Solutions of the Betchov-Da Rios soliton equation: A Lorentzian approach

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Abstract

The purpose of this paper is to find out explicit solutions of the Betchov-Da Rios soliton equation in 3-dimensional Lorentzian space forms. We start with non null curves and obtain solutions living in certain flat ruled surfaces in $\mathbb{L}^3$ and $\mathbb{H}^3_1$, as well as in $\mathbb{R}^3$ and $\mathbb{S}^3$. Next we take a null curve and have got solutions lying in the associated B-scrolls into $\mathbb{L}^3$, $\mathbb{S}^3_1$ and $\mathbb{H}^3_1$. It shoul be pointed out that we extend previous results already obtained and, as far as we know, this is the first time that solutions in the De Sitter 3-space appear in the literature. Soliton solutions are characterized as null geodesics in B-scrolls.

1. Introduction

Physical systems such as vortex filaments in perfect fluids, one-dimensional classical continuum Heisenberg chains and elastic strings can be thought of as one-dimensional extended objects, the support of which, their centerline, may be mathematically modeled by a generally twisted space curve $\gamma \subset \mathbb{R}^3$. Regardless of the physical properties that actually characterize the dynamics of such systems, for the moment let us take into consideration the pure kynematics of curves as an idealization of the general evolution of these systems. Let us identify the thin filament with the vortex line γ, smooth and free from self-intersections. The velocity induced by the vortex line γ at an external point was obtained by Da Rios by means of the so-called localized induction aproximation (LIA). By using the Biot-Savart integral to express that velocity and ignoring finite contributions, Da Rios found out that the asymptotic velocity contribution is given along binormal direction, say $\mathbf{v} = \kappa\mathbf{B}$, κ being the curvature of the vortex axis. Hence, under LIA (neglecting long-distance effects and self-interaction), vortex filaments move simply in the binormal direction with speed proportional to the curvature (see [5], [6] and [7] for more details)

The intrinsic equations governing vortex motion were also derived by Da Rios. They are given in terms of time derivatives of curvature and torsion of γ. Let $\gamma(s,t)$ be time variations of $\gamma(s)$, s being the arc length parameter. Let $\{\kappa, \tau, \gamma' = T, N, B\}$ be the Frenet-Serret apparatus along γ. Let us write the velocity as $\frac{\partial \gamma}{\partial t} = \dot{\gamma} = v_T T + v_N N + v_B B$, where v_T, v_N and v_B are regular functions of s and t. The intrinsic equations are given in terms of v_N', $\dot{\kappa}$ and $\dot{\tau}$, where overdots and primes denote partial derivatives with respect to t and s, respectively. The first one writes down as $v_N' = \kappa v_N$ which gives a necessary and sufficient condition for inextensibility of γ and can be regarded as a congruence condition for material points of the curve. Actually, that means simply that γ is arc-length parametrized. Under LIA $v_T = v_N = 0$ and $v_B = \kappa$, so that the two remainder equations are reduced to the so-called Betchov-Da Rios equations $\dot{\kappa} = -\kappa\tau' - 2\kappa'\tau$ and $\dot{\tau} = (\kappa''/\kappa - \tau^2)' + \kappa\kappa'$. These equations prescribe (up to rigid motion) the evolution of the vortex filament in an infinite domain of $\mathbb{R}^3$ for given initial conditions $\kappa(s,0)$ and $\tau(s,0)$.

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Otherwise, under LIA, the motion of a thin vortex is governed by the simple equation (also called the Betchov-Da Rios equation)

$$\frac{\partial\gamma}{\partial t}\wedge\overline{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial\gamma}{\partial t}=\frac{\partial\gamma}{\partial s},$$

$\overline{\nabla}$ being the Levi-Civita connection of the space. It is worth noting that, under LIA, total length, total squared curvature and total torsion are conserved quantities in time (see also [7]).

In [1] and [2] we have found solutions of the Betchov-Da Rios in the Riemannian 3-sphere and in the 3-dimensional anti De Sitter space. In the first case, by using the Hopf fibration, we have got a nice geometric characterization of those solutions: they are helices in $\mathbb{S}^3$ and geodesics of Hopf cylinders in $\mathbb{S}^3$. In the second one, we have also obtained that solutions are the geodesics of B-scrolls into $\mathbb{H}^3_1$. Closed solutions are found in any case. To point out the chief difference between Riemannian and pseudo-Riemannian situations, we have to mention that soliton solutions appear in $\mathbb{H}^3_1$ and they are characterized as null geodesics of B-scrolls into $\mathbb{H}^3_1$.

The purpose of this paper is to state and find out solutions of the Betchov-Da Rios soliton equation in 3-dimensional Lorentzian space forms. Actually, we give explicit examples of surfaces in $\mathbb{L}^3$, as well as in the De Sitter $\mathbb{S}^3_1$ and anti De Sitter $\mathbb{H}^3_1$ worlds, where the solutions are lying. To do that we start with non null curves and get solutions living in certain flat ruled surfaces in $\mathbb{L}^3$ and $\mathbb{H}^3_1$, as well as in $\mathbb{R}^3$ and $\mathbb{S}^3$. In a second step, we take a null curve and find out solutions lying in the associated B-scrolls into $\mathbb{L}^3$, $\mathbb{S}^3_1$ and $\mathbb{H}^3_1$. Several interesting facts should be pointed out. First, we extend the results already obtained in [1] and [2]. Secondly, as far as we know, this is the first time that solutions in the De Sitter 3-space appear in the literature.

2. Setup

Let $\bar{M}^3_\nu(c)$ be a 3-dimensional pseudo-Riemannian space form of curvature c and index $\nu=0,1$. As usual, $\bar{M}^3_\nu(c)$ is either the pseudo-Euclidean space $\mathbb{R}^3_\nu$, or the pseudo-sphere $\mathbb{S}^3_\nu(c)\subset\mathbb{R}^4_\nu$, or the pseudo-hyperbolic space $\mathbb{H}^3_\nu(c)\subset\mathbb{R}^4_{\nu+1}$, according to $c=0$, $c>0$ or $c<0$, respectively. For the sake of simplicity, and provided that we need explicitly mention neither curvature c nor index ν, we will simply write down $\bar{M}$ instead of $\bar{M}^3_\nu(c)$.

Let $\alpha:I\subset\mathbb{R}\to\bar{M}$ be an immersed curve and let V be a vector field along α in $\bar{M}$. Let us consider the ruled surface M_α in $\bar{M}$, generated by α and V, defined by

$$\begin{aligned} X:I\times(-a,a) &\to \bar{M}\\ (s,t) &\to X(s,t)=\exp_{\alpha(s)}(tV(s)).\end{aligned}$$

For each fixed s, the curve $\gamma_s(t)$, defined by $t\to\gamma_s(t)=X(s,t)$, is the geodesic of $\bar{M}$ uniquely determined by the initial conditions $\gamma_s(0)=\alpha(s)$ and $\gamma_s'(0)=V(s)$. Let $\{X_s,X_t\}$ be the frame defined by

$$X_s(s,t)=dX_{(s,t)}\left(\frac{\partial}{\partial s}\right) \;=\; (\mathrm{dexp}_{\alpha(s)})_{tV(s)}(\alpha'(s)+tV'(s))$$

and

$$X_t(s,t)=dX_{(s,t)}\left(\frac{\partial}{\partial t}\right) \;=\; (\mathrm{dexp}_{\alpha(s)})_{tV(s)}(V(s)),$$

$V'(s)$ being the covariant derivative along α of $V(s)$. Observe that, at $t=0$, $X_s(s,0)=\alpha'(s)$ and $X_t(s,0)=V(s)$, so that $X(s,t)$ will define a regular pseudo-Riemannian surface into $\bar{M}$ whenever $\alpha'(s)$ and $V(s)$ are linearly independent and the plane $\Pi=\mathrm{span}\{\alpha',V\}$ is non degenerate in $\bar{M}$. According to the causal character of α' and V, there are four possibilities:

(1) α' and V are non-null and linearly independent.

(2) α' is null and V is non-null with $\langle\alpha',V\rangle\neq 0$.

(3) α' is non-null and V is null with $\langle\alpha',V\rangle\neq 0$.

(4) α' and V are null with $\langle\alpha',V\rangle\neq 0$.

It is easy to see that, with an appropriate change of the curve α, cases (2) and (3) reduce to (1) and (4), respectively.

Now we are going to do a detailed study of this kind of surfaces. To compute the metric induced on M_α, we apply the Gauss lemma to see that

$$\begin{array}{rcl}\langle X_s,X_t\rangle = & \langle\alpha'+tV',V\rangle & =\langle\alpha',V\rangle,\\ \langle X_t,X_t\rangle = & \langle V,V\rangle & =\varepsilon_V,\end{array}$$

where $\varepsilon_V\in\{-1,0,1\}$. Note that, for each fixed s, the vector field X_s is a Jacobi vector field along $\gamma_s(t)$ with initial conditions $X_s(0)=\alpha'(s)$ and $X_s'(0)=V'(s)$. As $\bar{M}$ is a space of constant curvature, we can write $X_s(s,t)=f(t)T_s(t)+g(t)Q_s(t)$, $T_s(t)$ and $Q_s(t)$ being parallel translation vector fields along $\gamma_s(t)$ of vectors $\alpha'(s)$ and $V'(s)$, respectively. Furthermore, the differentiable functions f and g satisfy the following differential equations

$$\begin{array}{lll}f''+\varepsilon_V cf=0, & f(0)=1, & f'(0)=0,\\ g''+\varepsilon_V cg=0, & g(0)=0, & g'(0)=1.\end{array}$$

Observe that, under above conditions, the functions f and g must satisfy the following system of ordinary differential equations

$$\begin{array}{rcl}f' & = & -\varepsilon_V cg,\\ g' & = & f,\\ f^2+\varepsilon_V cg^2 & = & 1.\end{array}$$

Then we have

$$\begin{array}{rcl}\langle X_s,X_s\rangle & = & f^2\langle T_s,T_s\rangle+2fg\langle T_s,Q_s\rangle+g^2\langle Q_s,Q_s\rangle\\ & = & f^2\langle\alpha',\alpha'\rangle+2fg\langle\alpha',V'\rangle+g^2\langle V',V'\rangle.\end{array}$$

Hence the matrix (G_{ij}) of the induced metric on M_α states as follows

$$\left(\begin{array}{cc}f^2\langle\alpha',\alpha'\rangle+2fg\langle\alpha',V'\rangle+g^2\langle V',V'\rangle & \langle\alpha',V\rangle\\ \langle\alpha',V\rangle & \varepsilon_V\end{array}\right).$$

Assume now that we have choosen an orientation on $\bar{M}$. Then a volume element ω is determined on $\bar{M}$ by the condition $\omega(X,Y,Z)=(-1)^\nu$, for any positively oriented orthonormal frame $\{X,Y,Z\}$. Therefore, for any couple X and Y of tangent vectors to $\bar{M}$, the vector product $X\wedge Y$ is the unique tangent vector to $\bar{M}$ such that $\langle X\wedge Y,Z\rangle=\omega(X,Y,Z)$, for any tangent vector Z.

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It is well known that vector product of parallel vector fields is again a parallel vector field, so that a vector field ξ normal to M_α in $\bar{M}$ can be given in terms of $X_s \wedge X_t$ and therefore we can write

$$\xi(s,t) = X_s \wedge X_t = f(t)\hat{P}_s(t) + g(t)\hat{Q}_s(t),$$

where $\hat{P}_s(t)$ and $\hat{Q}_s(t)$ are parallel translation vector fields along $\gamma_s(t)$ of $(\alpha' \wedge V)(s)$ and $(V' \wedge V)(s)$, respectively. Bearing in mind that $\langle X \wedge Y, X \wedge Y\rangle = (-1)^\nu(\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2)$, we see that $\langle \xi, \xi\rangle = (-1)^\nu \det(G_{ij})$.

When $\alpha(s)$ is a non null curve, if we take the vector field V orthogonal to α' then the metric (G_{ij}) writes down as

$$\begin{pmatrix} G_{11} & 0 \\ 0 & \varepsilon_V \end{pmatrix}.$$

The unit normal vector field is given by $\eta = e\xi$, where the function e is obtained from $e^2 = \varepsilon_\alpha / G_{11}$. The shape operator S is easily computed from

$$\begin{aligned} \langle SX_s, X_s\rangle &= -e\langle X_s, \xi_s\rangle = e\langle X_{ss}, \xi\rangle, \\ \langle SX_s, X_t\rangle &= -e\langle X_s, \xi_t\rangle = e\langle X_{st}, \xi\rangle, \\ \langle SX_t, X_t\rangle &= -e\langle X_t, \xi_t\rangle = e\langle X_{tt}, \xi\rangle = 0, \end{aligned}$$

and the curvature of the surface is given by

$$K = c - \left(\frac{\varepsilon_\alpha \varepsilon_V}{G_{11}}\right) e^2 \langle X_{st}, \xi\rangle^2.$$

As $X_s = fT_s + gQ_s$, we have that $X_{st} = f'T_s + g'Q_s$ and therefore

$$\begin{aligned} \langle X_{st}, \xi\rangle &= ff'\langle \alpha', \alpha' \wedge V\rangle + fg'\langle \alpha' \wedge V, V'\rangle + f'g\langle \alpha', V' \wedge V\rangle + g^2\langle V', V \wedge V'\rangle \\ &= (fg' - f'g)\omega(\alpha', V, V') \\ &= \langle \alpha' \wedge V, V'\rangle. \end{aligned}$$

Thus

$$K = c - (-1)^\nu \left(\frac{\langle \alpha' \wedge V, V'\rangle}{G_{11}}\right)^2. \tag{1}$$

3. Solutions in flat ruled surfaces

Let $\alpha : I \to \bar{M}$ be an immersed unit speed curve in $\bar{M}$. Let $\{T, N, B\}$ be the Frenet frame along α. The Frenet equations relative to this frame write down as follows

$$\begin{aligned} \overline{\nabla}_T T &= \varepsilon_2 \kappa N, \\ \overline{\nabla}_T N &= -\varepsilon_1 \kappa T - \varepsilon_3 \tau B, \\ \overline{\nabla}_T B &= \varepsilon_2 \tau N, \end{aligned}$$

where $\varepsilon_1 = \langle T, T\rangle$, $\varepsilon_2 = \langle N, N\rangle$ and $\varepsilon_3 = \langle B, B\rangle$ stand for the causal characters of T, N and B, respectively; κ and τ being the curvature and torsion functions of α.

Consider the surface M_α into $\bar{M}$ parametrized by

$$X(s,t) = \exp_{\alpha(s)}(tB(s)).$$

Then we have

$$\begin{aligned} X_s(s,t) &= (\mathrm{dexp}_{\alpha(s)})_{tB(s)}(T(s)+\varepsilon_2\tau(s)tN(s)),\\ X_t(s,t) &= (\mathrm{dexp}_{\alpha(s)})_{tB(s)}(B(s)). \end{aligned}$$

Since X_s is a Jacobi vector field along the geodesic $\gamma_s(t)$, we can write $X_s(s,t) = f(t)T_s(t) + \varepsilon_2\tau(s)g(t)N_s(t)$, $T_s(t)$ (resp. $N_s(t)$) being the parallel translation of $T(s)$ (resp. $N(s)$) along $\gamma_s(t)$ and the functions f and g are determined as above.

Assume now that M_α is flat. From (1) this is equivalent to $\tau^2(s) = (-1)^\nu c$. The unit normal vector field to M_α can be written as

$$\eta(s,t) = f(t)N_s(t) - \varepsilon_1\tau(s)g(t)T_s(t).$$

For later use we have that $X_s \wedge X_t = \varepsilon_2\eta$, $X_s \wedge \eta = -\varepsilon_3 X_t$ and $X_t \wedge \eta = \varepsilon_1 X_s$. On the other hand, it is quite easy to get

$$\begin{aligned} \overline{\nabla}_{X_s}X_s &= \varepsilon_2\kappa\eta,\\ \overline{\nabla}_{X_s}X_t &= \varepsilon_2\tau\eta,\\ \overline{\nabla}_{X_t}X_s &= \varepsilon_2\tau\eta,\\ \overline{\nabla}_{X_t}X_t &= 0. \end{aligned}$$

It is a straightforward computation that, in general, the standard parametrization $X(s,t)$ of M_α is not a solution of the Betchov-da Rios equation. In view of the geometric structure of ruled surfaces, that was explicitly described in Section 2, it seems natural to seek for parametrizations of them being congruence solutions of Betchov-da Rios equation. A special case appears when α is a curve of constant curvature κ; in this case, it suffices to write $Y(s,t) = X(s,ct)$, $c = -\varepsilon_2\varepsilon_3\kappa$, to find out that Y is a solution of this equation. In the main result of this section we will show that all solutions can be essentially found in this way.

Let $h \in$Diff$(\mathbb{R}^2)$ and write $h(u,v) = (s(u,v), t(u,v))$, so that $s_u t_v - s_v t_u$ does not vanish anywhere. Now $Y(u,v) = X(s(u,v), t(u,v))$ is a solution of the Betchov-Da Rios equation if and only if $Y_u \wedge \overline{\nabla}_{Y_u}Y_u = Y_v$ and $\langle Y_u, Y_u\rangle = \varepsilon$, ε being the causal character of the u-curves. In particular, $\langle Y_u, Y_v\rangle = 0$. We put $Y_u = s_u X_s + t_u X_t$ and $Y_v = s_v X_s + t_v X_t$. A straightforward computation allows us to get

$$\begin{aligned} \overline{\nabla}_{Y_u}Y_u &= s_{uu}X_s + t_{uu}X_t + 2s_u t_u\overline{\nabla}_{X_s}X_t + s_u^2\overline{\nabla}_{X_s}X_s\\ &= s_{uu}X_s + t_{uu}X_t + \varepsilon_2(2\tau s_u t_u + \kappa s_u^2)\eta. \end{aligned}$$

We find that

$$Y_u \wedge \overline{\nabla}_{Y_u}Y_u = \varepsilon_1\varepsilon_2(\kappa s_u^2 t_u + 2\tau s_u t_u^2)X_s - \varepsilon_2\varepsilon_3(\kappa s_u^3 + 2\tau s_u^2 t_u)X_t + \varepsilon_2(s_u t_{uu} - s_{uu}t_u)\eta.$$

Therefore $Y(u,v)$ is a solution of the Betchov-Da Rios equation if and only if the following system of partial differential equations holds:

$$\begin{aligned} s_v &= \varepsilon_1\varepsilon_2 s_u t_u(\kappa s_u + 2\tau t_u),\\ t_v &= -\varepsilon_2\varepsilon_3 s_u^2(\kappa s_u + 2\tau t_u),\\ 0 &= s_u t_{uu} - s_{uu}t_u. \end{aligned}$$

It follows that $t_u = bs_u$, for a certain function b, only depending on v, which measures the slope of the u-curves (v constant). On the other hand, since $\langle Y_u, Y_u\rangle = \varepsilon_1 s_u^2 + \varepsilon_3 t_u^2 = \varepsilon$, we find that $\varepsilon = (\varepsilon_1 + \varepsilon_3 b^2)s_u^2$. Then s_u only depends on v, so that $s(u,v) = h_1(v)u + h_2(v)$, for certain differentiable functions h_1 and h_2. In particular, $s_{uu} = t_{uu} = 0$, and so we obtain

$$\overline{\nabla}_{Y_u} Y_u = \varepsilon_2 s_u^2(\kappa + 2b\tau)\eta.$$

On the way, the following claim has been proved: the u-curves are geodesics in the surface M_α. The curvature function of these curves in $\bar{M}$ is given by

$$\rho(u,v) = (\kappa + 2b\tau)s_u^2.$$

In particular it does not vanish anywhere.

Now we use the compatibility condition $s_{uv} = s_{vu}$ to get $\rho(u,v) = f_1(v)u + f_2(v)$, for certain functions f_1 and f_2 defined on the whole real line. Consequently, f_1 vanishes identically, because ρ does not vanish and is defined on the whole plane. This shows that $\rho_u = 0$ and from (2) we deduce that κ is constant. Now we use the other compatibility condition, namely $t_{uv} = t_{vu}$, to deduce that b, s_u and ρ are all constant. Consequently, we have proved the following

Theorem 3.1 *Let $\alpha(s)$ be an arclength parametrized curve in $\bar{M}$ with torsion $\tau^2 = (-1)^\nu c$ and M_α the flat ruled surface parametrized by $X(s,t)$. For any $h \in$ Diff$(\mathbb{R}^2)$ we consider $Y = X \circ h : \mathbb{R}^2 \to M_\alpha$. Then Y is a solution of the Betchov-Da Rios soliton equation in $\bar{M}$ if and only if*
(1) *α has constant curvature, say κ, in $\bar{M}$ and*
(2) *$h(u,v) = (s(u,v), t(u,v))$ is given by*

$$\begin{aligned} s(u,v) &= au + \varepsilon_1\varepsilon_2 ab\rho v + c_1, \\ t(u,v) &= abu - \varepsilon_2\varepsilon_3 a\rho v + c_2, \end{aligned}$$

where $a^2(\varepsilon_1 + \varepsilon_3 b^2) = \varepsilon = \pm 1$, $b \in \mathbb{R} \setminus \{-\kappa/2\tau\}$, $\rho = a^2(\kappa + 2b\tau)$ is the curvature of the u-curves in $\bar{M}$ and (c_1, c_2) is any couple of constants.

Next we are going to show that any solution obtained in this theorem is actually a helix in $\bar{M}$, whose evolution is made up by helices which are congruent to him. In order to clarify this fact, we only need to compute the torsion of the u-curves. Notice that the unit normal to those curves and η (the unitary normal to M_α in $\bar{M}$) agree, and the unit binormal is $(1/\rho)Y_v$. Therefore the torsion θ of the u-curves is

$$\theta = \langle \overline{\nabla}_{Y_u}\eta, \frac{1}{\rho}Y_v\rangle = \varepsilon\varepsilon_2 \frac{\varepsilon_3\tau - \varepsilon_1\kappa b - \varepsilon_1\tau b^2}{\varepsilon_1 + \varepsilon_3 b^2}.$$

The converse holds too. Given a helix β in $\bar{M}$ with curvature ρ and torsion θ, it can be regarded as a solution of the filament equation in $\bar{M}$ living in a certain flat ruled surface M_α. Indeed, just consider the ruled surface M_α over a curve α in $\bar{M}$ with constant curvature $\kappa = (\varepsilon/\rho)\{\varepsilon_1\rho^2 + \varepsilon_3\theta^2 - \varepsilon_3(-1)^\nu c\}$ and torsion $\tau^2 = (-1)^\nu c$, and then take a geodesic in M_α with slope $b = (1/\rho)(\varepsilon\varepsilon_3\tau - \varepsilon_1\varepsilon_2\theta)$.

We wish to point out that this theorem allows us to give explicit examples of solutions of the Betchov-Da Rios equation in the Riemannian space forms $\mathbb{R}^3$ and $\mathbb{S}^3$, as well as in the Lorentzian space forms $\mathbb{L}^3$ and $\mathbb{H}^3_1$ (see [2]).

Now, we exhibit some examples.

Example 3.2 (Solution lying in the Hopf lifting of a horocycle)
Let $\alpha : \mathbb{R} \to \mathbb{H}^3_1$ be the curve in $\mathbb{H}^3_1 \subset \mathbb{R}^4_2$ defined by

$$\alpha(s) = (\cos s + s \sin s, s \cos s - \sin s, s \sin s, s \cos s)$$

It should be noticed that this curve projects down, via the usual Hopf maps, in a horocycle of the hyperbolic plane ([2]). It is not difficult to see that the unit binormal of this curve is timelike and it is given by

$$B(s) = (\sin s - s \cos s, \cos s + s \sin s, -s \cos s, s \sin s)$$

Moreover, the curvature and torsion of α are computed to be $\kappa = 2$ and $\tau^2 = 1$. The B-scroll M_α can be parametrized by

$$X(s,t) = (\cos(s+t) + s \sin(s+t), -\sin(s+t) + s \cos(s+t), s \sin(s+t), s \cos(s+t))$$

As a consequence of Theorem 3.1, the solutions of the Betchov-Da Rios equation in $\mathbb{H}^3_1$ lying in the above ruled surface are given by $Y(u,v) = X(s(u,v), t(u,v))$, where $s(u,v) = a(u + b\rho v)$ and $t(u,v) = a(bu + \rho v)$ with $a^2(1-b^2) = \pm 1$ and $\rho = 2a^2(1 \pm b) \neq 0$.

Example 3.3 (Solution lying in the Hopf lifting of a geodesic circle)
Let $\alpha : \mathbb{R} \to \mathbb{H}^3_1$ be the curve in $\mathbb{H}^3_1 \subset \mathbb{R}^4_2$ defined by

$$\alpha(s) = (p \cos(\frac{q}{p}s), -p \sin(\frac{q}{p}s), q \sin(\frac{p}{q}s), q \cos(\frac{p}{q}s))$$

where $p = \sqrt{\frac{2r+1}{2}}$ and $q = \sqrt{\frac{2r-1}{2}}$, and r a real number with $4r^2 - 1 > 0$. It should be noticed that this curve projects down, via the usual Hopf maps, in a geodesic circle of the hyperbolic plane ([2]). It is not difficult to see that the unit binormal of this curve is timelike and it is given by

$$B(s) = (p \sin(\frac{q}{p}s), p \cos(\frac{q}{p}s), -q \cos(\frac{p}{q}s), q \sin(\frac{p}{q}s))$$

Moreover, the curvature and torsion of α are computed to satisfy $\kappa^2 = \frac{16r^2}{4r^2-1}$ and $\tau^2 = 1$. The B-scroll M_α may be parametrized as

$$X(s,t) = (p \cos(\frac{q}{p}s + t), -p \sin(\frac{q}{p}s + t), q \sin(\frac{p}{q}s + t), q \cos(\frac{p}{q}s + t))$$

Now we use Theorem 3.1 to see that the solutions of the Betchov-Da Rios equation in $\mathbb{H}^3_1$ which lie in the above ruled surface are given by $Y(u,v) = X(s(u,v), t(u,v))$, where $s(u,v) = a(u + b\rho v)$ and $t(u,v) = a(bu + \rho v)$ with $a^2(1-b^2) = \pm 1$ and $\rho = a^2(\frac{4r}{\sqrt{4r^2-1}} \pm 2b) \neq 0$.

Observe that when $\varepsilon_1 + \varepsilon_3 b^2 = 0$, $Y(u,v)$ parametrizes a null geodesic of M_α into $\mathbb{L}^3$ and $\mathbb{H}^3_1(c)$. A straightforward computation shows that this curve is a singular solution of the Betchov-Da Rios equation. So we have the following

Corollary 3.4 *Let M_α be a flat Lorentzian ruled surface into $\mathbb{L}^3$ or $\mathbb{H}^3_1(c)$ where α is of constant curvature. Then the only soliton solutions of the Betchov-Da Rios equation in $\mathbb{L}^3$ or $\mathbb{H}^3_1(c)$ lying in M_α are the null geodesics of M_α.*

4. Solutions in B-scrolls over null curves

Let $\alpha : I \to \bar{M}$ be an immersed null curve in $\bar{M}$ ($\nu = 1$) with associated Cartan frame $\{A = \alpha', B, C\}$, i.e., $\langle A, A\rangle = 0$, $\langle B, B\rangle = 0$, $\langle A, B\rangle = -1$, $\langle C, C\rangle = 1$, $\langle A, C\rangle = 0$ and $\langle B, C\rangle = 0$, satisfying the equations

$$\begin{aligned}\overline{\nabla}_A A &= kC,\\ \overline{\nabla}_A B &= \lambda C,\\ \overline{\nabla}_A C &= \lambda A + kB,\end{aligned}$$

$k = k(s) \neq 0$ being a function along the curve $\alpha(s)$ and λ a constant.

Let M_α be the surface in $\bar{M}$ parametrized by

$$X(s,t) = \exp_{\alpha(s)}(tB(s)).$$

This surface is called a B-scroll over the null curve α (see [3]). As above we have

$$\begin{aligned}X_s(s,t) &= (\mathrm{dexp}_{\alpha(s)})_{tB(s)}(A(s) + t\lambda C(s)),\\ X_t(s,t) &= (\mathrm{dexp}_{\alpha(s)})_{tB(s)}(B(s)).\end{aligned}$$

Since X_s is a Jacobi field along the null geodesic $\gamma_s(t) = X(s,t)$ and $\bar{M}$ is a space form, we can write

$$X_s(s,t) = A_s(t) + t\lambda C_s(t),$$

$A_s(t)$ (resp. $C_s(t)$) being the parallel translation of $A(s)$ (resp. $C(s)$) along $\gamma_s(t)$.

Let $\eta = X_s \wedge X_t$ be a unit normal vector field to M_α. A straightforward computation yields

$$\begin{aligned}\overline{\nabla}_{X_s} X_s &= \lambda^2 t X_s + \lambda^4 t^3 X_t + (k - \lambda^3 t^2)\eta,\\ \overline{\nabla}_{X_s} X_t &= \overline{\nabla}_{X_t} X_s = -\lambda^2 t X_t + \lambda\eta,\\ \overline{\nabla}_{X_t} X_t &= 0.\end{aligned}$$

We look for reparametrizations of X which are solutions of the Betchov-Da Rios equation. To do that let $h \in$Diff$(\mathbb{R}^2)$ and write $h(u,v) = (s(u,v), t(u,v))$. Then $Y = X \circ h$ is a solution if and only if

$$Y_u \wedge \overline{\nabla}_{Y_u} Y_u = Y_v,$$

and $\langle Y_u, Y_u\rangle = \delta = \pm 1$. Bearing in mind that $Y_u = s_u X_s + t_u X_t$ and $Y_v = s_v X_s + t_v X_t$, a simple computation leads to

$$\begin{aligned}\overline{\nabla}_{Y_u} Y_u &= s_{uu} X_s + t_{uu} X_t + 2 s_u t_u \overline{\nabla}_{X_s} X_t + s_u^2 \overline{\nabla}_{X_s} X_s\\ &= (s_{uu} + \lambda^2 t s_u^2) X_s + (t_{uu} - 2\lambda^2 t s_u t_u + \lambda^4 t^3 s_u^2) X_t\\ &\quad + (2\lambda s_u t_u + (k - \lambda^3 t^2) s_u^2)\eta.\end{aligned}$$

Now from

$$\begin{aligned}X_s \wedge \eta &= X_s + \lambda^2 t^2 X_t,\\ X_t \wedge \eta &= -X_t,\end{aligned}$$

we deduce that Y is a solution of the Betchov-Da Rios equation if and only if the following system of partial differential equations holds:

$$\begin{aligned}\lambda\left(\frac{t_u}{s_u}-\frac{t_v}{s_v}\right) &= \frac{s_v}{s_u^3}-k,\\ \frac{t_u}{s_u}+\frac{t_v}{s_v} &= \lambda^2 t^2,\\ \left(\frac{t_v}{s_v}\right)_u &= \lambda^2 t s_u\frac{t_v}{s_v}.\end{aligned}$$

To get solutions of this system, let α be a generalized null cubic, i.e., a null curve with a Cartan frame such that $\lambda=0$ (see [4]). Then it reduces to

$$\begin{aligned}s_v &= k s_u^3,\\ t_v &= -k s_u^2 t_u,\\ 0 &= s_{uu}t_u - s_u t_{uu}.\end{aligned}$$

Following a similar procedure to that in the previous section we deduce that $s_u=a$ and $t_u=ab$, a and b being both constant and related by $2ba^2=-\delta$. Moreover k is also a constant function and the u-curves are geodesics in M_α whose curvature in $\bar{M}$ is $\rho=ka^2$. On the other hand, since the vector field $\frac{1}{\rho}Y_v$ is the binormal to u-curves, we find that the torsion θ of the u-curves in $\bar{M}$ is given by

$$\theta=-\delta\langle\overline{\nabla}_{Y_u}\eta,\frac{1}{\rho}Y_v\rangle=\delta\rho.$$

So we have shown the following result:

Theorem 4.1 *Let $\alpha(s)$ be a generalized null cubic in $\bar{M}$ and M_α the B-scroll parametrized by $X(s,t)$. For any $h\in$Diff$(\mathbb{R}^2)$ we consider $Y=X\circ h:\mathbb{R}^2\to M_\alpha$. Then Y is a solution of the Betchov-Da Rios soliton equation in $\bar{M}$ if and only if*
(1) *the function k is constant and*
(2) *$h(u,v)=(s(u,v),t(u,v))$ is given by*

$$\begin{aligned}s(u,v) &= au+ka^3v+c_1,\\ t(u,v) &= abu-kba^3v+c_2,\end{aligned}$$

where $2ba^2=-\delta=\pm1$, δ is the causal character of the u-curves, $b\in\mathbb{R}\setminus\{0\}$ and (c_1,c_2) is any couple of constants. Moreover, the u-curves are helices in $\bar{M}$ with curvature $\rho=ka^2$ and torsion $\theta=\delta ka^2$.

It is worth noting that we have found out explicit examples of solutions of the Betchov-Da Rios equation in the three models of Lorentzian space forms $\mathbb{L}^3$, $\mathbb{S}^3_1$ and $\mathbb{H}^3_1$. The newness here is $\mathbb{S}^3_1$.

Finally, to illustrate the last theorem we exhibit some examples.

Example 4.2 Let $\alpha:\mathbb{R}\to\mathbb{L}^3$ be the curve in $\mathbb{L}^3$ defined by

$$\alpha(s)=k\left(\frac{s^3}{3}-\frac{s}{4},\frac{s^2}{2},\frac{s^3}{3}+\frac{s}{4}\right),\qquad k\neq0.$$

It is easy to see that this curve is a generalized null cubic in $\mathbb{L}^3$ with constant curvature k and Cartan frame given by

$$\begin{aligned} A(s) &= k\left(s^2-\frac{1}{4}, s, s^2+\frac{1}{4}\right), \\ B(s) &= \frac{2}{k}(1,0,1), \\ C(s) &= (2s,1,2s). \end{aligned}$$

The B-scroll M_α associated to α is parametrized by

$$X(s,t) = \left(k\left(\frac{s^3}{3}-\frac{s}{4}\right)+\frac{2t}{k}, k\frac{s^2}{2}, k\left(\frac{s^3}{3}+\frac{s}{4}\right)+\frac{2t}{k}\right).$$

As consequence of Theorem 4.1 the solutions of the Betchov-Da Rios equation in $\mathbb{L}^3$ lying in the B-scroll M_α are given by $Y(u,v) = (Y_1(u,v), Y_2(u,v), Y_3(u,v))$, where

$$\begin{aligned} Y_1(u,v) &= k\left(\frac{a^3}{3}(u+a^2kv)^3-\frac{a}{4}(u+a^2kv)\right)-\frac{\delta}{ak}(u-a^2kv), \\ Y_2(u,v) &= \frac{a^2k}{2}(u+a^2kv)^2, \\ Y_3(u,v) &= k\left(\frac{a^3}{3}(u+a^2kv)^3+\frac{a}{4}(u+a^2kv)\right)-\frac{\delta}{ak}(u-a^2kv), \end{aligned}$$

with $a \in \mathbb{R}\setminus\{0\}$ and $\delta = \pm 1$. The u-curves are helices in $\mathbb{L}^3$ with causal character δ, curvature $\rho = ka^2$ and torsion $\theta = k\delta a^2$.

Example 4.3 Let $\alpha : \mathbb{R} \to \mathbb{S}^3_1$ be the curve in $\mathbb{S}^3_1$ defined by

$$\alpha(s) = \frac{\sqrt{2}}{2}\left(\cos[\sqrt{k}s], \sin[\sqrt{k}s], \cosh[\sqrt{k}s], \sinh[\sqrt{k}s]\right), \qquad k>0.$$

This curve is a generalized null cubic in $\mathbb{S}^3_1$ with constant curvature k and Cartan frame given by

$$\begin{aligned} A(s) &= \frac{\sqrt{2k}}{2}\left(-\sin[\sqrt{k}s], \cos[\sqrt{k}s], \sinh[\sqrt{k}s], \cosh[\sqrt{k}s]\right), \\ B(s) &= \frac{\sqrt{2k}}{2k}\left(\sin[\sqrt{k}s], -\cos[\sqrt{k}s], \sinh[\sqrt{k}s], \cosh[\sqrt{k}s]\right), \\ C(s) &= \frac{\sqrt{2}}{2}\left(-\cos[\sqrt{k}s], -\sin[\sqrt{k}s], \cosh[\sqrt{k}s], \sinh[\sqrt{k}s]\right). \end{aligned}$$

The B-scroll M_α associated to α is parametrized by

$$\begin{aligned} X(s,t) = \frac{\sqrt{2}}{2}\Big(&\cos[\sqrt{k}s]+\frac{t}{\sqrt{k}}\sin[\sqrt{k}s], \sin[\sqrt{k}s]-\frac{t}{\sqrt{k}}\cos[\sqrt{k}s], \\ &\cosh[\sqrt{k}s]+\frac{t}{\sqrt{k}}\sinh[\sqrt{k}s], \sinh[\sqrt{k}s]+\frac{t}{\sqrt{k}}\cosh[\sqrt{k}s]\Big). \end{aligned}$$

Now the solutions of the Betchov-Da Rios equation in $\mathbb{S}^3_1$ lying in M_α are given by

$$Y(u,v) = \frac{\sqrt{2}}{2}(Y_1(u,v), Y_2(u,v), Y_3(u,v), Y_4(u,v)),$$

where

$$\begin{aligned}
Y_1(u,v) &= \cos[\sqrt{k}(au + ka^3v)] - \frac{\delta}{2\sqrt{k}}\left(\frac{u}{a} - kav\right)\sin[\sqrt{k}(au + ka^3v)],\\
Y_2(u,v) &= \sin[\sqrt{k}(au + ka^3v)] + \frac{\delta}{2\sqrt{k}}\left(\frac{u}{a} - kav\right)\cos[\sqrt{k}(au + ka^3v)],\\
Y_3(u,v) &= \cosh[\sqrt{k}(au + ka^3v)] - \frac{\delta}{2\sqrt{k}}\left(\frac{u}{a} - kav\right)\sinh[\sqrt{k}(au + ka^3v)],\\
Y_4(u,v) &= \sinh[\sqrt{k}(au + ka^3v)] - \frac{\delta}{2\sqrt{k}}\left(\frac{u}{a} - kav\right)\cosh[\sqrt{k}(au + ka^3v)],
\end{aligned}$$

with $a \in \mathbb{R} \setminus \{0\}$ and $\delta = \pm 1$.

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