2}g_1 + \varepsilon g_2.$$

Therefore, Willmore-Chen submanifolds in (M, g) and $(M, \tilde{g})$ agree. Moreover, we will profit by the pseudo-Riemannian product structure of $(M, \tilde{g})$. Let us denote by $\mathcal{N}$ the smooth manifold of $(n_2 + 1)$-dimensional compact submanifolds in $(M, \tilde{g})$. The Willmore-Chen functional on $\mathcal{N}$ writes down

$$\Omega(N) = \int_N (\langle H, H\rangle - \tau_e)^{\frac{n_2+1}{2}} dv,$$

H and τ_e standing for the mean curvature vector field and the extrinsic scalar curvature of N in $(M, \tilde{g})$, respectively, and dv is the volume element of N relative to the induced metric. Now set $\mathcal{N}_G$ the submanifold of $\mathcal{N}$ made up by those submanifolds which are G-invariant. By Proposition 6.1, we already know that $\mathcal{N}_G = \{\gamma \times_f (\varepsilon M_2) : \ \gamma$ is an immersed closed curve in $M_1\}$. Similarly, let $\mathcal{C}$ and $\mathcal{C}_G$ be the set of critical point of Ω on $\mathcal{N}$ (i.e., the set of Willmore-Chen submanifolds) and on $\mathcal{N}_G$, respectively. The principle of symmetric criticality of R.S. Palais, [21], can be applied here, because Ω is invariant under the action of G on (M, g). Observe that G acts through isometries, so that f has no influence. Hence $\mathcal{C} \cap \mathcal{N}_G = \mathcal{C}_G$. Now we are going to compute Ω on $\mathcal{N}_G$. First Ω writes down as

$$\Omega(\gamma \times_f (\varepsilon M_2)) = \int_{\gamma \times M_2} (\langle H, H\rangle - \tau_e)^{\frac{n_2+1}{2}} ds\ dv_2,$$

where ds stands for the arclength element of γ into $(M_1, \frac{1}{f^2}g_1)$ and dv_2 is the volume element of $(M_2, \varepsilon g_2)$. As $(M, \tilde{g})$ is a pseudo-Riemannian product, it is not difficult to see that τ_e vanishes identically and $\langle H, H\rangle = \frac{1}{(n_2+1)^2}\kappa^2$, κ being the curvature function of γ in $(M_1, \frac{1}{f^2}g_1)$. Thus we obtain

$$\Omega(\gamma \times_f (\varepsilon M_2)) = \frac{\mathrm{vol}(M_2, \varepsilon g_2)}{(n_2+1)^{n_2+1}} \int_\gamma (\kappa^2)^{\frac{n_2+1}{2}} ds,$$

which finishes the proof.

7. Some applications

7.1. Let g_1 be any conformally flat Lorentzian metric on a torus $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$. It is known that (T^2, g_1) is complete, which is not guaranteed, in the realm of Lorentzian geometry, from the compactness of T^2 (see for instance [23]). Let us denote by g_0 the conformal flat metric. There is a positive function, say $f : T^2 \to \mathbb{R}$, such that $g_1 = f^2 g_0$. Set $M^3 = T^2 \times \mathbb{S}^1$ endowed with the pseudo-Riemannian metric $g = g_1 + \varepsilon f^2 dt^2$, where $\varepsilon = \pm 1$ and dt^2 denotes the canonical metric on the unit circle $\mathbb{S}^1$. The following statement shows the existence of Willmore tori in the 3-dimensional pseudo-Riemannian manifold (M, g), which topologically is the product of three circles.

Corollary 7.1 *Let $M^3 = T^2 \times \mathbb{S}^1$ endowed with the metric $g = g_1 + \varepsilon f^2 dt^2$, where g_1 is any conformally flat Lorentzian metric on T^2 and f the positive function on T^2 giving this conformal flatness (g_0 being flat and $g_1 = f^2 g_0$). Then $\Upsilon = \gamma \times_f (\varepsilon\mathbb{S}^1)$ is a Willmore torus in (M^3, g) if and only if γ is a closed free elastica in the Lorentzian flat torus (T^2, g_0).*

The proof is a straightforward computation from Theorem 6.2. Furthermore, one can construct closed free elasticae in (T^2, g_0) from free elasticae in the Lorentz-Minkowski 2-plane $\mathbb{L}^2$ (see [6]).
7.2. Let $(\mathbb{H}^3_1, g_0)$ be the standard anti De Sitter 3-space. Given any positive function $f : \mathbb{H}^3_1 \to \mathbb{R}$, consider the metric $g_f = f^2 h_0$. Let (M, g) be the pseudo-Riemannian product manifold $M = \mathbb{H}^3_1 \times M_2$ endowed with the metric $g = g_f + f^2 \varepsilon g_2$, (M_2, g_2) being any compact homogeneous space. Then $\gamma \times M_2$ is a Willmore-Chen submanifold in (M, g) if and only if γ is a $\frac{n_2+1}{2}$-generalized closed free elastica in $(\mathbb{H}^3_1, g_0)$.

As above, one can find $\frac{n_2+1}{2}$-generalized closed free elasticae in $(\mathbb{H}^3_1, g_0)$ for any non zero rational number (see [6] again).
7.3. We will get 3-dimensional pseudo-Riemannian manifolds (either Riemannian or Lorentzian) admitting a foliation whose leaves are non trivial Willmore tori (either Riemannian or Lorentzian). These foliations will be called *Willmore foliations*. Also we will say that the pseudo-Riemannian manifold is *Willmore foliated*. To do that we start with an immersed plane curve $\gamma : I \subset \mathbb{R} \to \mathbb{R}^2$ and a pair of positive functions $f_1, f_2 : I \to \mathbb{R}$. Let $M = \gamma \times \mathbb{S}^1 \times \mathbb{S}^1$ endowed with the metric given by $g = ds^2 + f_1^2 dt_1^2 + \varepsilon f_2^2 dt_2^2$, keeping the above terminology. It is clear that g is conformal to the pseudo-Riemannian product metric $\tilde{g}$ defined by $\tilde{g} = g_0 + \varepsilon dt_2^2$ on the manifold $M = N \times \mathbb{S}^1$, where $g_0 = dt^2 + (\frac{f_1}{f_2})^2 dt_1^2$ and $N = \gamma \times \mathbb{S}^1$. Notice that we have reparametrized γ by $\frac{ds}{dt} = f_1(s)$. We can now make a suitable choice of both f_1 and f_2 along γ in order to view (N, g_0) as a surface of revolution in $\mathbb{R}^3$. On the other hand, the elasticity of parallels in a surface of revolution was yet discussed in [8]. There it was shown that, besides right cylinders (all whose parallels are geodesics and therefore trivial free elastic curves), the only surfaces whose parallels are all free elasticae are the trumpet surfaces (which are free of geodesic parallels, see [8] for details). Then we have:

Corollary 7.2 *Let (b, c) be a pair of real numbers, with $c > 0$. Set $I = (-\frac{2}{c}, \frac{2}{c}) - \{0\}$ and define $\gamma : I \subset \mathbb{R} \to \mathbb{R}^2$ by*

$$\gamma(s) = \left(\frac{c}{4}s^2, \frac{c}{2}\sqrt{1 - \frac{c^2}{4}s^2} - \frac{1}{c}\text{arc}\cos\frac{c}{2}s + b \right).$$

Let $f_1, f_2 : I \to \mathbb{R}^2$ be two positive functions satisfying $\frac{f_1}{f_2}(s) = \frac{c}{4}s^2$. Then $M = \gamma \times \mathbb{S}^1 \times \mathbb{S}^1$, endowed with the metric $g = ds^2 + f_1^2 dt_1^2 + \varepsilon f_2^2 dt_2^2$, admits a Willmore foliation which is either Riemannian or Lorentzian, according to ε is 1 or -1, respectively.

7.4. The construction we made in 7.1 can be extended as follows. Let (M, g) be any compact Riemannian manifold and f any positive smooth function on M. Let $N = M \times \mathbb{S}^1$ endowed with the metric $g_f = g + \varepsilon f^2 dt^2$. From Theorem 6.2 and a remarkable result of N.Koiso, [16], we have the following existence result for Willmore tori.

Corollary 7.3 *There exist Willmore tori in (N, g_f) for any positive smooth function f on M.*

Proof. We first apply Theorem 6.2 to (N, g_f). Given a closed curve γ immersed in (M, g), then $\gamma \times_f (\varepsilon\mathbb{S}^1)$ is a Willmore torus in (N, g_f) if and only if γ is an elastica into $(M, \frac{1}{f^2}g)$. Now the existence of these curves in any compact Riemannian manifold is guaranteed by Koiso's result [16].

It should be noticed that the elastica in $(M, \frac{1}{f^2}g)$ could be a closed geodesic. In fact, elasticae appear as stationary solutions of a parabolic partial differential equation. The existence of such a solution on the space of closed curves of fixed length is proved in [16]. This solution can be a geodesic since geodesics are singular, stationary solutions of that equation.

7.5. The last application will show once more how powerful is our method.

Corollary 7.4 *For any positive function f on a genus zero Riemann surface M, there exist at least three conformal minimal (maximal if $\varepsilon = -1$) Willmore tori in $M \times_f (\varepsilon\mathbb{S}^1)$ which are embedded.*

This assertion comes easily from Theorem 6.2 combined with the following new ingredient. A very classical result of L.Lusternik and L.Schnirelmann, [19], guarantees the existence of at least three geodesics without self-intersections on any simply connected Riemannian surface M. That means that $(M, \frac{1}{f^2}g)$ has at least three closed geodesics without self-intersections, which we will denote by $\gamma_i, i = 1, 2, 3$. Then $N_i = \gamma_i \times_f (\varepsilon\mathbb{S}^1), i = 1, 2, 3$ are embedded Willmore tori in $M \times_f (\varepsilon\mathbb{S}^1)$. Of course they are minimal (maximal if $\varepsilon = -1$) with regard to the pseudo-Riemannian metric $\frac{1}{f^2}g + \varepsilon dt^2$.

It should be noticed that this is the best possible result. In fact, just choose $(M, \frac{1}{f^2}g)$ to be an ellipsoid. Then it has exactly three embedded closed geodesics. In this case we can obtain exactly three conformal minimal (maximal) Willmore tori in $M \times_f (\varepsilon\mathbb{S}^1)$ which are embedded. Actually, we can obtain infinitely many others immersed Willmore tori.

We also observe that we have essentially covered the whole space of metrics on M. Indeed, a nice consequence of the Uniformization Theorem for Riemann surfaces ensures the existence of exactly one conformal structure and therefore only one conformal class of metrics.

Finally, recall that compact minimal (maximal) surfaces are always Willmore surfaces only if the ambient space has constant sectional curvature.

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