Biharmonic Hopf cylinders

Angel Ferrández, Pascual Lucas and Miguel Angel Meroño
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1. Introduction

This paper concerns curves and surfaces, into indefinite space forms, whose mean curvature vector field is in the kernel of certain elliptic differential operators. It has been inspired by the paper of M. Barros and O.J. Garay [**2**], where the Riemannian version of this question is solved. We first consider the Laplacian to study indefinite submanifolds with harmonic mean curvature vector field in the normal bundle. This problem is closely related to a conjecture of B.-Y. Chen [**5**], on Riemannian submanifolds, stated as follows: harmonicity of the mean curvature vector field implies harmonicity of the immersion. Submanifolds with harmonic mean curvature vector field were called by Chen biharmonic submanifolds. In the realm of indefinite submanifolds, counterexamples to that conjecture have been given by the two first authors (see [**1**]). Biharmonic submanifolds are a special class of submanifolds for which its mean curvature vector is an eigenvector of Δ, that is, $\Delta H = \lambda H$ for some real constant λ. First we describe the family of curves whose mean curvature vector field is proper for the Laplacian. This problem has been yet solved for Euclidean curves by M. Barros and O.J. Garay [**2**]. We have to think of a different Laplacian if we want to characterize curves others than those of both constant curvature and torsion. Since H is a normal vector field, it seems natural to consider the Laplacian associated to the connection in the normal bundle. Then we show that the indefinite Cornu spirals are the only non standard curves in a semi-Riemannian manifold that are biharmonic in the normal bundle. As for surfaces, we deal with the semi-Riemannian Hopf cylinders we introduced in [**4**]. Then we show that the biharmonicity of them strongly depends on the biharmonicity of the curves to which are associated. In fact, a non standard Hopf cylinder in $\mathbb{H}^3_1(-1)$ is biharmonic in the normal bundle if and only if it is associated to a Cornu spiral in $\mathbb{H}^2_s(-4)$. Then we extend the results in [**2**].

The second operator considered is the Jacobi operator, which was introduced by J. Simons [**9**] and involves the Laplacian in the normal bundle. A normal vector field is called a Jacobi field if it belongs to the kernel of the Jacobi operator. This operator appears when one studies the second variation of the area functional for compact Riemannian minimal submanifolds. It has been recently used by Barros and Garay [**3**] to classify Hopf cylinders into $\mathbb{S}^3$ with Jacobi mean curvature vector field. In [**4**] we have made, following [**7**], a qualitative description of elastic curves into indefinite space forms to be used as a tool to find Lorentzian Willmore tori in $\mathbb{H}^3_1(-1)$. Now the Jacobi operator allows us to get a characterization of elastic curves, as well as a characterization of semi-Riemannian Hopf cylinders in $\mathbb{H}^3_1(-1)$, in terms of elasticae in $\mathbb{H}^2_s(-4)$, $s = 0, 1$. We show that a curve in an indefinite real space form has Jacobi mean curvature vector field if and only if it is curvature homothetic to a free elastica. As before, this characterization leads to find Hopf cylinders into $\mathbb{H}^3_1(-1)$ with Jacobi mean curvature vector field.

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2. Curves with harmonic mean curvature vector field

Let γ be an arclength parametrized curve isometrically immersed in an indefinite real space form M^n_ν of constant curvature c. As usual, the metric on M^n_ν will be denoted by $\langle , \rangle$ and the Riemannian connection by ∇. Assume that γ does not lie in a 2-dimensional totally geodesic submanifold of M^n_ν. Let $\kappa > 0$ and τ be the curvature and torsion functions of γ and $\{T = \gamma', \xi_2, \xi_3\}$ a Frenet frame along γ. The Frenet equations for γ can be partially written as

$$\nabla_T T = \varepsilon_2 \kappa \xi_2, \tag{1}$$
$$\nabla_T \xi_2 = -\varepsilon_1 \kappa T - \varepsilon_3 \tau \xi_3, \tag{2}$$
$$\nabla_T \xi_3 = \varepsilon_2 \tau \xi_2 + \delta, \tag{3}$$

where $\delta \in \mathrm{span}\{T, \xi_2, \xi_3\}^\perp$ and $\varepsilon_i, i = 1, 2, 3$, are the causal characters of T, ξ_2 and ξ_3,respectively. The Laplacian operator along γ is given by $\Delta = -\varepsilon_1 \nabla_T \nabla_T$.

Let σ be the second fundamental form associated to γ. Then the mean curvature vector field H is defined by

$$H = \mathrm{tr}(\sigma) = \varepsilon_1 \sigma(T, T) = \varepsilon_1 \nabla_T T = \varepsilon_1 \varepsilon_2 \kappa \xi_2.$$

Taking covariant derivative of H with respect to T, and using the Frenet equations, we obtain

$$\nabla_T H = -\varepsilon_2 \kappa^2 T + \varepsilon_1 \varepsilon_2 \kappa' \xi_2 - \varepsilon_1 \varepsilon_2 \varepsilon_3 \kappa \tau \xi_3.$$

The second covariant derivative of H yields

$$\begin{aligned} \Delta H = {} & (3\varepsilon_1 \varepsilon_2 \kappa \kappa')T + (-\varepsilon_2 \kappa'' + \varepsilon_1 \kappa^3 + \varepsilon_3 \kappa \tau^2)\xi_2 \\ & + \varepsilon_2 \varepsilon_3 (2\kappa' \tau + \kappa \tau')\xi_3 + \varepsilon_2 \varepsilon_3 \kappa \tau \delta. \end{aligned} \tag{4}$$

Then $\Delta H = \lambda H$, $\lambda \in \mathbb{R}$, if and only if the following equations hold

$$\kappa \kappa' = 0, \tag{5}$$
$$\varepsilon_2 \kappa'' - \varepsilon_1 \kappa^3 - \varepsilon_3 \kappa \tau^2 + \lambda \varepsilon_1 \varepsilon_2 \kappa = 0. \tag{6}$$
$$2\kappa' \tau + \kappa \tau' = 0, \tag{7}$$
$$\kappa \tau \delta = 0. \tag{8}$$

From the assumption on γ, equation (8) implies that $\delta = 0$ and so the curve γ lies in a 3-dimensional totally geodesic submanifold of M^n_ν. Hence we can assume without loss of generality that $n = 2$ or $n = 3$. On the other hand, from (7) we deduce that $\kappa^2 \tau$ is a constant. As a consequence we have the following result.

Proposition 2.1 *Let γ be a unit speed curve in M^n_ν. Then $\Delta H = \lambda H$ if and only if one of the following statements holds:*
(1) γ *is a geodesic.*
(2) γ *is a small pseudocircle or pseudohyperbola in a 2-dimensional totally geodesic submanifold of M^n_ν.*
(3) γ *is a helix in a 3-dimensional totally geodesic submanifold of M^n_ν.*

Proof. From (5) we can assume that κ is constant. If $\kappa = 0$ then γ is a geodesic; otherwise, since $\kappa^2\tau$ is constant then τ is also constant. Therefore we obtain (2) or (3) according to $\tau = 0$ or $\tau \neq 0$, respectively. Conversely, it is easy to show that all curves in the proposition satisfy the required condition for appropriate λ.

This proposition shows that we must work with a different Laplacian if we want to characterize curves others than those of constant curvature. Since H is a normal vector field, it is natural to think of the Laplacian Δ^D associated to the connection D in the normal bundle, defined by $\Delta^D = -\varepsilon_1 D_T D_T$.

A straightforward computation leads to

$$\Delta^D H = (-\varepsilon_2\kappa'' + \varepsilon_3\kappa\tau^2)\xi_2 + \varepsilon_2\varepsilon_3(2\kappa'\tau + \kappa\tau')\xi_3 + \varepsilon_2\varepsilon_3\kappa\tau\delta. \tag{9}$$

Then $\Delta^D H = \lambda H$, $\lambda \in \mathbb{R}$, is equivalent to the set of equations (7), (8) and the following

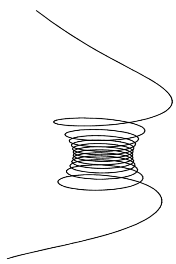
$$\varepsilon_2\kappa'' - \varepsilon_3\kappa\tau^2 + \varepsilon_1\varepsilon_2\lambda\kappa = 0.$$

Recall that a curve $\gamma : I \to M$ is said to be a *Cornu spiral* if its curvature κ is a non-constant linear function.

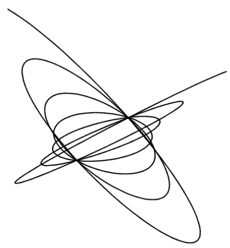
Proposition 2.2 *Let γ be a unit speed curve in M^2_ν. Then $\Delta^D H = \lambda H$, $\lambda \in \mathbb{R}$, if and only if one of the following statements holds:*
(1) $\lambda = 0$ *and κ is a linear function. So γ is a geodesic, a pseudocircle, a pseudohyperbola or a Cornu spiral.*
(2) $\varepsilon_1\lambda > 0$ *and κ is given by* $\kappa(s) = a\cos(\sqrt{\varepsilon_1\lambda}s) + b\sin(\sqrt{\varepsilon_1\lambda}s)$.
(3) $\varepsilon_1\lambda < 0$ *and κ is given by* $\kappa(s) = a\exp(\sqrt{-\varepsilon_1\lambda}s) + b\exp(-\sqrt{-\varepsilon_1\lambda}s)$.

Proof. Since the torsion $\tau = 0$, then $\Delta^D H = \lambda H$ if and only if $\kappa'' + \varepsilon_1\lambda\kappa = 0$. Then it suffices to integrate that differential equation.

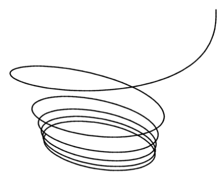
The behaviour of these curves in $\mathbb{H}^2_1(-1)$ can be sketched as follows.



Cornu spiral

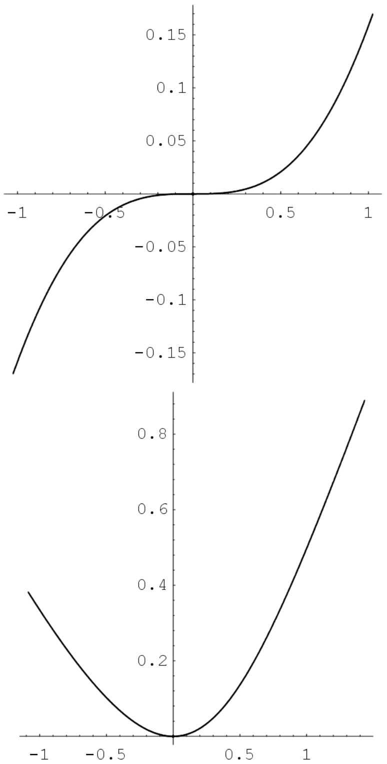


$\kappa(s) = \cos(s)$

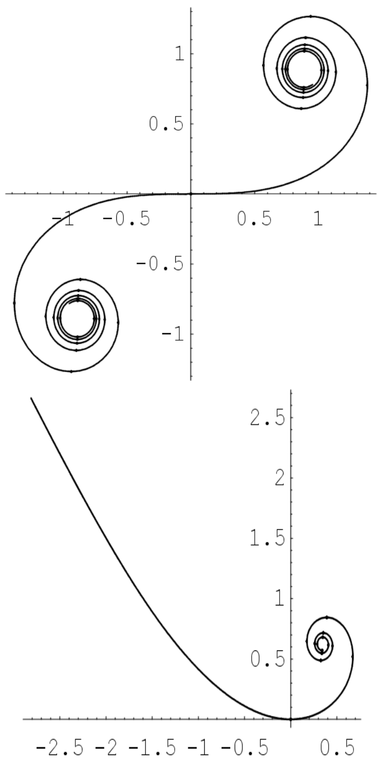


$\kappa(s) = \exp(s)$

The curves characterized in Proposition 2.2 are quite different in the Lorentzian plane $\mathbb{L}^2$ with respect to the Euclidean plane $\mathbb{R}^2$. In fact, the curvature function $\kappa(s)$ in $\mathbb{R}^2$ gives the rate of change of the Euclidean angle between the tangent vector and a fixed vector, whereas in $\mathbb{L}^2$ it gives the corresponding rate of change for the hyperbolic angle. The following pictures show those differences.



$\kappa(s) = \exp(s)$ in $\mathbb{L}^2$



$\kappa(s) = \exp(s)$ in $\mathbb{R}^2$

To solve the n-dimensional case, we can suppose that γ lies in a 3-dimensional totally geodesic submanifold of M^n_ν. Then if we take $u = k^2$, (10) can be rewritten as

$$(u')^2 + 4(\varepsilon_1\lambda u^2 - \varepsilon_2 bu + \varepsilon_2\varepsilon_3 a^2) = 0,$$

where b is a constant. By integrating (11) we obtain the following solutions:

i) If $\varepsilon_1\lambda < 0$, then u is a root of $Au^2+Bu+C = 0$, where $A = 48\lambda^2$, $B = 16\varepsilon_1\lambda(\exp(2\sqrt{-\varepsilon_1\lambda}s) - 3\varepsilon_2 b)$ and $C = (\exp(2\sqrt{-\varepsilon_1\lambda}s) - 4\varepsilon_2 b)^2 - 16\varepsilon_1\varepsilon_2\varepsilon_3 a^2\lambda$.

ii) If $\varepsilon_1\lambda < 0$, then u is a root of $Au^2 + Bu + C = 0$, where $A = 4\lambda^2(1 + \tan^2(2\sqrt{\varepsilon_1\lambda}s))$, $B = -4\varepsilon_1\varepsilon_2\lambda b(1 + \tan^2(2\sqrt{\varepsilon_1\lambda}s))$ and $C = b^2 + 4\varepsilon_1\varepsilon_2\varepsilon_3 a^2\lambda\tan^2(2\sqrt{\varepsilon_1\lambda}s)$.

iii) If $\lambda = 0$, then κ and τ are given by

$$\begin{aligned} \kappa(s) &= \sqrt{\varepsilon_2 bs^2 + \varepsilon_3\frac{a^2}{b}}, \\ \tau(s) &= \frac{\varepsilon_2 ab}{b^2s^2 + \varepsilon_2\varepsilon_3 a^2}. \end{aligned} \tag{12}$$

The integration of (11) shows that the case $\lambda = 0$ is the most interesting one, so that working on the equations in (12) we obtain

$$\frac{\varepsilon_2\varepsilon_3}{\kappa^2} + \frac{(\kappa')^2}{\tau^2\kappa^4} = \frac{\varepsilon_2 b}{a^2}, \tag{13}$$

provided that $\kappa\tau \neq 0$.

Lemma 2.3 *Let γ be a curve with curvature κ and torsion τ satisfying the equation (13). Then γ lies in a hyperquadric Q defined by the equation $\langle x - p_0, x - p_0\rangle = \varepsilon_2\varepsilon_3 b/a^2$.*

Proof. Let γ be the curve defined by

$$\gamma = \gamma + \frac{\varepsilon_1}{k}\xi_2 + \frac{\varepsilon_1\varepsilon_2\kappa'}{\tau\kappa^2}\xi_3.$$

Then we get

$$\gamma' = \varepsilon_1\left[\varepsilon_2\left(\frac{\kappa'}{\tau\kappa^2}\right)' - \varepsilon_3\frac{\tau}{\kappa}\right]\xi_3,$$

so that taking covariant derivative in (13) we deduce that $\gamma' = 0$. Hence there exists a point p_0 such that $\gamma(s) = p_0$, for any s. Now it is easy to show that $\gamma(s) \in Q$, for any s.

Proposition 2.4 *Let γ be a unit speed curve in M_ν^n. Then the curvature κ and the torsion τ of γ in M_ν^n are given by (12) if and only if γ is a Cornu spiral in a totally umbilical hypersurface of M_ν^n.*

Proof. Let P be the totally umbilical hypersurface of M_ν^n obtained by taking the intersection with the hyperquadric Q given in the above lemma. Then the shape operator S of P in M_ν^n is given by $S = a/\sqrt{|b|}I$. Let ρ denote the curvature of γ in P, that is, $\rho^2 = \iota\left\langle\nabla_T^P T, \nabla_T^P T\right\rangle$, where ∇^P stands for the Levi-Civita connection on P and ι denotes the causal character of $\nabla_T^P T$. The Gauss formula and (13) lead to

$$\rho = \frac{a}{\sqrt{|b|}}\,\frac{\kappa'}{\tau\kappa},$$

showing that $\rho(s)$ linearly depends on s.

Assume now that $\gamma : I \to P \subset M_\nu^n$ is a Cornu spiral. Let $Q = \{x : \langle x - p_0, x - p_0\rangle = \varepsilon r^2\}$, $\varepsilon = \pm 1$, be the hyperquadric such that $P = M_\nu^n \cap Q$ and put $\rho(s) = as + b$. Then we have (13) along with

$$\rho = \frac{1}{r}\,\frac{\kappa'}{\tau\kappa} \qquad \text{and} \qquad \varepsilon_2\kappa^2 = \iota\rho^2 + \frac{\varepsilon}{r^2}.$$

These equations imply (12).

The following theorem classifies all biharmonic curves in the normal bundle and extends a result in [**2**, Theorem 1].

Theorem 2.5 *Let γ be a unit speed curve in M_ν^n. Then the mean curvature field of γ is harmonic in the normal bundle if and only if one of the following statements holds:*
(1) $n = 2$ *and γ is a geodesic.*
(2) $n = 2$ *and γ is a pseudocircle or a pseudohyperbola.*
(3) $n = 2$ *and γ is a Cornu spiral.*
(4) $n = 3$ *and γ is a Cornu spiral in a totally umbilical surface.*

Proof. If γ lies in a 2-dimensional totally geodesic submanifold, then it reduces to Proposition 2.2. Otherwise, we may assume $n = 3$. Moreover, the curvature κ and the torsion τ are given by (12), so we apply Proposition 2.4. The converse is clear.

To finish this section, let $x_i : M_i \to \tilde{M}_i$, $i = 1, \dots, m$, be isometric immersions and consider $x = x_1 \times \cdots \times x_m : M = \prod_i M_i \to \prod_i \tilde{M}_i$ the product isometric immersion with mean curvature vector field H. Let Δ_i^D be the Laplacian in the normal bundle associated to x_i and consider Δ^D the corresponding operator for x. Then we have

$$\Delta^D(\xi_1, \dots, \xi_m) = (\Delta_1^D\xi_1, \dots, \Delta_m^D\xi_m),$$

where ξ_i is a normal vector field to M_i in $\tilde{M}_i$. As a consequence of Theorem 2.5 we have the following.

Corollary 2.6 *Let $\gamma_i : I_i \to M^{n_i}_{\nu_i}$, $i = 1, \ldots, m$, be unit speed curves in indefinite real space forms and consider the isometric immersion $x = \gamma_1 \times \cdots \times \gamma_m$. Then H is harmonic in the normal bundle if and only if, for any index i, we have either $n_i = 2$ and γ_i is a geodesic, a pseudocircle, a pseudohyperbola or a Cornu spiral, or $n_i = 3$ and γ_i is a Cornu spiral in a totally umbilical surface of $M^3_{\nu_i}$.*

Proof. It reduces to show that $mH = (H_1, \ldots, H_m)$, H_i being the mean curvature vector field associated to γ_i.

Now we are going to characterize the hypercylinders with harmonic mean curvature field H in the normal bundle, which is an extension of [**2**, Corollary 1].

Corollary 2.7 *Let $\gamma : I \to M^n_\nu$ be a unit speed curve and consider the hypercylinder $x = \gamma \times Id : I \times \mathbb{R}^m_t \to M^n_\nu \times \mathbb{R}^m_t$. Then H is harmonic in the normal bundle if and only if either $n \leqslant 3$ and γ is a geodesic, a pseudocircle, a pseudohyperbola, a Cornu spiral, or a Cornu spiral in a totally umbilical surface.*

3. Curves with Jacobi mean curvature vector field

Let $\gamma : I \to M^n_\nu$ be as in the above section and consider the following functional

$$\mathfrak{F}^\mu(\gamma) = \int_0^L (\langle \nabla_T T, \nabla_T T \rangle + \mu) ds,$$

where μ, L and ds stand for a real constant, the length and the arclength on γ, respectively.

Definition 3.1 Let γ be a unit-speed curve in M^n_ν. γ is said to be an *elastica* (or elastic curve) if it is an extremal point of the functional $\mathfrak{F}^\mu$ for some μ. It is called a *free elastica* if $\mu = 0$.

The Euler-Lagrange equation associated to the variational problem given by $\mathfrak{F}^\mu$ is

$$2\nabla^3_T T + \varepsilon_1 \nabla_T((3\varepsilon_2\kappa^2 - \mu)T) - 2R(\nabla_T T, T)T = 0,$$

where R stands for the curvature tensor, provided that γ is closed or satisfies given first order boundary data (see [**4**, **7**] for details). Since M^n_ν is of constant curvature c, the Euler-Lagrange equation can be rewritten as follows

$$2\varepsilon_2\kappa'' + \varepsilon_1\kappa^3 - 2\varepsilon_3\kappa\tau^2 + \varepsilon_1\varepsilon_2(2c - \mu)\kappa = 0,$$

along with (7) and (8). From these equations we can assume without loss of generality that $n = 2$ or $n = 3$.

Let P^m_μ be a semi-Riemannian submanifold of M^n_ν and denote by $\mathfrak{T}P$ and $\mathfrak{N}P$ the tangent and normal bundles on P^m_μ, respectively. Consider the Simon operator $\widetilde{A} : \mathfrak{N}P \to \mathfrak{N}P$ defined by $\langle \widetilde{A}\xi, \zeta \rangle = \text{trace}(S_\xi \circ S_\zeta)$, where S_ξ is the shape operator associated with ξ. Let $R^* : \mathfrak{N}P \to \mathfrak{N}P$ be the transformation given by $R^*\xi = \sum_{i=1}^n \varepsilon_i (R_{E_i\xi}E_i)^\perp$, where $\{E_i\}$ is a local orthonormal frame, with $\varepsilon_i = \langle E_i, E_i \rangle$, and $()^\perp$ denotes normal component. The *Jacobi operator* is the second order differential operator defined by

$$J : \mathfrak{N}P \to \mathfrak{N}P, \qquad J\xi = (\Delta^D - \widetilde{A} + R^*)\xi.$$

When P is compact then J arises from the second variation formula. A normal vector field $\xi \in \mathfrak{N}P$ is said to be a *Jacobi field* if it belongs to the kernel of J.

A straightforward computation yields

$$\Delta\xi = (\Delta\xi)^T + J\xi + 2\widetilde{A}\xi - R^*\xi,$$

where $()^T$ denotes tangential component. Since M^n_ν is of constant curvature c, then $R^* = -mcI$, I being the identity on $\mathfrak{N}P$.

In this section we want to characterize the curves γ in M^n_ν whose mean curvature vector field is a Jacobi field. More generally, we are going to classify those curves with mean curvature vector field proper for the Jacobi operator, that is, $JH = \lambda H$, $\lambda \in \mathbb{R}$. A straightforward computation leads to

$$\widetilde{A}H = \varepsilon_1\kappa^3\xi_2.$$

From here and (9), $JH = \lambda H$ if and only if the set of equations (7), (8) and

$$\varepsilon_2\kappa'' + \varepsilon_1\kappa^3 - \varepsilon_3\kappa\tau^2 + \varepsilon_1\varepsilon_2(c+\lambda)\kappa = 0,$$

holds. As before, we can assume without loss of generality that $n = 2$ or $n = 3$. The following definition is given in [**3**].

Definition 3.2 Let $\gamma(s)$ and $\widetilde{\gamma}(s)$ be two unit speed curves in M^n_ν with curvature functions κ and $\widetilde{\kappa}$, respectively. γ and $\widetilde{\gamma}$ are *curvature homothetic* if there is a constant a such that $\widetilde{\kappa}(s) = a\kappa(s)$.

Proposition 3.3 *Let $\gamma : I \to M^n_\nu$ be a unit speed curve. Then $JH = \lambda H$, for some real constant λ, if and only if $n \leqslant 3$ and γ is curvature homothetic to an elastica with $a = \sqrt{2}$.*

Proof. Let $\widetilde{\gamma}$ be a curve with curvature $\widetilde{\kappa}(s) = \sqrt{2}\kappa(s)$ and torsion $\widetilde{\tau}(s) = \tau(s)$. Then γ satisfies $JH = \lambda H$ if and only if $\widetilde{\gamma}$ is an elastica. This completes the proof.

As a consequence, we classify the curves in M^n_ν whose mean curvature vector field is a Jacobi field.

Proposition 3.4 *Let $\gamma : I \to M^n_\nu$ be a unit speed curve. Then $JH = 0$ if and only if $n \leqslant 3$ and γ is curvature homothetic to a free elastica with $a = \sqrt{2}$.*

Let x_i and $x = x_1 \times \cdots \times x_m$ be as in Section 2. Let J_i be the Jacobi operator in the normal bundle associated to x_i and consider J the corresponding operator for x. Then we have

$$J(\xi_1, \ldots, \xi_m) = (J_1\xi_1, \ldots, J_m\xi_m),$$

where ξ_i is a normal vector field to M_i in $\tilde{M}_i$. As a consequence of Proposition 3.3 we have the following.

Corollary 3.5 *Let $\gamma_i : I_i \to M^{n_i}_{\nu_i}$, $i = 1, \ldots, m$, be unit speed curves in indefinite real space forms and consider the isometric immersion $x = \gamma_1 \times \cdots \times \gamma_m$. Then H is a Jacobi vector field if and only if $n_i \leqslant 3$ and γ_i is curvature homothetic to a free elastica in $M^{n_i}_{\nu_i}$, for any index i.*

The hypercylinders with Jacobi mean curvature vector field are characterized as follows. For Euclidean curves that characterization can be found in [**3**].

Corollary 3.6 *Let $\gamma : I \to M^n_\nu$ be a unit speed curve and consider the hypercylinder $x = \gamma \times Id : I \times \mathbb{R}^m_t \to M^n_\nu \times \mathbb{R}^m_t$. Then H is a Jacobi vector field if and only if $n \leqslant 3$ and γ is curvature homothetic to a free elastica.*

4. Hopf cylinders with proper mean curvature vector field

In [**4**] we have just constructed a new class of submanifolds in $\mathbb{H}^3_1(-1)$, the so-called *semi-Riemannian Hopf cylinders*, defined by means of two semi-Riemannian submersions $\pi_s : \mathbb{H}^3_1(-1) \to \mathbb{H}^2_s(-4)$, $s = 0, 1$. Let us recall how those surfaces were defined. First we identify $\mathbb{H}^3_1(-1)$ with an appropriate subset of maps $\mathbb{R}^4_2 \to \mathbb{R}^4_2$. To do that, let P be a 2-dimensional subspace in $\mathbb{R}^4_2$ and $\{x, y\}$ an orthonormal basis of P. We define the following maps:

$$
\begin{aligned}
f &: P \to P, \qquad f(x) = y,\ f(y) = -x,\\
g &: P \to P, \qquad g(x) = y,\ g(y) = x,\\
h &: P \to P, \qquad h(x) = -y,\ h(y) = -x,
\end{aligned}
$$

that will be called *rotation, first reflection* and *second reflection* on P, respectively. Let $\{e_1, e_2, e_3, e_4\}$ be the canonical basis of $\mathbb{R}$, for which the matrix of the metric is given by $(g_{ij}) = \mathrm{diag}[-1, -1, 1, 1]$. Let P_i, $i = 2, 3, 4$, be the 2-dimensional linear space spanned by $\{e_1, e_i\}$, so that $\mathbb{R}^4_2 = P_i \oplus P_i^\perp$. Consider the following maps:

$$
\begin{aligned}
\rho &= f \times f : P_2 \oplus P_2^\perp \to P_2 \oplus P_2^\perp,\\
\sigma &= g \times h : P_3 \oplus P_3^\perp \to P_3 \oplus P_3^\perp,\\
\iota &= g \times g : P_4 \oplus P_4^\perp \to P_4 \oplus P_4^\perp,
\end{aligned} \tag{1}
$$

and let $1 : \mathbb{R}^4_2 \to \mathbb{R}^4_2$ denote the identity map. It is clear that the set $\mathcal{F} = \mathrm{span}\{1, \rho, \sigma, \iota\}$ is a 4-dimensional vector space over $\mathbb{R}$ and the following identities hold:

$$
\begin{array}{lll}
\rho^2 = -1, & \sigma\rho = -\iota, & \iota\rho = \sigma,\\
\rho\sigma = \iota, & \sigma^2 = 1, & \iota\sigma = \rho,\\
\rho\iota = -\sigma, & \sigma\iota = -\rho, & \iota^2 = 1.
\end{array}
$$

This shows that $\mathcal{F}$ is closed under composition.

Now, let $\varphi : \mathcal{F} \to \mathbb{R}^4_2$ the isomorphism given by $\varphi(1) = e_1$, $\varphi(\rho) = e_2$, $\varphi(\sigma) = e_3$, $\varphi(\iota) = e_4$. Then φ becomes an isometry when $\mathcal{F}$ is endowed with the metric $\varphi^*(g_0)$, g_0 being the standard scalar product on $\mathbb{R}^4_2$. Throughout this paper, both metrics will be denoted by $\langle , \rangle$.

Let $\omega = a + b\rho + c\sigma + d\iota$ be an element of $\mathcal{F}$, where we write a for $a \cdot 1$, a, b, c and d being real numbers. Then we define $\overline{\omega} = -a + b\rho + c\sigma + d\iota$ and it is easy to show that $\langle \omega, \omega \rangle = \omega\overline{\omega} = \overline{\omega}\omega$. In general, $\langle \omega_1, \omega_2 \rangle = p_1(\omega_1 \overline{\omega}_2)$, p_1 denoting the projection over the subspace spanned by the identity map. As an immediate consequence we deduce $\overline{\omega_1\omega_2} = -\overline{\omega_2}\,\overline{\omega_1}$ and so $\langle \omega_1\omega_2, \omega_1\omega_2 \rangle = -\langle \omega_1, \omega_1 \rangle \langle \omega_2, \omega_2 \rangle$.

Now, we identify $\mathbb{H}^3_1(-r^2)$ with the set $\{\omega \in \mathcal{F} : \omega\overline{\omega} = -1/r^2\}$, $\mathbb{H}^2(-r^2)$ with the subset of $\mathbb{H}^3_1(-r^2)$ spanned by $\{1, \sigma, \iota\}$, and $\mathbb{H}^2_1(-r^2)$ with the subset of $\mathbb{H}^3_1(-r^2)$ spanned by $\{1, \rho, \sigma\}$.

Define $\pi_s : \mathbb{H}^3_1(-1) \to \mathbb{H}^2_s(-4)$ by

$$\pi_s(\omega) = \frac{1}{2}\widetilde{\omega}\omega,$$

where $\omega \to \widetilde{\omega}$ denotes the antiautomorphism of $\mathcal{F}$ given by

$$\widetilde{\omega} = a - b\rho + c\sigma + d\iota, \qquad \text{or} \qquad \widetilde{\omega} = a + b\rho + c\sigma - d\iota,$$

according to the base manifold is $\mathbb{H}^2(-4)$ or $\mathbb{H}^2_1(-4)$, respectively. It is easy to show that $\widetilde{\omega_1\omega_2} = \widetilde{\omega_2}\,\widetilde{\omega_1}$, $\langle \omega, \omega\rangle = \langle \tilde{\omega}, \tilde{\omega}\rangle$ and π is surjective. Moreover, $\pi_0(e^{\rho x}\omega) = \pi_0(\omega)$ and $\pi_1(e^{\iota x}\omega) = \pi_1(\omega)$ for all $\omega \in \mathbb{H}^3_1(-1)$, $x \in \mathbb{R}$. As usual, we define $e^{\theta x}$, $\theta \in \mathcal{F}$, by $\cos x + \sin x\,\theta$ if $\theta^2 = -1$, and $\cosh x + \sinh x\,\theta$ if $\theta^2 = 1$. That means that the fibers are topologically $\mathbb{S}^1$ and $\mathbb{H}^1$, respectively.

Remark 4.1 Notice that if in (1) we put $\sigma = f \times f$ and $\iota = f \times f$, then we obtain in the Euclidean space $\mathbb{R}^4$ the standard quaternionic structure, which was already used by U. Pinkall (see [**8**]) to describe the usual Hopf fibration of $\mathbb{S}^3(1)$ over $\mathbb{S}^2(1)$.

By pulling back via π_s a non-null curve γ in $\mathbb{H}^2_s(-4)$ we get the total horizontal lift of γ, which is an immersed flat surface M_γ in $\mathbb{H}^3_1(-1)$, that will be called the *semi-Riemannian Hopf cylinder associated to* γ. Notice that if $s = 0$, M_γ is a Lorentzian surface, whereas if $s = 1$, M_γ is Riemannian or Lorentzian, according to γ be spacelike or timelike, respectively.

Let $\gamma : I \to \mathbb{H}^2_s(-4)$ be a unit speed curve with Frenet frame $\{T, \xi_2\}$ and curvature function κ. Let $\bar{\gamma}$ be a horizontal lift of γ to $\mathbb{H}^3_1(-1)$ with Frenet frame $\{\bar{T}, \bar{\xi}_2, \xi^*_3\}$, curvature $\bar{\kappa} = \kappa \circ \pi_s$ and torsion $\tau = 1$. Recall that ξ^*_3 is nothing but the unit tangent vector field to the fibers along $\bar{\gamma}$. Then the Hopf Cylinder M_γ can be orthogonally parametrized as

$$X(t,z) = \begin{cases} \cos(z)\bar{\gamma}(t) + \sin(z)\xi^*_3(t), \text{ if } s = 0, \\ \cosh(z)\bar{\gamma}(t) + \sinh(z)\xi^*_3(t), \text{ if } s = 1. \end{cases}$$

Setting, as usual, $X_t = \frac{\partial X}{\partial t}$ and $X_z = \frac{\partial X}{\partial z}$, then $\{X_t, X_z\}$ is an orthonormal frame of $T_{X(t,z)}M_\gamma$ along X. A direct computation shows that the shape operator S of M_γ in this frame can be written as

$$\begin{aligned} S(X_t) &= \bar{\kappa}X_t - \varepsilon X_z, \\ S(X_z) &= X_t, \end{aligned}$$

where $\varepsilon = -1$ if M_γ is Riemannian and $\varepsilon = +1$ if M_γ is Lorentzian.

Notice that a unit normal vector field to M_γ into $\mathbb{H}^3_1(-1)$ is obtained from the complete horizontal lift of ξ_2 and it is, of course, $\bar{\xi}_2$ along each horizontal lift of γ. As a consequence we have that M_γ is a flat surface and its mean curvature function α is given by $\alpha = \varepsilon\frac{1}{2}\bar{\kappa}$. Then a Hopf cylinder in $\mathbb{H}^3_1(-1)$ associated to a curve γ in $\mathbb{H}^2_s(-4)$ is isoparametric if and only if γ is of constant curvature. So as a consequence of [**1**, Lemma 2.1] we have

Proposition 4.2 *Let $\gamma : I \to \mathbb{H}^2_s(-4)$ be an immersed curve and $\pi_s : \mathbb{H}^3_1(-1) \to \mathbb{H}^2_s(-4)$ the Hopf fibrations. Let H be the mean curvature vector of the Hopf cylinder $M_\gamma = \pi_s^{-1}(\gamma)$ associated to the curve γ. Then $\Delta H = \lambda H$ if and only if γ is of constant curvature in $\mathbb{H}^2_s(-4)$.*

All Hopf cylinders classified in that proposition are of constant mean curvature in $\mathbb{H}_1^3(-1)$. So if we want to obtain Hopf cylinders with non constant mean curvature we must work with the Laplacian Δ^D associated to the normal connection D, which is given by $\Delta^D\xi = -\sum_{i=1}^n \varepsilon_i(D_{E_i}D_{E_i}\xi - D_{\nabla_{E_i}E_i}\xi)$, where $\{E_i\}$ is a local orthonormal frame, with $\varepsilon_i = \langle E_i, E_i\rangle$. As for semi-Riemannian hypersurfaces M_ν^n, let N be a unit vector field normal to M_ν^n and let α denote the mean curvature function with respect to N. Then a straightforward computation shows $\Delta^D H = (\Delta\alpha)N$, where the Laplacian Δ is given in (t,z)-coordinates by $\Delta = (-1)^s\left\{-\varepsilon\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial z^2}\right\}$. Hence the following result is clear.

Proposition 4.3 *Let $\gamma : I \to \mathbb{H}_s^2(-4)$ be an immersed curve and $\pi_s : \mathbb{H}_1^3(-1) \to \mathbb{H}_s^2(-4)$ the Hopf fibrations. Let H be the mean curvature vector field of the Hopf cylinder $M_\gamma = \pi_s^{-1}(\gamma)$ associated to the curve γ. Then $\Delta^D H = \lambda H$ if and only if one of the following conditions holds:*
(1) $\lambda = 0$ *and* $\kappa(t) = at + b$.
(2) $\theta = (-1)^s\varepsilon\lambda > 0$ *and* $\kappa(t) = a\cos(\sqrt{\theta}t) + b\sin(\sqrt{\theta}t)$.
(3) $\theta < 0$ *and* $\kappa(t) = a\exp(\sqrt{-\theta}t) + b\exp(-\sqrt{-\theta}t)$.

Proof. It is a straightforward computation, because $\Delta^D H = \lambda H$ is equivalent to the differential equation $\kappa'' + (-1)^s\varepsilon\lambda\kappa = 0$.

As a special consequence of that result, we obtain that H is in the kernel of Δ^D if and only if the curvature function $k(t)$ is a linear function. So we have proved the following.

Theorem 4.4 *Let $\gamma : I \to \mathbb{H}_s^2(-4)$ be an immersed curve and $\pi_s : \mathbb{H}_1^3(-1) \to \mathbb{H}_s^2(-4)$ the Hopf fibrations. Then the mean curvature vector field H of the Hopf cylinder M_γ is harmonic in the normal bundle if and only if one of the following statements holds:*
(1) γ *is a geodesic.*
(2) γ *is a pseudocircle or a pseudohyperbola.*
(3) γ *is a Cornu spiral.*

That theorem has been yet obtained in [**2**] by using the usual Hopf fibration $\mathbb{S}^3 \to \mathbb{S}^2$.

Now we want to get another relation between the Hopf cylinders M_γ and the curves γ to which they are associated. To do that, let Δ_γ^D and H_γ be the normal Laplacian and the mean curvature vector of the curve γ in $\mathbb{H}_s^2(-4)$, respectively.

Proposition 4.5 *Let $\pi_s : \mathbb{H}_1^3(-1) \to \mathbb{H}_s^2(-4)$ and $\gamma : I \to \mathbb{H}_s^2(-4)$ be as before. Then $\Delta^D H = \lambda H$ if and only if $\Delta_\gamma^D H_\gamma = \lambda H_\gamma$.*

Proof. Let $\gamma : I \to \mathbb{H}_s^2(-4)$ be a unit speed curve and suppose $\langle\gamma', \gamma'\rangle = \varepsilon_1$. Then $\Delta_\gamma^D H_\gamma = \lambda H_\gamma$ if and only if $\kappa'' + \varepsilon_1\lambda\kappa = 0$, κ being the curvature function of the curve γ in $\mathbb{H}_s^2(-4)$. So the result follows from Proposition 4.3.

To finish this section we are going to characterize the Hopf cylinders whose mean curvature vector field is an eigenvector of the Jacobi operator. Before that, we first state a general result.

Let P_μ^{n-1} be a hypersurface of M_ν^n. Let N denote a unit normal vector field to P_μ^{n-1} in M_ν^n and let α be the mean curvature with respect to N, so the mean curvature vector field can be written as $H = \alpha N$. A direct computation shows the following.

Proposition 4.6 *Let $x : P_\mu^{n-1} \to M_\nu^n$ be a hypersurface in an indefinite space form of constant curvature c. Then H is an eigenvector of J, that is, $JH = \lambda H$, for some real number λ, if and only if $\Delta\alpha = (\lambda + \varepsilon tr(S^2) + (n-1)c)\alpha$, where $\varepsilon = \langle N, N\rangle$ and S stands for the shape operator associated to N.*

As a consequence we deduce that

Proposition 4.7 *Let $\pi_s : \mathbb{H}^3_1(-1) \to \mathbb{H}^2_s(-4)$ be the Hopf fibrations and $\gamma : I \to \mathbb{H}^2_s(-4)$ an immersed curve. Let H be the mean curvature vector field of M_γ in $\mathbb{H}^3_1(-1)$. Then $JH = \lambda H$, $\lambda \in \mathbb{R}$, if and only if the following differential equation holds*

$$\kappa'' + (-1)^s\{\kappa^3 + \varepsilon(\lambda - 4)\kappa\} = 0. \tag{2}$$

The differential equation given in this proposition can be solved by standard techniques in terms of elliptic functions (see [**6**] for more details).

These curves are related to elastic curves in $\mathbb{H}^2_s(-4)$ in the following sense. Bearing in mind the differential equation of the elastic curves in $\mathbb{H}^2_s(-4)$, if γ is a curve satisfying (2), then a curve $\widetilde{\gamma}$ with $\widetilde{\kappa}(t) = \sqrt{2}\kappa(t)$ is an elastica. In particular, if the constant λ in (2) is equal to 0, then $\widetilde{\gamma}$ is a free elastica. Hence Proposition 4.7 can be rewritten as follows.

Theorem 4.8 *Let $\pi_s : \mathbb{H}^3_1(-1) \to \mathbb{H}^2_s(-4)$ be the Hopf fibrations and $\gamma : I \to \mathbb{H}^2_s(-4)$ an immersed curve. Let H be the mean curvature vector field of M_γ in $\mathbb{H}^3_1(-1)$. Then $JH = \lambda H$, $\lambda \in \mathbb{R}$, if and only if γ is curvature homothetic to an elastica $\widetilde{\gamma}$ with $\widetilde{\kappa}(t) = \sqrt{2}\kappa(t)$. In particular, H is a Jacobi vector field if and only if γ is curvature homothetic to a free elastica.*

As a consequence of Proposition 4.7 we obtain the following result, whose proof is similar to that of Proposition 4.5.

Proposition 4.9 *Let $\pi_s : \mathbb{H}^3_1(-1) \to \mathbb{H}^2_s(-4)$ and $\gamma : I \to \mathbb{H}^2_s(-4)$ be as before. Then $JH = \lambda H$ if and only if $J_\gamma H_\gamma = \lambda H_\gamma$, where J_γ and H_γ stand for the Jacobi operator and the mean curvature vector of γ.*

Bibliography

[**1**] L. J. Alías, A. Ferrández and P. Lucas. 2-type surfaces in $\mathbb{S}^3_1$ and $\mathbb{H}^3_1$. *Tokyo J. Math.*, **17** (1994), 447–454.

[**2**] M. Barros and O. J. Garay. On submanifolds with harmonic mean curvature. To appear in Proc. AMS, 1994.

[**3**] M. Barros and O. Garay. Isometric immersions with Jacobi mean curvature vector field and related problems. In F. Dillen, I. V. de Woestijne and L. Verstraelen, editors, *Geometry and Topology of Submanifolds, VI*, pages 23–42. World Sci. Co., 1994.

[**4**] M. Barros, A. Ferrández, P. Lucas and M. A. Meroño. Hopf cylinders and filament equation in the three dimensional anti-de sitter space. Preprint, 1995.

[**5**] B. Y. Chen. Some open problems and conjectures on submanifolds of finite type. *Soochow J. Math.*, **17** (1991), 169–188.

[**6**] H. T. Davis. *Introduction to non-linear differential and integral equations*. Dover Publ. Inc., 1960.

[7] J. Langer and D. Singer. The total squared curvature of closed curves. *J. Differential Geometry*, **20** (1984), 1–22.

[8] U. Pinkall. Hopf tori in $\mathbb{S}^3$. *Inventiones mathematicae*, **81** (1985), 379–386.

[9] J. Simons. Minimal varieties in Riemannian manifolds. *Ann. of Math.*, **88** (1968), 62–105.