Solutions of the Betchov-Da Rios soliton equation in the anti-De Sitter 3-space

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En 'New Approaches in Nonlinear Analysis', ed. Th. M. Rassias, Hadronic Press Inc., Palm Harbor, Florida, pp. 51–71, 1999. ISBN: 1-57485-042-3/pbk

(Partially supported by DGICYT grant PB94-0750 and Fundación Séneca PIB95-34)

Abstract

In the three-dimensional anti-De Sitter space we study solutions of the Betchov-Da Rios soliton equation and find that they are helices sweeping out a B-scroll. The soliton solutions are precisely the null geodesics of that B-scroll. Closed solutions are also given.

1. Introduction

The Betchov-Da Rios equation, otherwise known as the localized induction equation, $X_z = X_t \wedge X_{tt}$, is a soliton equation for space curves $X(t, z)$, best known as a model for the behaviour of thin vortex tubes in an incompressible, inviscid, three-dimensional fluid. It is also seen as the global setting of the filament equation in the evolution context (see [**7**], [**15**], [**16**] and [**21**]). In [**5**] we bring that equation to a unit three-sphere and we succeed in getting a two-parameter family of solutions in $\mathbb{S}^3(1)$. It was obtained by using Pinkall's Hopf cylinders as a tool (see [**20**]). The solutions are actually helices lying in certain Hopf cylinders of constant mean curvature. We also gave a rational one-parameter family of closed helices living in a Hopf torus. It is worth pointing out that no soliton solution living in a Hopf torus can be found. Indeed, we got a congruence solution, i.e., X moves without changing shape, only position. This is the reason why in this paper we undertake to check the Betchov-Da Rios equation (or the filament equation) in a Lorentzian space form.

In dealing with isometric immersions between Lorentz spaces L. Graves, [**13**], introduced a special class of surfaces, the so called B-scrolls, that played an important role in the classification of flat surfaces isometrically immersed in $\mathbb{H}^3_1(-1)$ obtained by M. Dajczer and K. Nomizu, [**10**]. Those surfaces have also been the key to solve some others interesting classification problems as well as to check a Chen's conjecture brought to indefinite space forms (see [**2**], [**3**], [**4**] and [**12**]).

The aim of this paper is threefold. First, following Pinkall's idea, [**20**], we construct the family of semi-Riemannian Hopf cylinders in $\mathbb{H}^3_1(-1)$ by means of certain semi-Riemannian submersions of $\mathbb{H}^3_1(-1)$ over $\mathbb{H}^2_s(-1/4)$, $s = 0, 1$. Then we show that a B-scroll is a Lorentzian Hopf cylinder. Some examples are given in the last section. Secondly, in view of the geometric structure of Hopf cylinders, we check the Betchov-Da Rios soliton equation in $\mathbb{H}^3_1(-1)$. Indeed, that equation, when considered as an evolution equation, is nothing but the filament equation, so that we find solutions lying in B-scrolls. We also give closed solutions. Thirdly, the solitons solutions of the filament equation in $\mathbb{H}^3_1(-1)$, living in a B-scroll, are its null geodesics. Therefore, as far as we know, this is a nice approach to get a physical interpretation of a B-scroll.

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Finally, we ask for Hopf torus which are Willmore surfaces in $\mathbb{H}_1^3(-1)$. The answer should be supplied by Hopf torus over closed elasticae in $\mathbb{H}^2(-1/4)$. Since the Lagrange multiplier λ has to be -4, no Lorentzian Willmore Hopf torus can be found in $\mathbb{H}_1^3(-1)$.

Part of results in this paper were advanced in [**6**].

2. Semi-Riemannian Hopf cylinders and B-scrolls

In order to introduce a Hopf fibration $\pi_s : \mathbb{H}_1^3(-1) \to \mathbb{H}_s^2(-1/4)$, $s = 0, 1$, we need to identify $\mathbb{H}_1^3(-1)$ with an appropriate subset of maps $\mathbb{R}_2^4 \to \mathbb{R}_2^4$. To do that, let P be a 2-dimensional subspace in $\mathbb{R}_2^4$ and $\{x, y\}$ an orthonormal basis of P. We define the following maps:

$$\begin{aligned}
&f : P \to P, \qquad f(x) = y,\ f(y) = -x,\\
&g : P \to P, \qquad g(x) = y,\ g(y) = x,\\
&h : P \to P, \qquad h(x) = -y,\ h(y) = -x,
\end{aligned}$$

that will be called *rotation, first reflection* and *second reflection* on P, respectively. Let $\{e_1, e_2, e_3, e_4\}$ be the canonical basis on $\mathbb{R}_2^4$, for which the matrix of the metric is given by $(g_{ij}) = \mathrm{diag}[-1, -1, 1, 1]$. Let P_i, $i = 2, 3, 4$, be the 2-dimensional linear space spanned by $\{e_1, e_i\}$, so that $\mathbb{R}_2^4 = P_i \oplus P_i^\perp$. Consider the following maps:

$$\begin{aligned}
\rho &= f \times f : P_2 \oplus P_2^\perp \to P_2 \oplus P_2^\perp,\\
\sigma &= g \times h : P_3 \oplus P_3^\perp \to P_3 \oplus P_3^\perp,\\
\iota &= g \times g : P_4 \oplus P_4^\perp \to P_4 \oplus P_4^\perp,
\end{aligned} \tag{1}$$

and let $1 : \mathbb{R}_2^4 \to \mathbb{R}_2^4$ denote the identity map. It is clear that the set $\mathcal{F} = \mathrm{span}\{1, \rho, \sigma, \iota\}$ is a 4-dimensional vector space over $\mathbb{R}$ and the following identities hold:

$$\begin{aligned}
&\rho^2 = -1, \quad \sigma\rho = -\iota, \quad \iota\rho = \sigma,\\
&\rho\sigma = \iota, \qquad \sigma^2 = 1, \qquad \iota\sigma = \rho,\\
&\rho\iota = -\sigma, \quad \sigma\iota = -\rho, \quad \iota^2 = 1.
\end{aligned}$$

This shows that $\mathcal{F}$ is closed under composition.

Now, let $\varphi : \mathcal{F} \to \mathbb{R}_2^4$ the isomorphism given by $\varphi(1) = e_1$, $\varphi(\rho) = e_2$, $\varphi(\sigma) = e_3$, $\varphi(\iota) = e_4$. Then φ becomes an isometry when $\mathcal{F}$ is endowed with the metric $\varphi^*(g_0)$, g_0 being the standard scalar product on $\mathbb{R}_2^4$. Throughout this paper, both metrics will be denoted by $\langle , \rangle$.

Let $\omega = a + b\rho + c\sigma + d\iota$ be an element of $\mathcal{F}$, where we write a for $a \cdot 1$, a, b, c and d being real numbers. Then we define $\overline{\omega} = -a + b\rho + c\sigma + d\iota$ and it is easy to show that $\langle \omega, \omega \rangle = \omega\overline{\omega} = \overline{\omega}\omega$. In general, $\langle \omega_1, \omega_2 \rangle = p_1(\omega_1\overline{\omega}_2)$, p_1 denoting the projection over the subspace spanned by the identity map. As an immediate consequence we deduce $\overline{\omega_1\omega_2} = -\overline{\omega_2}\,\overline{\omega_1}$ and so $\langle \omega_1\omega_2, \omega_1\omega_2 \rangle = -\langle \omega_1, \omega_1 \rangle \langle \omega_2, \omega_2 \rangle$.

Now, we identify $\mathbb{H}_1^3(-r^2)$ with the set $\{\omega \in \mathcal{F} : \omega\overline{\omega} = -r^2\}$, $\mathbb{H}^2(-r^2)$ with the subset of $\mathbb{H}_1^3(-r^2)$ spanned by $\{1, \sigma, \iota\}$, and $\mathbb{H}_1^2(-r^2)$ with the subset of $\mathbb{H}_1^3(-r^2)$ spanned by $\{1, \rho, \sigma\}$.

Define $\pi_s : \mathbb{H}_1^3(-1) \to \mathbb{H}_s^2(-1/4)$ by

$$\pi_s(\omega) = \frac{1}{2}\widetilde{\omega}\omega,$$

where $\omega \to \widetilde{\omega}$ denote the antiautomorphism of $\mathcal{F}$ given by

$$\widetilde{\omega} = a - b\rho + c\sigma + d\iota, \qquad \text{or} \qquad \widetilde{\omega} = a + b\rho + c\sigma - d\iota,$$

according to the base manifold be $\mathbb{H}^2(-1/4)$ or $\mathbb{H}^2_1(-1/4)$, respectively. It is easy to show that $\widetilde{\omega_1\omega_2} = \widetilde{\omega_2}\ \widetilde{\omega_1}$, $\langle \omega, \omega \rangle = \langle \tilde{\omega}, \tilde{\omega} \rangle$ and π_s is surjective. Moreover, $\pi_0(e^{\rho x}\omega) = \pi_0(\omega)$ and $\pi_1(e^{\iota x}\omega) = \pi_1(\omega)$ for all $\omega \in \mathbb{H}^3_1(-1)$, $x \in \mathbb{R}$. As usual, we define $e^{\theta x}$, $\theta \in \mathcal{F}$, by $\cos(x) + \sin(x)\theta$ if $\theta^2 = -1$, and $\cosh(x) + \sinh(x)\theta$ if $\theta^2 = 1$. That means that the fibers are topologically $\mathbb{S}^1$ and $\mathbb{H}^1$, respectively.

Remark 2.1 Notice that if in (1) we put $\sigma = f \times f$ and $\iota = f \times f$, then we obtain in the Euclidean space $\mathbb{R}^4$ the standard quaternionic structure, which was already used by U. Pinkall (see [**20**]) to describe the usual Hopf fibration $\mathbb{S}^3(1) \to \mathbb{S}^2(1)$.

It is not difficult to see that the above defined $\pi_s : \mathbb{H}^3_1(-1) \to \mathbb{H}^2_s(-1/4)$, $s = 0, 1$, are semi-Riemannian submersions and so we will follow the notation and terminology of [**19**]. Therefore given $\omega \in \mathbb{H}^3_1(-1)$ one has the splitting of $T_w\mathbb{H}^3_1(-1)$ into the horizontal subspace $\mathcal{H}_\omega$ and the vertical line $\mathcal{V}_\omega$ spanned by a vector V_ω. Recall that $\mathcal{V}_\omega = \ker(d(\pi_s)_\omega)$ and $d(\pi_s)_\omega$ restricted to $\mathcal{H}_\omega$ gives an isometry between $\mathcal{H}_\omega$ and $T_{\pi_s(\omega)}\mathbb{H}^2_s(-1/4)$.

Let $\overline{\nabla}$ and ∇ be the semi-Riemannian connections of $\mathbb{H}^3_1(-1)$ and $\mathbb{H}^2_s(-1/4)$, respectively, and denote by overbars the lifts of corresponding objects on the base $\mathbb{H}^2_s(-1/4)$. Then, we have

$$\begin{aligned} \overline{\nabla}_{\overline{X}}\overline{Y} &= \overline{\nabla_X Y} + (-1)^s(\langle JX, Y \rangle \circ \pi_s)V, \\ \overline{\nabla}_{\overline{X}}V &= \overline{\nabla}_V\overline{X} = \theta\overline{X}, \\ \overline{\nabla}_V V &= 0, \end{aligned}$$

where J denotes the standard complex structure of $\mathbb{H}^2_s(-1/4)$ and $\theta = \rho$ when $s = 0$ or $\theta = \iota$ when $s = 1$.

Let $\beta : I \to \mathbb{H}^2_s(-1/4)$ be a unit speed curve with Frenet frame $\{T, \xi_2\}$ and curvature κ. Consider a horizontal lift $\overline{\beta} : I \to \mathbb{H}^3_1(-1)$ of β with Frenet frame $\{\overline{T}, \xi^*_2, \xi^*_3\}$ and curvatures κ^* and τ^*.

Now, from the Frenet equations, we can deduce that $\xi^*_2 = \overline{\xi_2}$. In particular ξ^*_2 lies in the horizontal distribution along $\overline{\beta}$ and it has the same causal character as ξ_2. Also it is not difficult to see that $\tau^* = \pm 1$ and $\xi^*_3 = \pm V$, that is, the binormal ξ^*_3 of $\overline{\beta}$ coincides with the unit tangent to the fibers through each point of $\overline{\beta}$.

Then, using the terminology in [**10**], we have proved the following.

Lemma 2.2 (i) *The horizontal lifts of unit speed curves in* $\mathbb{H}^2(-1/4)$ *are spacelike Frenet curves in* $\mathbb{H}^3_1(-1)$ *with torsion* ± 1.
(ii) *The horizontal lifts of unit speed timelike curves in* $\mathbb{H}^2_1(-1/4)$ *are timelike Frenet curves in* $\mathbb{H}^3_1(-1)$ *with torsion* ± 1.

By pulling back via π_s a non-null curve β in $\mathbb{H}^2_s(-1/4)$ we get the total horizontal lift of β, which is a flat immersed surface M_β in $\mathbb{H}^3_1(-1)$, that will be called the *semi-Riemannian Hopf cylinder associated to* β. Notice that if $s = 0$, M_β is a Lorentzian surface, whereas if $s = 1$, M_β is Riemannian or Lorentzian, according to β be spacelike or timelike, respectively.

In [**10**] M. Dajczer and K. Nomizu studied certain flat Lorentzian surfaces immersed in $\mathbb{H}^3_1(-1)$, the so called B-scrolls, as an extension of those first introduced by L. Graves [**13**]. Now we are going to give a relation between Hopf cylinders and B-scrolls.

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Theorem 2.3 *Let M be a Lorentzian surface immersed into $\mathbb{H}^3_1(-1)$. Then M is the semi-Riemannian Hopf cylinder in $\mathbb{H}^3_1(-1)$ associated to a unit speed curve β in $\mathbb{H}^2_s(-1/4)$ if and only if M is the B-scroll of any horizontal lift $\overline{\beta}$ of β.*

Proof. Let $\overline{\beta}$ be a horizontal lift of β. Then M_β can be parametrized as $X(t,z) = e^{\theta z}\overline{\beta}(t)$, θ being ρ or ι, according to s be 0 or 1, respectively. Thus we find that

$$X(t,z) = \begin{cases} \cos(z)\overline{\beta}(t) + \sin(z)\rho\overline{\beta}(t), & \text{if } s = 0 \\ \cosh(z)\overline{\beta}(t) + \sinh(z)\iota\overline{\beta}(t), & \text{if } s = 1 \end{cases}$$

Observe that $\theta\overline{\beta}(t)$ is the unit tangent vector field to the fibers along $\overline{\beta}$, which is nothing but the binormal of $\overline{\beta}$. Then if M_β is Lorentzian, this proves that M_β is the B-scroll associated to $\overline{\beta}$. A similar argument works to prove the converse.

Let $\beta : I \to \mathbb{H}^2_s(-1/4)$ be a unit speed curve with Frenet frame $\{T, \xi_2\}$ and curvature function κ. Let $\bar{\beta}$ be a horizontal lift of β to $\mathbb{H}^3_1(-1)$ with Frenet frame $\{\bar{T}, \bar{\xi}_2, \xi^*_3\}$ and curvature $\bar{\kappa} = \kappa \circ \pi_s$ and $\tau = 1$. Recall that ξ^*_3 is nothing but the unit tangent vector field to the fibers along $\bar{\beta}$. Then the Hopf cylinder M_β can be orthogonally parametrized as

$$X(t,z) = \begin{cases} \cos(z)\overline{\beta}(t) + \sin(z)\xi^*_3(t), & \text{if } s = 0 \\ \cosh(z)\overline{\beta}(t) + \sinh(z)\xi^*_3(t), & \text{if } s = 1 \end{cases}$$

Setting, as usual, $X_t = \frac{\partial X}{\partial t}$ and $X_z = \frac{\partial X}{\partial z}$, then $\{X_t, X_z\}$ is an orthonormal frame of $T_{X(t,z)}M_\beta$ along X and a direct computation shows that the shape operator S of M_β in this frame can be written as

$$\begin{aligned} S(X_t) &= \overline{\kappa} X_t + \varepsilon X_z, \\ S(X_z) &= X_t, \end{aligned}$$

where $\varepsilon = +1$ if M_β is Riemannian and $\varepsilon = -1$ if M_β is Lorentzian.

Notice that a unit normal vector field to M_β into $\mathbb{H}^3_1(-1)$ is obtained from the complete horizontal lift of ξ_2 and it is, of course, $\bar{\xi}_2$ along each horizontal lift of β. As a consequence we have that M_β is a flat surface, as we said before, and its mean curvature function α is given by $\alpha = \bar{\kappa}/2$.

According to the description of curves with constant curvature in $\mathbb{H}^2_s(-1/4)$ (see, for instance, [**8**, p. 178 and ff]) and using the terminology of [**1**], [**3**] and [**18**] we can give the following description of Hopf cylinders of constant mean curvature.

Proposition 2.4 *Let β be a unit speed curve in $\mathbb{H}^2_s(-1/4)$ with constant curvature κ. Then one of the following statements holds*

(1) *M_β is a minimal complex circle ($\kappa = 0$).*

(2) *M_β is a non-minimal complex circle ($0 < \kappa^2 < 4$).*

(3) *M_β is the Hopf cylinder over the horocycle ($s = 0$, $\kappa^2 = 4$) or over the pseudo-horocycle ($s = 1$, $\kappa^2 = 4$).*

(4) *M_β is one of the following semi-Riemannian products*

(4.1) $\mathbb{H}^1_1(-r^2) \times \mathbb{S}^1(r^2 - 1)$ *if $s = 0$ and $\kappa^2 > 4$,*

(4.2) $\mathbb{H}^1(-r^2) \times \mathbb{S}^1_1(r^2 - 1)$ *if $s = 1$ and $\kappa^2 > 4$.*

(5) *M_β is the Riemannian product $\mathbb{H}^1(-r^2) \times \mathbb{H}^1(-1 + r^2)$ with r satisfying*

$$\frac{1 - 2r^2}{r\sqrt{1 - r^2}} = \kappa.$$

It should be noticed that the above cases (1) through (4) correspond to the Lorentzian character of M_β and so, according to Theorem 2.3, it can be considered as the classification of B-scrolls with constant mean curvature in $\mathbb{H}^3_1(-1)$. The remainder case corresponds with the Riemannian character of M_β.

3. Lorentzian Hopf tori

We know, from the above section, that Hopf surfaces in $\mathbb{H}^3_1(-1)$ shaped on closed curves in $\mathbb{H}^2(-1/4)$ are Lorentzian flat tori. Now we want to determine the isometry group of these surfaces.

To do that we first define a connection in the circle bundle $\pi_0 : \mathbb{H}^3_1(-1) \to \mathbb{H}^2(-1/4)$ by attaching to each $\omega \in \mathbb{H}^3_1(-1)$ the horizontal 2-plane $\mathcal{H}_\omega$. The Lie algebra of $\mathbb{S}^1 = \mathcal{U}(1)$ is identified to $u(1) = \mathbb{R}$ and so $\mathcal{V}$ is the fundamental vector field 1^* corresponding to $1 \in u(1)$. In particular, the connection form Γ is the $u(1)$-valued 1-form over $\mathbb{H}^3_1(-1)$ defined by $\Gamma(\mathcal{H}_\omega) = 0$ and $\Gamma(\mathcal{V}_\omega) = 1$, for any $\omega \in \mathbb{H}^3_1(-1)$.

Let $\bar{\Omega}$ be the curvature 2-form of the connection. It is well known that there exists a unique $\mathbb{R}$-valued 2-form Ω on $\mathbb{H}^2(-1/4)$ such that $\bar{\Omega} = \pi_0^*(\Omega)$. The standard volume form dV of $\mathbb{H}^2(-1/4)$ satisfies that $dV(X, JX) = 1$, for any unit vector field X on $\mathbb{H}^2(-1/4)$.

Now we use standard computations involving the structure equations of the induced connection and the formulae in [**14**, Vol. I, p. 146] to get $\Omega = -2\, dV$.

Given a closed embedded curve β in $\mathbb{H}^2(-1/4)$, we will show that the isometry type of M_β not only depends on the length of β, but also on the area enclosed by β, and only on both of them. Actually we have the following.

Theorem 3.1 *Let β be a closed embedded curve in $\mathbb{H}^2(-1/4)$ of length L enclosing an area A. Then M_β is isometric to $\mathbb{L}^2/\Lambda$, Λ being the lattice in the Lorentzian plane $\mathbb{L}^2$ generated by the vectors $(2\pi, 0)$ and $(2A, L)$.*

Proof. Let $\bar{\beta}$ be any horizontal lift of β and $X : \mathbb{L}^2 \to M_\beta \subset \mathbb{H}^3_1(-1)$ the semi-Riemannian covering defined by $X(z, t) = e^{iz}\bar{\beta}(t)$. The lines parallel to the z-axis in $\mathbb{L}^2$ are mapped by X onto the fibres of π_0, whereas those parallel to the t-axis are mapped onto the horizontal lifts of β. Of course, these curves are not closed. However, following [**14**, Vol. II, p. 293], there exists a fixed number $\delta \in [-\pi, \pi)$ such that $\bar{\beta}(L) = e^{i\delta}\bar{\beta}(0)$. The group of deck transformations of the covering X is then generated by the translations $(2\pi, 0)$ and (δ, L). To get an explicit computation of the holonomy number δ we come back to the circle bundle $\pi_0 : \mathbb{H}^3_1(-1) \to \mathbb{H}^2(-1/4)$. The curvature form measures the non-closedness of the horizontal lifts of closed curves. In our context, that means that (see again [**14**, Vol. II, p. 293])

$$\delta = -\int_C \Omega,$$

where C is any 2-chain on $\mathbb{H}^2(-1/4)$ such that $\partial C = \beta$. From here and $\Omega = -2\, dV$ we find that $\delta = 2A$.

Remark 3.2 It is worth noting that $(2A, L)$ is only constrained by the isoperimetric inequality in $\mathbb{H}^2(-1/4)$ (see [**22**])

$$L^2 \geqslant 4\pi A + 4A^2.$$

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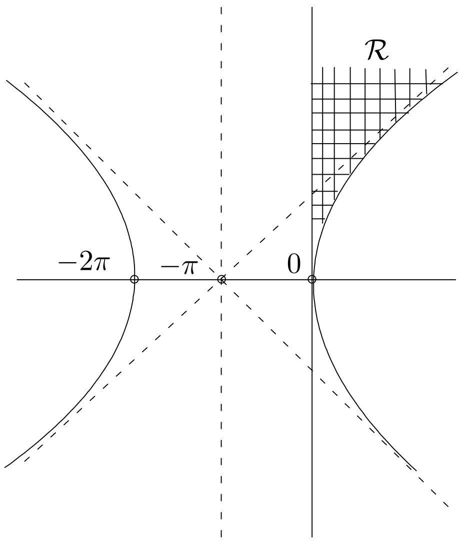
Hence the vector $(2A, L)$ must be spacelike. Observe that the above inequality can be rewritten as $(2A+\pi)^2 - L^2 \leqslant \pi^2$. Therefore $(2A, L)$ lies in the shaded region $\mathcal{R}$



Note that equality holds at the boundary points of $\mathcal{R}$. They correspond with lattices giving constant mean curvature Hopf tori, i.e., Hopf tori over geodesic circles in $\mathbb{H}^2(-1/4)$. Moreover, for each point in $\mathcal{R}$ there exists a unique (up to rigid motions in $\mathbb{L}^2$) lattice in $\mathbb{L}^2$ producing a Hopf torus in $\mathbb{H}^3_1(-1)$. Therefore we have got a sort of rigidity result for constant mean curvature Hopf tori. This can be viewed as follows: although a Hopf torus coming from an interior point of $\mathcal{R}$ could be shaped on two non-congruent closed embedded curves in $\mathbb{H}^2(-1/4)$, this can not occur for the boundary points of $\mathcal{R}$.

4. Semi-Riemannian Hopf cylinders and solitons

The Betchov-Da Rios equation, also called localized induction equation in 3-dimensional hydrodynamics,

$$\frac{\partial Y}{\partial t} \wedge \overline{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial Y}{\partial t} = \frac{\partial Y}{\partial z}$$

is a soliton equation for space curves $Y(t, z)$, $\overline{\nabla}$ being the Levi-Civita connection of the space. This can be rewritten as $Y_z = \kappa B$ (the “filament equation”), where κ and B stand for the curvature and the binormal of Y, respectively. The evolution of Y governed by this equation of motion can be viewed as an idealization of the motion of a thin vortex cylinder (see [**15**] and [**16**] for details).

In view of the geometric structure of semi-Riemannian Hopf cylinders in $\mathbb{H}^3_1(-1)$, that was explicitly described in Section 2, it seems natural to seek for parametrizations of them being congruence solutions of (2). It is a straigthforward computation that, in general, the standard parametrization $X(t, z)$ of M_β is not a solution of (2). From now on, $X(t, z)$, given in Theorem 2.3, will be called the standard covering of $\mathbb{R}^2$ over M_β. Then we set up the following question: let $\mathrm{Diff}(\mathbb{R}^2)$ be the group of diffeomorphisms of $\mathbb{R}^2$. For any $h \in \mathrm{Diff}(\mathbb{R}^2)$, define the covering map $Y = X \circ h : \mathbb{R}^2 \to M_\beta \subset \mathbb{H}^3_1(-1)$, where β is an arc-length parametrized curve in $\mathbb{H}^2_s(-1/4)$

and M_β its Hopf cylinder in $\mathbb{H}^3_1(-1)$. We ask for the classification of $h \in \mathrm{Diff}(\mathbb{R}^2)$ in order to Y be a solution of the Betchov-Da Rios equation in $\mathbb{H}^3_1(-1)$.

In this paper we completely solve this problem. To do that we first recall the Frenet equations in $\mathbb{H}^3_1(-1)$ of the lift $\bar\beta$ of β:

$$\begin{array}{rcl} \overline{\nabla}_{\bar T}\bar T &=& (-1)^s\varepsilon_1\bar\kappa\xi_2^*, \\ \overline{\nabla}_{\bar T}\bar\xi_2^* &=& -\varepsilon_1\bar\kappa\bar T + (-1)^s\xi_3^*, \\ \overline{\nabla}_{\bar T}\bar\xi_3^* &=& (-1)^s\varepsilon_1\xi_2^*. \end{array}$$

Let η be a unit normal vector field to M_β in $\mathbb{H}^3_1$(-1). Then η can be written as follows:

$$\eta = \left\{ \begin{array}{ll} -\sin(z)\bar T(t) + \varepsilon_1\cos(z)\xi_2^*(t), & s=0, \\ -\sinh(z)\bar T(t) + \varepsilon_1\cosh(z)\xi_2^*(t), & s=1. \end{array} \right.$$

Let $h \in \mathrm{Diff}(\mathbb{R}^2)$ and set $h(u,v) = (t(u,v), z(u,v))$. Write $Y = X \circ h : \mathbb{R}^2 \to M_\beta$ as $Y(u,v) = X(t(u,v), z(u,v)$, so that $t_u z_v - t_v z_u$ does not vanish anywhere. To see that Y is a solution of the Betchov-Da Rios soliton equation one only need check that $Y_u \wedge \overline{\nabla}_{Y_u}Y_u = Y_v$, where $\overline{\nabla}$ is the semi-Riemannian connection of $\mathbb{H}^3_1(-1)$.

The cross product in the tangent space $T_p\mathbb{H}^3_1(-1)$, in any point $p \in \mathbb{H}^3_1(-1) \subset \mathbb{R}^4_2$ is defined as follows. In $T_p\mathbb{H}^3_1(-1)$ there is a natural orientation: an ordered basis X, Y, Z in $T_p\mathbb{H}^3_1(-1)$ is positively oriented if det$[pXYZ] > 0$, where $[pXYZ]$ is the matrix with $p, X, Y, Z \in \mathbb{R}^4_2$ as row vectors. Now let ω be the volume element on $\mathbb{H}^3_1(-1)$ given by $\omega(X,Y,Z) = \mathrm{d}et[pXYZ]$. Then for $X, Y \in T_p\mathbb{H}^3_1(-1)$, the cross product $X \wedge Y$ is the unique vector in $T_p\mathbb{H}^3_1(-1)$ such that $\langle X \wedge Y, Z\rangle = \omega(X,Y,Z)$, for any $Z \in T_p\mathbb{H}^3_1(-1)$ (see [**10**]).

As for the covariant derivative, a straightforward computation yields

$$\overline{\nabla}_{Y_u}Y_u = t_{uu}X_t + z_{uu}X_z + t_u^2\overline{\nabla}_{X_t}X_t + t_u z_u\overline{\nabla}_{X_z}X_t + t_u z_u\overline{\nabla}_{X_t}X_z.$$

By using the Frenet equations it is easy to see that

$$\begin{array}{rcl} \overline{\nabla}_{X_t}X_t &=& (-1)^s\bar\kappa\eta, \\ \overline{\nabla}_{X_z}X_t &=& (-1)^s\eta, \\ \overline{\nabla}_{X_t}X_z &=& (-1)^s\eta. \end{array}$$

Then by choosing $\omega(X_t, X_z, \eta) = 1$, we have $X_t \wedge X_z = (-1)^s\varepsilon_1\eta$, $X_z \wedge \eta = \varepsilon_1 X_t$ and $X_t \wedge \eta = (-1)^s X_z$. Thus we find that

$$\begin{array}{rcl} Y_u \wedge \overline{\nabla}_{Y_u}Y_u &=& (-1)^s\varepsilon_1 t_u z_u(t_u\bar\kappa + 2z_u)X_t \\ && + \, t_u^2(t_u\bar\kappa + 2z_u)X_z \\ && + \, (-1)^2\varepsilon_1(t_u z_{uu} - z_u t_{uu})\eta. \end{array}$$

Therefore $Y(u,v)$ is a solution of the Betchov-Da Rios equation if and only if the following PDE system holds:

$$\begin{array}{rcl} t_v &=& (-1)^s\varepsilon_1 t_u z_u(t_u\bar\kappa + 2z_u), \\ z_v &=& t_u^2(t_u\bar\kappa + 2z_u), \\ 0 &=& t_u z_{uu} - z_u t_{uu}. \end{array}$$

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It follows that $z_u = gt_u$, for a certain function g, only depending on v, which measures the slope of the u-curves. On the other hand, since $\langle Y_u, Y_u\rangle = \varepsilon_1 t_u^2 - (-1)^s z_u^2 = \varepsilon$, ε being the causal character of the u-curves, we find that $\varepsilon = (\varepsilon_1 - (-1)^s g^2)t_u^2$. Then t_u only depends on v, so that $t(u,v) = h_1(v)u + h_2(v)$, for certain differentiable functions h_1 and h_2. In particular, $t_{uu} = 0 = z_{uu}$, and so we obtain

$$\overline{\nabla}_{Y_u} Y_u = (-1)^s(\bar{\kappa} + 2g)t_u^2 \eta.$$

On the way, the following claim has been proved: the u-curves are geodesics in the semi-Riemannian Hopf cylinder M_β.

Now the curvature function in $\mathbb{H}_1^3(-1)$ ρ of these curves is given by

$$\rho = \varepsilon_1 t_u^2(\bar{\kappa} + 2g).$$

As $t_u z_v - t_v z_u = \varepsilon\varepsilon_1 t_u^2(\bar{\kappa} + 2g) \neq 0$, we deduce that ρ does not vanish anywhere. From the compability condition $t_{uv} = t_{vu}$ we easily get that $\rho(u,v) = f_1(v)u + f_2(v)$, for certain differentiable real functions that only depend on v. Then f_1 vanishes identically and so $\rho_u = \bar{\kappa}_u = 0$. Let α be the mean curvature of M_β in $\mathbb{H}_1^3(-1)$. Since $2\alpha = \bar{\kappa}$, we have $\alpha_u = 0$, so that α is a constant function along the fibers. On the other hand, the linear independence of Y_u and X_z shows that α and $\bar{\kappa}$ are actually constant functions on M_β. By using now the remainder compatibility condition $z_{uv} = z_{vu}$, we find that g, t_u and ρ are constant functions.

Summarizing, we have proved the following

Theorem 4.1 *Let β be an arc length parametrized curve in $\mathbb{H}_s^2(-1/4)$ and M_β its Hopf cylinder in $\mathbb{H}_1^3(-1)$. For any $h \in \mathrm{Diff}(\mathbb{R}^2)$, take $Y = X \circ h : \mathbb{R}^2 \to M_\beta$, X being the standard covering of $\mathbb{R}^2$ over M_β. Then Y is a solution of the Betchov-Da Rios soliton equation in $\mathbb{H}_1^3(-1)$ if and only if the following statements hold:*
(i) *β has constant curvature, say κ, in $\mathbb{H}_s^2(-1/4)$;*
(ii) *$h(u,v) = (t(u,v), z(u,v))$ is given by*

$$\begin{aligned} t(u,v) &= au + (-1)^s ag\rho v + c_1, \\ z(u,v) &= agu + \varepsilon_1 a\rho v + c_2, \end{aligned}$$

where $(\varepsilon_1 - (-1)^s g^2)a^2 = \varepsilon$, ε_1 being the causal character of β, ε the causal character of the u-curves, $g \in \mathbb{R} - \{-\kappa/2\}$, $\rho = \varepsilon_1(\kappa + 2g)a^2$ is the curvature of the u-curves in $\mathbb{H}_1^3(-1)$ and a, c_1, c_2 are arbitrary constants.

A sharper description of the u-curves solution of the Betchov-Da Rios equation can be given. Looking at the proof of the above theorem we find that $\overline{\nabla}_{Y_u} Y_u = (-1)^s \varepsilon_1 \rho\eta$ and $\langle Y_v, Y_v\rangle = -\varepsilon\varepsilon_1(-1)^s\rho^2$. Therefore the torsion of the u-curves is given by $\nu = \langle \overline{\nabla}_{Y_u}\eta, \frac{1}{\rho}Y_v\rangle$. Then, from Section 2 we get

$$\begin{aligned} \nu &= -\frac{1}{\rho}\langle S_\eta Y_u, Y_v\rangle \\ &= -\frac{1}{\rho}\left\{t_u t_v \langle S_\eta X_t, X_t\rangle + (t_u z_v + t_v z_u)\langle S_\eta X_t, X_z\rangle + z_u z_v \langle S_\eta X_z, X_z\rangle\right\} \\ &= -\frac{\varepsilon\varepsilon_1(-1)^s}{\varepsilon_1 - (-1)^s g^2}(g^2 + \bar{\kappa}g + \varepsilon_1(-1)^s), \end{aligned}$$

which is constant. Therefore, the solution we have just found is actually a helix in $\mathbb{H}_1^3$(-1), lying in a Hopf cylinder, whose evolution through the filament flow is made up by helices congruent to him.

Conversely, given an helix γ of constant curvature functions ρ and ν, a straightforward computation yields

$$\begin{aligned} g &= \frac{-(-1)^s(\varepsilon\varepsilon_1+\nu)}{\rho}, \\ k &= \frac{\varepsilon\rho^2+(-1)^s\varepsilon\varepsilon_1(1-\nu^2)}{\rho}. \end{aligned}$$

Then γ determines a solution of the filament equation in $\mathbb{H}_1^3$(-1) lying in the Hopf cylinder. Indeed, take the Hopf cylinder M_β over a curve β in $\mathbb{H}_s^2(-1/4)$ of constant curvature κ and then a u-geodesic in M_β with slope g.

Let $I=\{g\in\mathbb{R} : \varepsilon_1-(-1)^s g^2\neq 0 \text{ and } \kappa+2g\neq 0\}$ and consider the one parameter family $\{Y_g = X\circ h_g : g\in I\}$ of congruence solutions of the Betchov-Da Rios equation. When $\varepsilon_1-(-1)^s g^2=0$, Y_g parametrizes a null geodesic of M_β and a straightforward computation shows that this curve is a singular solution of the Betchov-Da Rios equation. Summing up we have

Corollary 4.2 *Let M_β be a Lorentzian Hopf cylinder in $\mathbb{H}_1^3(-1)$ of constant mean curvature. Then the only soliton solutions of the Betchov-Da Rios equation in $\mathbb{H}_1^3(-1)$ lying in M_β are the null geodesics of M_β.*

From Theorem 3.1 it is easy to find closed solutions. Actually we have

Corollary 4.3 *Let β be a closed curve of constant curvature in $\mathbb{H}^2(-1/4)$ with length L enclosing an oriented area A. Then for any rational number q, the slope*

$$g=\frac{2\pi}{L}\left(q+\frac{A}{\pi}\right)$$

defines a unique closed helix in $\mathbb{H}_1^3(-1)$ and therefore a closed solution of the Betchov-Da Rios equation in $\mathbb{H}_1^3(-1)$ living in the Hopf torus M_β. Furthermore, the closed solution is either spacelike, or timelike or null according to $q\in(q_1,q_2)$, $q\in\mathbb{R}-(q_1,q_2)$, $q\in\{q_1,q_2\}$, respectively, where $q_1=-\frac{A}{\pi}-\frac{L}{2\pi}$ and $q_2=-\frac{A}{\pi}+\frac{L}{2\pi}$.

5. Looking for Willmore Hopf tori in $\mathbb{H}_1^3(-1)$

Let M_ν^n be a semi-Riemannian manifold and consider immersed curves $\gamma : I\to M_\nu^n$. As usual, the metric will be denoted by $\langle,\rangle$ and the Riemannian connection by ∇. Let $V(t)$ be the tangent vector to γ at $\gamma(t)$ and $T(t)$ the unit tangent vector, so we have $\gamma'(t)=v(t)T(t)$, where $v(t)=(\varepsilon_1\langle V(t),V(t)\rangle)^{1/2}$ is the speed of γ and $\varepsilon_1=\langle T,T\rangle$ denotes its causal character. The curvature $\kappa(t)$ of γ is given by $\kappa(t)^2=\varepsilon_2\langle\nabla_T T,\nabla_T T\rangle$, ε_2 being the causal character of $\nabla_T T$.

A unit-speed curve γ in M_ν^n is said to be an *elastica* (or *elastic curve*) if it is an extremal point of the functional

$$\mathfrak{F}^\lambda(\gamma)=\int_0^L(\langle\nabla_T T,\nabla_T T\rangle+\lambda)ds=\int_0^1(\langle\nabla_T T,\nabla_T T\rangle+\lambda)v dt,$$

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for some λ, where ds and L stand for the arclength on γ and the length of γ, respectively. It is called a *free elastica* if $\lambda = 0$.

In order to obtain the variation formula of $\mathfrak{F}^\lambda$, we will keep the notation and computations in [**17**, Section 1]. When the curve γ is closed or satisfies given first order boundary data, the Euler-Lagrange equation associated to our variational problem is

$$2\nabla_T^3 T + \varepsilon_1 \nabla_T((3\varepsilon_2\kappa^2 - \lambda)T) - 2R(\nabla_T T, T)T = 0.$$

The Frenet equations for γ can be partially written as

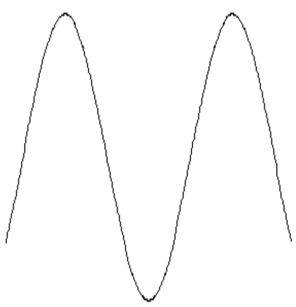
$$\begin{aligned} \nabla_T T &= \varepsilon_2 \kappa \xi_2, & (3)\\ \nabla_T \xi_2 &= -\varepsilon_1 \kappa T - \varepsilon_3 \tau \xi_3, & (4)\\ \nabla_T \xi_3 &= \varepsilon_2 \tau \xi_2 + \delta, & (5) \end{aligned}$$

where $\delta \in \operatorname{span}\{T, \xi_2, \xi_3\}^\perp$, $\langle \xi_i, \xi_i \rangle = \varepsilon_i$ and τ is the torsion function (the second curvature if $n > 3$). Assume now that M_ν^n is of constant curvature c. Then the Euler-Lagrange equation can be rewritten as follows

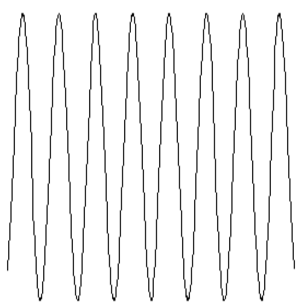
$$\begin{aligned} &2\varepsilon_2\kappa'' + \varepsilon_1\kappa^3 - 2\varepsilon_3\kappa\tau^2 + \varepsilon_1\varepsilon_2(2c - \lambda)\kappa = 0, & (6)\\ &2\kappa'\tau + \kappa\tau' = 0, & (7)\\ &\kappa\tau\delta = 0. & (8) \end{aligned}$$

If γ does not lie in a 2-dimensional totally geodesic submanifold of M_ν^n, then the equation (8) implies that $\delta = 0$ and so the curve γ lies in a 3-dimensional totally geodesic submanifold of M_ν^n. Hence we can assume without loss of generality that $n = 2$ or $n = 3$. On the other hand, from (7) we deduce that $\kappa^2\tau = a$ is constant.

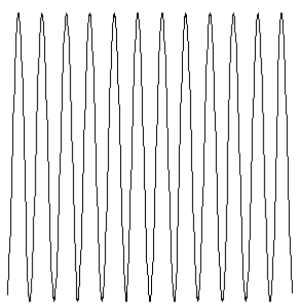
Taking $u = \kappa^2$ the equation (6) can be solved by standard techniques in terms of elliptic functions (see [**11**] for a more detailed discussion on this subject). For instance, a qualitative description of elasticae in the Lorentz-Minkowski plane $\mathbb{L}^2$ is given as follows. In general, the elasticae in $\mathbb{L}^2$ are curves which oscillates around a geodesic, so that the parameter λ, in some sense, could be viewed as the wavelength. That length increases or decreases according to $\varepsilon_1\lambda$ does. In the following we skecht some of these curves.



$\varepsilon_1\lambda > 0$

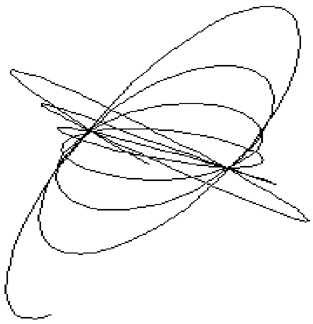


$\lambda = 0$

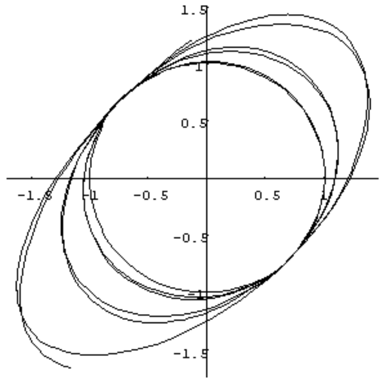


$\varepsilon_1\lambda < 0$

As for the pseudo-hyperbolic plane $\mathbb{H}_1^2(-1)$ the behaviour of the elastic curves is essentially the same as in $\mathbb{L}^2$, they also oscillate around geodesics. In particular, we can draw a free elastica oscillating around the central circle in $\mathbb{H}_1^2(-1)$.



Free elastica



Projection on xy-plane

Let M_s^2 be a surface in an indefinite 3-space $\tilde{M}_\mu^3$ of constant curvature c, and let H denote its mean curvature vector field. We define the operator W over sections of the normal bundle of M_s^2 into $\tilde{M}_\mu^3$ as follows

$$W : \mathfrak{N}M \to \mathfrak{N}M, \qquad W(\xi) = (\Delta^D + 2\,\langle H, H\rangle\, I - \widetilde{A})\xi,$$

where $\widetilde{A}$ denotes the Simon operator, [**23**]. A cross section ξ will be called a Willmore section if $W(\xi) = 0$. Suppose that M is compact and consider the Willmore functional (see, for example, [**9**], [**24**] and [**25**])

$$\mathcal{W}(M) = \int_M (\langle H, H\rangle + c)dv.$$

Then the operator W naturally appears provided that one computes the first variation formula for $\mathcal{W}$. That can be obtained in a similar way to that given by J. L. Weiner (see [**24**]) in the definite case. Now Willmore surfaces are nothing but the extremal points of the Willmore functional and they are characterized from the fact that their mean curvature vector fields are Willmore fields.

Proposition 5.1 *Let $\pi_s : \mathbb{H}_1^3(-1) \to \mathbb{H}_s^2(-1/4)$ and $\beta : I \to \mathbb{H}_s^2(-1/4)$ be as before. Then the Hopf cylinder M_β satisfies $WH = \mu H$, $\mu \in \mathbb{R}$, if and only if β is an elastica in $\mathbb{H}_s^2(-1/4)$.*

Proof. Let α be the mean curvature of M_β in $\mathbb{H}_1^3(-1)$. Then $WH = \mu H$ if and only if $\Delta\alpha + 2\varepsilon\alpha^3 - (\mu + \varepsilon \mathrm{tr}(S^2))\alpha = 0$, ε and S being the sign of M_β and its shape operator, respectively. By using the usual local coordinates $\{t, z\}$ we have that $\Delta = (-1)^{s+1}\left\{\varepsilon\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2}\right\}$. Bearing in mind that $\alpha = \varepsilon\kappa/2$ and $\mathrm{tr}(S^2) = \kappa^2 - 2\varepsilon$ we deduce $WH = \mu H$ if and only if $2(-1)^{s+1}\kappa'' - \kappa^3 + 2\varepsilon(2-\mu)\kappa = 0$, that is, β is an elastica with $\lambda = -(4 + 2\mu)$.

Observe that the fibers of $\pi_0 : \mathbb{H}_1^3(-1) \to \mathbb{H}^2(-1/4)$ are circles, and so compact whereas the fibers of $\pi_1 : \mathbb{H}_1^3(-1) \to \mathbb{H}_1^2(-1/4)$ are not compact. Therefore to find compact Hopf surfaces we have to consider Hopf torus shaped on closed curves in $\mathbb{H}^2(-1/4)$. In fact, U. Pinkall, [**20**], exhibited an infinite series of Willmore tori in the unit 3-sphere $\mathbb{S}^3(1)$ obtained as Hopf tori, via the usual Hopf fibration of $\mathbb{S}^3(1)$ over $\mathbb{S}^2(1/4)$, over elasticae in $\mathbb{S}^2(1/4)$. In his family, Pinkall shows that the Clifford torus in $\mathbb{S}^3(1)$ is the only member with constant mean curvature (actually minimal). That is, it is the unique Willmore-Hopf tori coming from an elastica in $\mathbb{S}^2(1/4)$ with constant curvature (actually geodesic). In the anti-De Sitter world, we know from Proposition 5.1 that a Hopf torus M_β is a Willmore surface in $\mathbb{H}_1^3(-1)$ if and only if β is an elastica in $\mathbb{H}^2(-1/4)$ with $\lambda = -4$. However we have recently known from D. Singer (private communication) that cannot be hold. Thus *there is no (Lorentzian) Willmore Hopf torus in $\mathbb{H}_1^3(-1)$.*

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6. Some examples

In this section we are going to describe a method to find explicit orthogonal parametrization of Hopf cylinders in $\mathbb{H}^3_1(-1) \subset \mathbb{R}^4_2$ associated to curves in $\mathbb{H}^2_s(-1/4)$.

For the sake of simplicity our computations will be refered to the fibration $\pi_0 : \mathbb{H}^3_1(-1) \to \mathbb{H}^2(-1/4)$. It is worth noticing that our method also works for the fibration $\pi_1 : \mathbb{H}^3_1(-1) \to \mathbb{H}^2_1(-1/4)$, as well as for the usual Hopf fibration $\pi : \mathbb{S}^3 \to \mathbb{S}^2$, making suitable changes.

Let ω be any point in $\mathbb{H}^3_1(-1) \subset \mathbb{R}^4_2$, which is written as $\omega = x_1 + x_2\rho + x_3\sigma + x_4\iota$. Throughout this section we will identify ω with a point $(x_i)_{1\leqslant i\leqslant 4}$ in $\mathbb{R}^4$. In this sense $\pi_0(\omega) = \frac{1}{2}\widetilde{\omega}\omega$ is nothing but $(A, 0, B, C)$ where

$$A = \frac{1}{2}\sum_{i=1}^{4} x_i^2, \qquad B = x_1x_3 + x_2x_4, \qquad C = x_1x_4 - x_2x_3.$$

Now, given a point $p = (A, 0, B, C)$ in $\mathbb{H}^2(-1/4)$ it is not difficult to see that the fiber $\pi_0^{-1}(p)$ is given by

$$\pi_0^{-1}(p) = \left\{ (D\cos\alpha, D\sin\alpha, \frac{1}{D}(B\cos\alpha - C\sin\alpha), \frac{1}{D}(C\cos\alpha + B\sin\alpha)) : \alpha \in \mathbb{R} \right\},$$

where $D = \frac{\sqrt{2A+1}}{\sqrt{2}}$.

Let $\beta : I \to \mathbb{H}^2(-1/4)$ be a regular curve given by $\beta(t) = (A(t), 0, B(t), C(t))$. A straightforward computation shows that all horizontal lifts $\overline{\beta}$ of β to $\mathbb{H}^3_1(-1)$ are obtained as

$$\begin{aligned} \overline{\beta}(t) \;&=\; (D(t)\cos\alpha(t), D(t)\sin\alpha(t), \frac{1}{D(t)}(B(t)\cos\alpha(t) - C(t)\sin\alpha(t)), \\ &\quad\; \frac{1}{D(t)}(C(t)\cos\alpha(t) + B(t)\sin\alpha(t))), \end{aligned}$$

where $\alpha(t)$ is given by

$$\alpha(t) = 2\int \frac{B(t)C'(t) - C(t)B'(t)}{2A(t)+1}dt.$$

Thus an orthogonal parametrization $X(t,z) = e^{z\rho}\overline{\beta}(t)$ of the Hopf cylinder M_β associated to β is written as

$$\begin{aligned} X(t,z) \;&=\; (D(t)\cos(\alpha(t)+z), D(t)\sin(\alpha(t)+z), \\ &\quad\; \frac{1}{D(t)}(B(t)\cos(\alpha(t)+z) - C(t)\sin(\alpha(t)+z)), \\ &\quad\; \frac{1}{D(t)}(C(t)\cos(\alpha(t)+z) + B(t)\sin(\alpha(t)+z))), \end{aligned}$$

where we have chosen a suitable constant to determine α and fix the lift of β.

Now we will apply this method to some special curves in $\mathbb{H}^2(-1/4)$.

5.1 The Hopf cylinder associated to a horocycle.

In studying curves with constant curvature in the Poincaré half-plane $\mathbb{H}^2(-1/4)$, it appears a circle tangent to the x-axis whose curvature satisfies $\kappa^2 = 4$. This is the so-called horocycle, which can be parametrized as follows

$$\beta(t) = (t^2 + \frac{1}{2}, 0, t^2, t), \quad t \in \mathbb{R}.$$

The lift $\overline{\beta}$ of β to $\mathbb{H}^3_1(-1)$ with $\alpha(0) = 0$ is obtained by computing $D(t) = \sqrt{t^2+1}$ and $\alpha(t) = \arctan t - t$. Therefore $\cos\alpha(t) = \frac{1}{\sqrt{t^2+1}}(\cos t + t\sin t)$ and $\sin\alpha(t) = \frac{1}{\sqrt{t^2+1}}(t\cos t - \sin t)$, and so

$$\overline{\beta}(t) = (\cos t + t\sin t, t\cos t - \sin t, t\sin t, t\cos t).$$

Hence an orthogonal parametrization of M_β in $\mathbb{H}^3_1(-1)$ is given by

$$X(t,z) = (\cos(t-z) + t\sin(t-z), -\sin(t-z) + t\cos(t-z), t\sin(t-z), t\cos(t-z)).$$

5.2 The Hopf cylinder associated to a geodesic circle.

We find that geodesic circles in $\mathbb{H}^2(-1/4)$ have constant curvature satisfying $\kappa^2 > 4$. They are parametrized as

$$\beta(t) = \left(a, 0, \sqrt{a^2 - \frac{1}{4}}\cos t, \sqrt{a^2 - \frac{1}{4}}\sin t\right), \quad t \in \mathbb{R}.$$

To compute the lift $\overline{\beta}$ of β with $\alpha(0) = 0$ in $\mathbb{H}^3_1(-1) \subset \mathbb{R}^4_2$, we have $D(t) = \sqrt{\frac{2a+1}{2}}$ and $\alpha(t) = \frac{2a-1}{2}t$. Then

$$\begin{aligned} \overline{\beta}(t) &= \left(\sqrt{\frac{2a+1}{2}}\cos\left(\frac{2a-1}{2}t\right), \sqrt{\frac{2a+1}{2}}\sin\left(\frac{2a-1}{2}t\right),\right. \\ &\left.\sqrt{\frac{2a-1}{2}}\cos\left(\frac{2a+1}{2}t\right), \sqrt{\frac{2a-1}{2}}\sin\left(\frac{2a+1}{2}t\right)\right) \end{aligned}$$

and the Hopf cylinder M_β can be orthogonally parametrized as

$$\begin{aligned} X(t,z) &= \left(\sqrt{\frac{2a+1}{2}}\cos\left(\frac{2a-1}{2}t + z\right), \sqrt{\frac{2a+1}{2}}\sin\left(\frac{2a-1}{2}t + z\right),\right. \\ &\left.\sqrt{\frac{2a-1}{2}}\cos\left(\frac{2a+1}{2}t + z\right), \sqrt{\frac{2a-1}{2}}\sin\left(\frac{2a+1}{2}t + z\right)\right). \end{aligned}$$

5.3 The Hopf cylinder associated to some hyperbolic cyclide.

For a non-zero real number r, let $A(t)$ be a non-constant differentiable function satisfying $A(t) > 2r^2 + \frac{1}{2}$ and define $\beta(t)$ in $\mathbb{H}^2(-1/4) \subset \mathbb{R}^4_2$ to be

$$\beta(t) = \left(A(t), 0, r\sqrt{2A(t)+1}, \sqrt{\left(A(t) + \frac{1}{2}\right)\left(A(t) - 2r^2 - \frac{1}{2}\right)}\right).$$

Notice that β can be obtained by cutting the hyperboloid

$$\left\{(A,0,B,C)\in\mathbb{R}^4 : -A^2+B^2+C^2=-\frac{1}{4}, A>0\right\}$$

in $\mathbb{R}^4$ with the parabolic cylinder $B^2 = r^2(2A+1)$ and in this sense we call β a hyperbolic cyclide.

To get an orthogonal parametrization of M_β in $\mathbb{H}^3_1(-1)\subset\mathbb{R}^4_2$ we have $D(t)=\sqrt{\frac{2A(t)+1}{2}}$ and $\alpha(t)=r\sqrt{2A(t)-4r^2-1}$.Then M_β can be parametrized by

$$\begin{aligned} X(t,z) \;=\; & \left(\sqrt{\frac{2A(t)+1}{2}}\cos\left(r\sqrt{2A(t)-(4r^2+1)}+z\right),\right. \\ & \sqrt{\frac{2A(t)+1}{2}}\sin\left(r\sqrt{2A(t)-(4r^2+1)}+z\right), \\ & \sqrt{2}r\cos\left(r\sqrt{2A(t)-(4r^2+1)}+z\right)-\sqrt{A(t)-\left(2r^2+\frac{1}{2}\right)}\sin\left(r\sqrt{2A(t)-(4r^2+1)}+z\right), \\ & \left.\sqrt{2}r\sin\left(r\sqrt{2A(t)-(4r^2+1)}+z\right)+\sqrt{A(t)-\left(2r^2+\frac{1}{2}\right)}\cos\left(r\sqrt{2A(t)-(4r^2+1)}+z\right)\right). \end{aligned}$$

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