# Conformal tension in string theories and M-theory

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**Abstract**

This paper deals with string theories and M-theories on backgrounds of the form $AdS \times M$, $M$ being a compact principal $U(1)$-bundle. These configurations are the natural settings to study Hopf T-dualities [15], and so to define duality chains connecting different string theories and M-theories. There is an increasing great interest in studying those properties (physical or geometrical) which are preserved along the duality chains. For example, it is known that Hopf T-dualities preserve the black hole entropies [15]. In this paper we consider a two-parameter family of actions which constitutes a natural variation of the conformal total tension action (also known as Willmore-Chen functional in Differential Geometry). Then, we show that the existence of wide families of solutions (in particular compact solutions) for the corresponding motion equations is preserved along those duality chains. In particular, we exhibit ample classes of Willmore-Chen submanifolds with a reasonable degree of symmetry in a wide variety of conformal string theories and conformal M-theories, that in addition are solutions of a second variational problem known as the area-volume isoperimetric problem. These are good reasons to refer those submanifolds as the best worlds one can find in a conformal universe. The method we use to obtain this invariant under Hopf T-dualities is based on the principle of symmetric criticality. However, it is used in a two-fold sense. First to break symmetry and so to reduce variables. Second to gain rigidity in direct approaches to integrate the Euler-Lagrange equations. The existence of generalized elastic curves is also important in the explicit exhibition of those configurations. The relationship between solutions and elasticae can be regarded as a holographic property.

Anti-de Sitter space; String theories; M-theories; T-dualities; Conformal total tension actions; Generalized elasticae.

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## 1. Introduction and set up

There is an increasing interest in the interplay between bulk and boundary dynamics. The two main directions being explored presently are:

1. The conjecture of J.Maldacena [30], that concerns with the string theory or M-theory on certain backgrounds of the form $AdS_p \times M_{D-p}$. Here, $AdS_p$ is the anti-de Sitter space of dimension $p$ and $M_{D-p}$ is a compact space with dimension $D-p$. Now, $D$ is 10 or 11 depending on whether we are doing string theory or M-theory, respectively. This conjecture postulates that the quantum string or the M-theory on this background is mathematically equivalent to an ordinary but conformally invariant quantum field theory in a space time of dimension $p-1$ which plays the part of *boundary* of $AdS_p$. This relationship is known as $AdS/CFT$ correspondence.

2. The holographic hypothesis (see [24] and references therein) states that all information of a theory in the bulk of a bounded region is available, in some sense, on the boundary of the region. However, the above *some sense* means that, as far as we know, there are at least four possible non-equivalent definitions of what holography may mean. For instance, the $AdS/CFT$ correspondence can be viewed as an example of the holographic hypothesis [41].

Consequently, the $AdS/CFT$ correspondence has revived a great interest in gauged extended supergravities which arise as the massless sector of the Kaluza-Klein compactifications of $D = 11$ supergravity, such as $AdS_4 \times \mathbb{S}^7$ and $AdS_7 \times \mathbb{S}^4$, as well as in Type IIB supergravity, such as $AdS_5 \times \mathbb{S}^5$ (see, for example, [12, 14, 31]). In [14], M.J.Duff, H.Lü and C.N.Pope considered that $\mathbb{S}^5$ is a $U(1)$ bundle over $\mathbb{CP}^2$ (by means of the usual Hopf map) to define an unconventional type of T-duality. This is applied to construct the duality chain [$n = 4$ Yang-Mills]$\longrightarrow$[Type IIB string on $AdS_5 \times \mathbb{S}^5$]$\longrightarrow$[Type IIA string on $AdS_5 \times \mathbb{CP}^2 \times \mathbb{S}^1$]$\longrightarrow$[M-theory on $AdS_5 \times \mathbb{CP}^2 \times T^2$]. This kind of duality can be extended to more general contexts including backgrounds as $AdS \times M$, where $M$ are Einstein spaces that are not necessarily round spheres but still $U(1)$ fibrations. Notice that these spaces naturally appear as the near horizon geometries of supermembranes with fewer Killing spinors [16].

Hopf T-duality drastically changes the topology of the compactification space $\mathbb{S}^5$ by untwisting it to $\mathbb{CP}^2 \times \mathbb{S}^1$. However, it preserves the black hole entropies. The purpose of this paper is to show an interesting property which remains invariant along the Hopf T-duality chain. In fact, we prove the existence of rational one-parameter families of Willmore-Chen submanifolds (i.e. critical points of the conformal total tension action) at any stage of the duality chain. This is done in sections 4 and 5. The case of Type IIB theory on the background $AdS_5 \times \mathbb{S}^5$ is particularly interesting. The high rigidity of the round five sphere allows us to get a wide variety of solutions, having three quite different families of them. The first one contains solutions which are obtained via a standard use of the principle of symmetric criticality [36], when lifting elasticae in the complex projective plane $\mathbb{CP}^2$ by the usual Hopf mapping. In particular, many properties of these solutions are reflected, by a kind of holographic principle, in those elasticae (for example, they have constant mean curvature if and only if they come from elasticae which have constant curvature in $\mathbb{CP}^2$). Other two classes of configurations are obtained by exploiting the principle of symmetric criticality in a different approach, which is used to gain rigidity when integrating the corresponding motion equations. Therefore, we show that minimal flat tori are always solutions, which seems reasonable and however it is not an immediate result. This allows us to give a second family of explicit solutions by considering results of [26, 27]. We still have a third family by using the Chen finite type theory of submanifolds [3, 11]. Namely, we get explicit solutions constructed in the corresponding Euclidean space through the eigenfunctions of the Laplacian associated with exactly two different eigenvalues. Of course these three classes of solutions can be projected down to any $5$-dimensional lens space.

The above stated results are extended in section 6 to other configurations of the form $AdS_5 \times M_5$, where $M_5$ is a principal $U(1)$-bundle over a compact $4$-dimensional manifold $N$ which admits an Einstein metric with positive scalar curvature. These $U(1)$-bundles are classified by the cohomology group $H^2(N, \mathbb{Z})$. Therefore, as an illustration we consider the Stiefel manifold regarded as the principal $U(1)$-bundle on the complex quadric $N = \mathbb{S}^2 \times \mathbb{S}^2$ associated with a suitable multiple of its first Chern class.

In section 7, we investigate the above invariant along the Hopf T-duality chains generated by Type IIB theories on backgrounds carrying both NS-NS and R-R electric and magnetic 3-form charges, and whose near horizon geometry contains $AdS_3 \times \mathbb{S}^3$. This case is richer than those we have considered above where only R-R charges were supported. In this case, we show the existence of ample classes of solutions along any duality chain. In particular, we prove:
*The conformal structure associated with the string-frame in the near horizon limit of any non dilatonic black hole in $D = 5$ and $D = 4$ admits a rational one-parameter class of $4$-dimensional Willmore-Chen configurations that have constant mean curvature in the original string metric. However, the degree of symmetry of these solutions is preserved only when NS-NS charges appear while that is decreased when R-R charges are carried.*

All these results are preceded by a symmetry breaking method which is developed through two general settings in section 2. The algorithm we use is based on the principle of symmetric criticality, rather in a formulation of this principle due to R.S.Palais [36]. Certainly this method can be applied to other backgrounds different from those considered in this paper, and also it is open to be extended to other contexts. For example, we do not know if the statement of Theorem 2.4 works with no assumption on the flatness of the concerned gauge potential. The paper is completed with sections 3 and 9. The former is dedicated to exploit the nice geometry of $AdS_3$ in order to obtain wide classes of elastic helices (in particular closed elastic helices, see Theorem 3.2) which generate solutions of the motion equations in any configuration of the form $AdS_3 \times M$ (Corollaries 3.4 and 3.5). In section 8, we show how our results can be extended to higher-dimensional theories, including F-theory.

Now we give here a generic formulation of our method.

Let $I(Q, (L, ds^2))$ be the space of immersions of a compact manifold $Q$ in a pseudo-Riemannian manifold $(L, ds^2)$. Define a real two-parameter family of actions

$$\left\{ \mathcal{W}_{ar} : I(Q, (L, ds^2)) \longrightarrow \mathbb{R} : a, r \in \mathbb{R} \right\},$$

by

$$\mathcal{W}_{ar}(\phi) = \int_Q (\alpha^2 - a\tau_e)^{r/2} dv,$$

for all $\phi \in I(Q, (L, ds^2))$, where $\alpha$ and $\tau_e$ stand for the mean curvature and the extrinsic scalar curvature functions of $\phi$, respectively. Also $dv$ is the volume element of $\phi^*(ds^2)$ on $Q$. It should be noticed that when $a = 1$ and $r$ is the dimension of $Q$, then we get the Willmore-Chen functional, which provides a variational problem of great interest in Differential Geometry, due in part to its invariance under conformal changes of the surrounding metric $ds^2$, also known as the conformal total tension action. Moreover, if $r = 2$, we have the Willmore functional which formally coincides with the Canham-Helfrich bending energy of fluid membranes and lipid vesicles [8, 23], and amazingly also with the Polyakov extrinsic action in the bosonic string theory [37]. It should be also observed that if the dimension of $Q$ is one (i.e. we are talking about immersed curves), then $\tau_e$ vanishes identically and $\alpha$ is nothing but the curvature function $\kappa$ of the curve $\gamma$ in $(L, ds^2)$. In this case the above family of actions reduces to a one-parameter one of elastic energy functionals, $\mathcal{F}_r = \int_\gamma \kappa^r ds$, acting on the space of closed curves $\gamma$ in $(L, ds^2)$.

Now, all these actions naturally appear from the own Willmore-Chen functional. This may be showed via an interesting argument, involving the principle of symmetric criticality [36], as well as the Kaluza-Klein inverse mechanism, which allows us to obtain the above actions by a reduction of symmetry process from the Willmore-Chen action in the conformal Kaluza-Klein ansatz.

## 2. Symmetry breaking phase transitions

Let $\omega$ be a principal connection on a principal fibre bundle $P(M, G)$, $G$ being an $m$-dimensional, compact Lie group endowed with a bi-invariant metric $d\sigma^2$. We denote by $\mathcal{M}$ and $\bar{\mathcal{M}}$ the spaces of pseudo-Riemannian metrics on $M$ and $P$, respectively. For $\varepsilon \in \{-1, +1\}$, let $\Phi_\varepsilon : \mathcal{M} \to \bar{\mathcal{M}}$ be the mapping defined by

$$\Phi_\varepsilon(g) = \pi^*(g) + \varepsilon\omega^*(d\sigma^2),$$

where $\pi : P \to M$ stands for the projection map of the principal fibre bundle.

A pseudo-Riemannian metric $\bar{g} \in \bar{\mathcal{M}}$ is called a *Kaluza-Klein metric* on $P$ if it belongs to the image of $\Phi_\varepsilon$. These metrics are also known as "bundle like" metrics and they are the natural ones working on a unified theory, collecting the gravitation $g$ with the gauge potential $\omega$, in the sense of Kaluza-Klein.

It is obvious that $\pi : (P, \bar{g}) \to (M, g)$ is a pseudo-Riemannian submersion whose leaves are the fibres, and so they are diffeomorphic to the structure group $G$. It is also evident that the natural action of $G$ on $P$ is carried out by isometries of $(P, \bar{g})$.

Let $\mathcal{V}$ be the vertical distribution of the pseudo-Riemannian submersion and $\mathcal{H}$, the horizontal one, its complementary $\bar{g}$-orthogonal distribution. Note that while $\mathcal{V}$ is involutive, whose leaves are the fibres, $\mathcal{H}$ is not integrable, in general. Let $N$ be a $q$-dimensional submanifold in $(M, g)$, then one can define a distribution $\bar{D}$ along $\pi^{-1}(N)$ by $\bar{D}_{\bar{p}} = (d\pi_{\bar{p}})^{-1}(T_{\pi(\bar{p})}N)$, for any $\bar{p} \in P$. Notice that $\mathcal{V}_{\bar{p}} \subset \bar{D}_{\bar{p}}$, so $\bar{D}$ is an $(m+q)$-dimensional distribution which is integrable. Furthermore $\bar{N} = \pi^{-1}(N)$ is a $G$-invariant submanifold of dimension $m+q$. The converse is also true. Assume $\bar{N}$ is an $(m + q)$-dimensional submanifold of $(P, G)$, which is $G$-invariant. Then $\bar{N}$ is foliated by the fibres and so the horizontal vectors tangent to $\bar{N}$ define a $q$-dimensional distribution in $\bar{N}$. Notice that this is not involutive though it projects down to an integrable distribution $D$ in $(M, g)$. Hence, $\bar{N} = \pi^{-1}(N)$, where $N$ is a leaf of $D$. Summarizing we have

**Lemma 2.1** *Let $N$ be a $q$-dimensional submanifold of $(M, g)$, then $\bar{N} = \pi^{-1}(N)$ is an $(m + q)$-dimensional submanifold of $(P, \bar{g})$ which is $G$-invariant. Conversely, if $\bar{N}$ is a $d$-dimensional submanifold of $(P, \bar{g})$ which is $G$-invariant, then $\bar{N} = \pi^{-1}(N)$, where $N$ is a $(d-m)$-dimensional submanifold of $(M, g)$.*

From now on, we will denote by overbars the lifts of corresponding objects on the base. In particular, $\bar{X}$ will denote the horizontal lift of a vector field $X$ on $M$.

**Proposition 2.2** *Let $\sigma$ and $\bar{\sigma}$ be the second fundamental forms on $N$ in $(M, g)$ and $\bar{N} = \pi^{-1}(N)$ in $(P, \bar{g})$, respectively. Denote by $\alpha$ and $\bar{\alpha}$ the corresponding mean curvature functions, then we have*

1. $\bar{\sigma}(V, W) = 0$*, for vertical vector fields $V$ and $W$.*
2. $\bar{\sigma}(\bar{X}, \bar{Y}) = \overline{\sigma(X, Y)}$.
3. $\bar{\alpha}^2 = \frac{q^2}{(m+q)^2}\alpha^2 \circ \pi$
4. *If $\omega$ is flat, then* $\bar{\sigma}(\bar{X}, V) = 0$.

**Proof.** To show the first claim, just notice that the fibres are totally geodesic not only in $(P, \bar{g})$, but also in $\bar{N}$, with respect to the $\bar{g}$-induced metric. To prove the second statement observe that $\bar{\sigma}(\bar{X}, \bar{Y})$ is horizontal. The third one is now clear. The last assumption is also evident because the O'Neill invariant, which measures the obstruction to the integrability of the horizontal distribution, vanishes identically if $\omega$ is assumed to be flat.

**Proposition 2.3** *Let $\tau_e$ and $\bar{\tau}_e$ be the extrinsic scalar curvature functions of $N$ in $(M, g)$ and $\bar{N} = \pi^{-1}(N)$ in $(P, \bar{g})$, respectively. If $\omega$ is flat, then we have*

$$\bar{\tau}_e = \frac{q(q-1)}{(m+q)(m+q-1)} \tau_e \circ \pi$$

**Proof.** Let $\{V_a; 1 \leqslant a \leqslant m\}$ be the fundamental vector fields in $P$ associated with a frame of unit left-invariant vector fields in $(G, d\sigma^2)$. Let $\{X_i; 1 \leqslant i \leqslant q\}$ be a local orthonormal frame on $(N, g')$, $g'$ being the $g$-induced metric on $N$. Then we have

$$\begin{aligned} \bar{\tau}_e &= \frac{1}{(m+q)(m+q-1)} \left[ \sum_{i,j=1}^{q} (\bar{K}'(\bar{X}_i, \bar{X}_j) - \bar{K}(\bar{X}_i, \bar{X}_j) \right. \\ &\quad + \sum_{i=1}^{q} \sum_{a=1}^{m} (\bar{K}'(\bar{X}_i, V_a) - \bar{K}(\bar{X}_i, V_a) \\ &\quad \left. + \sum_{a,b=1}^{m} (\bar{K}'(V_b, V_a) - \bar{K}(V_b, V_a) \right], \end{aligned}$$

where $\bar{K}$ and $\bar{K}'$ stand for the sectional curvature functions of $(P, \bar{g})$ and $\bar{N}$ with respect to the $\bar{g}$-induced metric. Now, we use Proposition 2.2 jointly with the Gauss equations of $\bar{N}$ in $(P, \bar{g})$ and $N$ in $(M, g)$, respectively, to get the result.

For a compact, $q$-dimensional manifold $N$, we write $I_G(N \times G, (P, \bar{g})) = \{\bar{\phi} \in I(N \times G, (P, \bar{g})) : \bar{\phi} \,\text{is}\, G-\text{invariant}\}$. Since $G$ is compact, then $I_G(N \times G, (P, \bar{g}))$ is a submanifold of $I(N \times G, (P, \bar{g}))$, which according to Lemma 2.1 can be identified with $I(N, (M, g))$. That is, $\bar{\phi} \in I_G(N \times G, (P, \bar{g}))$ if and only if there exists $\phi \in I(N, (M, g))$ such that $\bar{\phi}(N \times G) = \pi^{-1}(\phi(N))$. Then we use Propositions 2.2 and 2.3 to obtain

$$\mathcal{W}_{ar}(\bar{\phi}) = \left( \frac{q}{m+q} \right)^r \mathrm{vol}(G, d\sigma^2) \mathcal{W}_{br}(\phi), \tag{1}$$

where $b = \frac{(q-1)(m+q)}{q(m+q-1)} a$.

Now, given $(a, r) \in \mathbb{R}^2$ we denote by $\Sigma^{ar}$, $\bar{\Sigma}^{ar}$ and $\bar{\Sigma}^{ar}_G$ the sets of critical points of the functional $\mathcal{W}_{ar}$ on $I(N, (M, g))$, $I(N \times G, (P, \bar{g}))$ and $I_G(N \times G, (P, \bar{g}))$, respectively. In other words, they are the sets of configurations which are solutions of the motion equations for the $\mathcal{W}_{ar}$-dynamics on $I(N, (M, g))$, $I(N \times G, (P, \bar{g}))$ and $I_G(N \times G, (P, \bar{g}))$, respectively. Then the principle of symmetric criticality [36] assures us that

$$\bar{\Sigma}^{ar}_G = \bar{\Sigma}^{ar} \cap I_G(N \times G, (P, \bar{g})).$$

Said otherwise, to obtain critical points of $\mathcal{W}_{ar}$ on $I(N \times G, (P, \bar{g}))$ which do not break the natural $G$-symmetry of the problem, we only need to compute $\mathcal{W}_{ar}$ restricted to $I_G(N \times G, (P, \bar{g}))$. Since we have already computed this restriction (see (1)), then we obtain the following result, which can be regarded as a criterion for reduction of variables (in the sense of Palais) in the variational problems associated with the functionals $\mathcal{W}_{ar}$.

**Theorem 2.4** *$\bar{\phi} \in \bar{\Sigma}^{ar}$ if and only if $\phi \in \Sigma^{br}$, with $b = \frac{(q-1)(m+q)}{q(m+q-1)}a$.*

**Remark 2.5** Roughly speaking, the last result reduces the search for symmetric solutions of the $\mathcal{W}_{ar}$-dynamics in a unified theory $(P, \bar{g})$ to that for solutions of the $\mathcal{W}_{br}$-dynamics in its gravitatory component $(M, g)$, $a$ and $b$ being related as above. It should be noticed, in addition, that the best worlds to live in a conformal unified universe $(P, [\bar{g}])$, $[\bar{g}]$ denoting the conformal structure associated with $\bar{g}$, are the Willmore-Chen submanifolds which preserve the internal symmetry. We use the term "best" in a two-fold sense. First, because they preserve the above mentioned symmetry, and secondly because they support the smallest global tension possible from the surrounding universe, and consequently they must be Willmore-Chen submanifolds. For these worlds, we obtain the following gravitatory characterization.

**Corollary 2.6** *$\bar{\phi} \in \bar{\Sigma}^{1,m+q}$, that is, $\bar{\phi}$ is a $G$-invariant Willmore-Chen submanifold in $(P, [\bar{g}])$ if and only if $\phi \in \Sigma^{b,m+q}$, with $b = \frac{(q-1)(m+q)}{q(m+q-1)}$.*

It should be pointed out that the constancy of the mean curvature (tension) is preserved in this symmetry breaking phenomenon, which can be viewed as an interesting holographic property when we reflect that constancy in the gravitatory component. Furthermore, this provides another reason to use the term "best" to name those solutions, since they are also solutions of the well known area-volume isoperimetric problem.

Next we exhibit a second framework where one can reduce symmetry. The background is globally a product and so it could be considered as a particular case of the above one, for example when $M$ is simply connected. However, the metric (or the conformal structure) involves a warping function and the fibre part is provided by a compact homogeneous space. Therefore it can be regarded as a complementary of the first stated setting.

To describe this new situation let $(M_1, g_1)$ and $(M_2, g_2)$ be a compact homogeneous space with group of isometries $H$ and a pseudo-Riemannian manifold, respectively. We recall that $M = M_1 \times_f M_2$ is endowed with the metric $g = g_1 + f^2 g_2$ (with the obvious meaning), where $f$ is a positive smooth function on $M_1$. It is usually called the warped product with warping function $f$ (see [6]) for details).

Let $S$ be a $d$-dimensional submanifold in $M_2$ with second fundamental form $h$. Then $N = M_1 \times_f S$ is an $H$-invariant $(n_1 + d)$-dimensional submanifold of $M$, $n_1$ being the dimension of $N_1$. Moreover, every $H$-invariant submanifold of $M$ is obtained in this way. To do clear this claim, just notice that every $H$-invariant submanifold $\tilde{N}$ of $M$ is foliated by leaves $\{M_1 \times \{p\}; p \in \tilde{N}\}$ and so it projects down, via the second projection, in a $(\mathrm{dim}\tilde{N} - n_1)$-dimensional submanifold of $M_2$.

The volume form of the $g$-induced metric on $M_1 \times_f S$ is given by

$$dv = f dv_1 ds,$$

where $dv_1$ and $ds$ stand for the volume forms of $(M_1, g_1)$ and $S$ with the $g_2$-induced metric, respectively. Now we are going to compute the second fundamental form $\sigma$ of $M_1 \times_f S$ in $M$.

**Proposition 2.7** *The following statements hold:*

1. $\sigma(X,Y)=0$ *for* $X$ *and* $Y$ *tangent to* $M_1$*;*
2. $\sigma(X,V)=0$ *for* $X$ *tangent to* $M_1$ *and* $V$ *tangent to* $S$*;*
3. $\sigma(V,W)=h(V,W)$ *for* $V$ *and* $W$ *tangent to* $S$.

**Proof.** The first one follows from the fact that the leaves are totally geodesic not only in $M$, but also in $M_1\times_f S$. Let $\xi$ be any vector field normal to $M_1\times_f S$ in $M$, then it can be viewed as a normal one to $S$ in $M_2$. On the other hand, since each fibre $\{p\}\times M_2$ is totally umbilical in $M$, then its second fundamental form satisfies $b(V,\xi)=0$. Consequently $g(\sigma(X,V),\xi)=0$, which shows the second statement. The last one follows from [33].

**Theorem 2.8** *If* $S$ *is compact, then* $M_1\times_f S$ *is in* $\Sigma^{br}$ *if and only if* $S$ *is in* $\Sigma^{cr}$*, where* $c=b\frac{(d-1)(n_1+d)}{d(n_1+d-1)}$.

**Proof.** Let $H$ and $H_2$ be the mean curvature vector fields of $M_1\times_f S$ in $M$ and $S$ in $M_2$, respectively. We use Proposition 2.2 to see that

$$H=\frac{d}{n_1+d}\frac{1}{f^2}H_2,$$

and so the corresponding mean curvature functions are related by

$$\alpha^2=\frac{d^2}{(n_1+d)^2}\frac{1}{f^2}\alpha_2^2.$$

On the other hand, the extrinsic scalar curvature functions $\tau$ and $\tau_2$ of $M_1\times_f S$ in $M$ and $S$ in $M_2$, respectively, are related by

$$\tau=\frac{d(d-1)}{(n_1+d)(n_1+d-1)}\frac{1}{f^2}\tau_2.$$

Now the restriction of $\mathcal{W}_{br}$ to the space of $H$-invariant submanifolds, $M_1\times_f S$, in $M$ is given by

$$\mathcal{W}_{br}(M_1\times_f S)=\left(\frac{d}{n_1+d}\right)^r\left(\int_{M_1}\frac{1}{f^{r-1}}dv_1\right)\int_S\left(\alpha_2^2-b\frac{(d-1)(n_1+d)}{d(n_1+d-1)}\right)^{r/2}ds.$$

Once more the principle of symmetric criticality allows us to obtain the statement.

The next result is obtained when $S$ is chosen to be a curve.

**Corollary 2.9** *Let* $\gamma$ *be a closed curve in* $M_2$*. Then* $M_1\times_f\gamma$ *is in* $\Sigma^{br}$ *if and only if* $\gamma$ *is a critical point of the elastic energy functional* $\mathcal{F}$ *given by*

$$\mathcal{F}^r(\gamma)=\int_\gamma\kappa^r ds,$$

*acting on closed curves in* $(M_2,g_2)$*, where* $\kappa$ *denotes the curvature function of* $\gamma$.

**Remark 2.10** Closed curves which are critical points of $\mathcal{F}^r$ are called *generalized elasticae* [4]. In particular, those which are critical points of $\mathcal{F}^2$ are called *free elasticae* [28].

## 3. Some $AdS$-geometry

The role of the anti-de Sitter ($AdS$) geometry in the high energy physics increased due in part to both the Maldacena conjecture and the holographic hypothesis. Furthermore, $AdS$-geometry plays a very important role in the theory of higher spin gauge fields where iterations contain negative powers of the cosmological constant [19]. That theory may be considered [43] as a candidate for a more symmetric phase of string theory. The group manifold case $AdS_3$ is of special interest in many subjects (see [38] for some of them, including the study of conserved currents of arbitrary spin built from massless scalar and spinor fields in $AdS_3$). Next, we study another interesting property emanating from the geometry of $AdS$ which also corroborates the importance of $AdS_3$. In fact, we go to classify all generalized elasticae with constant curvature in $AdS$, for arbitrary dimension. To do that, one first uses a standard argument, which involves several integrations by parts, to compute the Euler-Lagrange equations associated with the elastic energy functional $\mathcal{F}^r$. An early consequence obtained from these equations is that any generalized elasticae in $AdS$ must lie fully in some $AdS_3$ totally geodesic in $AdS$. We can also read from those equations that a generalized elastica with constant curvature in $AdS_3$ also has constant torsion and so it is a helix in $AdS_3$. This is a nice reason to study the geometry of helices in $AdS_3$.

The 4-dimensional pseudo-Euclidean space with index 2, $\mathbb{R}^4_2$, can be identified with $\mathbb{C}^2 = \{z = (z_1, z_2) : z_1, z_2 \in \mathbb{C}\}$ endowed with the usual inner product $\langle z, w\rangle = \mathrm{Re}(z_1\bar{w}_1 - z_2\bar{w}_2)$. The 3-dimensional anti De Siter space is the hyperquadric $AdS_3 = \{z \in \mathbb{R}^4_2 : \langle z, z\rangle = -1\}$, and the induced metric defines a Lorentzian structure, with constant sectional curvature $-1$, on $AdS_3$. The circle of radius one $\mathbb{S}^1$, regarded as the set of unit complex numbers, acts naturally (multiplication coordinate to coordinate) on $AdS_3$. The space of orbits, under this action, can be identified with the hyperbolic 2-plane $\mathbb{H}^2(-4)$ of Gaussian curvature $-4$. The natural projection $\Pi : AdS_3 \longrightarrow \mathbb{H}^2(-4)$ gives a semi-Riemannian submersion.

A global unit timelike vector field $V$ can be defined on $AdS_3$ by putting $V_z = iz$, for all $z \in AdS_3$ (as usual $i = \sqrt{-1}$). The $V$ flow is made up by fibres, which are unit circles with negative definite metric. We will use the standard notation and terminology of [33], relative to semi-Riemannian submersions. In particular, one has the splitting $T_z = \mathcal{V}_z \oplus \mathcal{H}_z$, $z \in AdS_3$, where $T_z$ is the tangent 3-space to $AdS_3$ in $z$, $\mathcal{V}_z = \mathrm{span}(V_z)$ is the vertical line and $\mathcal{H}_z$ is the horizontal subspace ($i\mathcal{H}_z = \mathcal{H}_z$). Recall that $\mathcal{V}_z = \ker(d\Pi_z)$ and $d\Pi_z$ restricted to $\mathcal{H}_z$ gives an isometry between $\mathcal{H}_z$ and the tangent plane to $\mathbb{H}^2(-4)$ at $\Pi(z)$. Overbars will denote the horizontal lifts of corresponding objects on $\mathbb{H}^2(-4)$. The semi-Riemannian connections $\bar{\nabla}$ and $\nabla$ of $AdS_3$ and $\mathbb{H}^2(-4)$, respectively, satisfy

$$\bar{\nabla}_{\bar{X}}\bar{Y} = \overline{\nabla_X Y} + (g_o(JX, Y) \circ \Pi)V, \tag{1}$$

$$\bar{\nabla}_{\bar{X}}V = \bar{\nabla}_V\bar{X} = i\bar{X}, \tag{2}$$

$$\bar{\nabla}_V V = 0, \tag{3}$$

here $J$ and $g_o$ denote the standard complex structure and metric of $\mathbb{H}^2(-4)$, respectively. Notice that the third equation gives the geodesic character of the fibres in $AdS_3$.

The mapping $\Pi : AdS_3 \longrightarrow \mathbb{H}^2(-4)$ is also a principal fibre bundle over $\mathbb{H}^2(-4)$ with structure group $\mathbb{S}^1$ (a *circle bundle*). We define a connection on this bundle by assigning to each $z \in AdS_3$ the horizontal 2-plane $\mathcal{H}_z$. The Lie algebra $u(1)$ of $\mathbb{S}^1 = U(1)$ is identified with $\mathbb{R}$, so $V$ is the *fundamental* vector field $1^*$ corresponding to $1 \in u(1)$.

We denote by $\omega$ and $\Omega$ the connection 1-form and the curvature 2-form of this connection, respectively. It is well known that there is a unique $\mathbb{R}$-valued 2-form $\Theta$ on $\mathbb{H}^2(-4)$ such that

$\Omega = \Pi^*(\Theta)$. We also put $dA$ to denote the canonical volume form on $\mathbb{H}^2(-4)$, in particular $dA(X, JX) = 1$ for any unit vector field $X$ on $\mathbb{H}^2(-4)$. It is clear that $\Theta(X, JX) = \Omega(\bar{X}, i\bar{X})$ and so we can use the structure equation, the horizontality of $\bar{X}$, and $i\bar{X}$ and the formula (1) to obtain

$$\Omega(\bar{X}, i\bar{X}) = d\omega(\bar{X}, i\bar{X}) = -\omega([\bar{X}, i\bar{X}]) = -2\omega(V) = -2,$$

and consequently

$$\Theta = -2dA.$$

Let $\alpha : [0, L] \longrightarrow \mathbb{H}^2(-4)$ be an immersed curve with length $L > 0$. We always assume that $\alpha$ is parametrized by the arclength. The complete lift $C_\alpha = \Pi^{-1}(\alpha)$ will be called the *Hopf tube* associated with $\alpha$ and it can be parametrized as follows. We start from a horizontal lift $\bar{\alpha} : [0, L] \longrightarrow AdS_3$ of $\alpha$ and then we get all the horizontal lifts of $\alpha$ by acting $\mathbb{S}^1$ over $\bar{\alpha}$. Therefore we have $\Psi : [0, L] \times \mathbb{R} \longrightarrow AdS_3$ with

$$\Psi(s, t) = e^{it}\bar{\alpha}(s).$$

It is not difficult to see that $C_\alpha$ is a Lorentzian flat surface which is isometric to $[0, L] \times \mathbb{S}^1$ (where the second factor is endowed with its negative definite standard metric). In particular, if $\alpha$ is closed, then $C_\alpha$ is a Lorentzian flat torus (the *Hopf torus* associated with $\alpha$). It will be embedded in $AdS_3$ if $\alpha$ so is in $\mathbb{H}^2(-4)$, and its isometry type depends not only on $L$ but also on the area $A > 0$ in $\mathbb{H}^2(-4)$ enclosed by $\alpha$.

**Theorem 3.1** *Let $\alpha$ be a closed immersed curve in $\mathbb{H}^2(-4)$ of length $L$ and enclosing an area $A$. The corresponding Hopf torus $C_\alpha$ is isometric to $\mathbb{L}^2/\Gamma$, where $\Gamma$ is the lattice in the Lorentzian plane $\mathbb{L}^2 = \mathbb{R}^2_1$, generated by $(0, 2\pi)$ and $(L, 2A)$.*

**Proof.** Let $\bar{\alpha}$ be any horizontal lift of $\alpha$ and $\Psi : \mathbb{L}^2 \longrightarrow C_\alpha \subset AdS_3$ the semi-Riemannian covering defined by $\Psi(s, t) = e^{it}\bar{\alpha}(s)$. The lines parallel to the $t$ axis in $\mathbb{L}^2$ are mapped by $\Psi$ onto the fibres of $\Pi$, while the lines parallel to the $s$ axis in $\mathbb{L}^2$ are mapped by $\Psi$ onto the horizontal lifts of $\alpha$. These curves are not closed because of the holonomy of the involved connection, which was defined above. However the non closeness of the horizontal lifts of closed curves is measured just for the curvature as follows (we will apply, without major details, a well known argument which is nicely exposed in [21, vol.II, p.293]). There exists $\delta \in [-\pi, \pi)$ such that $\bar{\alpha}(L) = e^{i\delta}\alpha(0)$ for any horizontal lift. The whole group of deck transformations of $\Psi$ is so generated by the translations $(0, 2\pi)$ and $(L, \delta)$. Finally we have $\delta = -\int_c \Theta$, where $c$ is any 2-chain on $\mathbb{H}^2(-4)$ with boundary $\partial c = \alpha$. In particular, from (4), we get $\delta = 2A$ and the proof finishes.

From now on, we assume that $\alpha$ is an arclength parametrized curve with constant curvature $\kappa$ in $\mathbb{H}^2(-4)$. Then $C_\alpha = \Pi^{-1}(\alpha)$ is a Lorentzian flat surface with constant mean curvature. Moreover it admits an obvious parametrization $\Psi(s, t)$ by means of fibres ($s$ constant) and horizontal lifts of $\alpha$ ($t$ constant). Let $\beta$ be a non-null geodesic of $C_\alpha$, that is determined from its *slope* $g$ (which is measured with respect to $\Psi$). It is not difficult to see that $\beta$ is a helix in $AdS_3$, with curvature and torsion given respectively by

$$\rho = \frac{\varepsilon(\kappa + 2g)}{1 - g^2}, \tag{5}$$

$$\nu = \frac{-\varepsilon(1 + g\kappa + g^2)}{1 - g^2}, \tag{6}$$

where $\varepsilon = \pm 1$ represents the causal character of $\beta$.

We also have a converse of this fact, namely, given any helix $\beta$ in $AdS_3$ with curvature $\rho$ and torsion $\nu$, then it can be regarded as a geodesic in a certain Hopf tube of $AdS_3$. Indeed, just consider the Hopf tube $C_\alpha = \Pi^{-1}(\alpha)$, where $\alpha$ is a curve in $\mathbb{H}^2(-4)$ with constant curvature $\kappa = \frac{\varepsilon(\rho^2-\nu^2+1)}{\rho}$, here $\varepsilon$ denotes the causal character of $\beta$, and then we choose a geodesic in $C_\alpha$ with slope $g = -\frac{\varepsilon+\nu}{\rho}$.

We suppose that $\alpha$ is closed, that is, it is a geodesic circle of a certain radius $r > 0$ in $\mathbb{H}^2(-4)$. Then its curvature is $\kappa = -2\coth 2r$ (notice we choose orientation to get negative values for curvature). The length of $\alpha$ is $L = \pi \sinh 2r$ and the enclosed area in $\mathbb{H}^2(-4)$ is $A = \frac{\pi}{2}(\cosh 2r - 1)$. As we already know the Hopf torus $C_\alpha = \Pi^{-1}(\alpha)$ comes from a lattice in $\mathbb{L}^2$ which is generated by $(0, 2\pi)$ and $(L, 2A)$. Now a geodesic $\beta(s)$ of $C_\alpha = \Pi^{-1}(\alpha)$ is closed if and only if there exists $s_o > 0$ such that $\Psi^{-1}(\beta(s_o)) \in \Gamma$. Consequently

$$g = \frac{2\pi}{L}\left(q + \frac{A}{\pi}\right), \tag{7}$$

where $q$ is a rational number.

The slope of closed helices can be also written in terms of $\kappa$ as follows

$$g = q\sqrt{\kappa^2-4} - \frac{1}{2}\kappa,$$

where $q \in \mathbb{Q} - \{0\}$.

The Euler-Lagrange equation for helices $\beta$ in $AdS_3$ of curvature $\rho > 0$ and torsion $\nu \neq 0$ being critical points of $\mathcal{F}^r$ is

$$(r-1)\rho^2 + r\nu^2 - r = 0,$$

that is, in the $(\rho, \nu)$ plane of helices in $AdS_3$, the action $\mathcal{F}^r$ has exactly one ellipse of critical points. To determine the closed helices in $AdS_3$, which are in the above ellipse, we use the discussion made above. In particular the Euler-Lagrange equation can be written in terms of $\kappa$ and the slope $g$ as follows

$$2r\kappa g^3 + \left(r\kappa^2 + 4(2r-1)\right)g^2 + 2(3r-2)\kappa g + (r-1)\kappa^2 = 0.$$

The following theorem shows the existence of wide families of generalized elasticae in $AdS_3$ for arbitrary degrees.

**Theorem 3.2** *For any non-zero rational number $q$ and an arbitrary natural number $r$, there exists a closed helix $\beta_{qr}$ in $AdS_3$ which is a generalized elastica in $AdS_3$, i.e. a critical point of $\mathcal{F}^r$.*

**Proof.** We manipulate equations (8) and (10) to see that a closed helix $\beta$ in $AdS_3$ is a critical point of $\mathcal{F}^r$ if and only if, regarded as a geodesic of *rational slope* $q$ in a Hopf torus on a geodesic circle $\alpha$ in $\mathbb{H}^2(-4)$ with curvature $\kappa$, then both parameters give a zero of the following function

$$F(\kappa, q) = r(4q^2+1)\kappa\sqrt{\kappa^2-4} - 4q\left(\kappa^2 - \frac{4r-2}{r}\right).$$

It is not difficult to see that for any non-zero rational number $q$, there exists a real number $\kappa \in (-\infty, -2)$ such that $F(\kappa, q) = 0$. We choose a geodesic circle in $\mathbb{H}^2(-4)$ with curvature $\kappa$ and a geodesic in $C_\alpha = \Pi^{-1}(\alpha)$ whose slope is $g = \frac{2\pi}{L}(q + \frac{A}{\pi})$, where $L$ and $A$ are, respectively, the length of $\alpha$ and the enclosed area by $\alpha$ in $\mathbb{H}^2(-4)$. Certainly $\beta$ is a closed helix in $AdS_3$ and its curvature and torsion satisfy the Euler-Lagrange equation associated with $\mathcal{F}^r$.

From now on we will denote by $\Lambda_{qr}$ the set of closed helices $\beta_{qr}$ in $AdS_3$ obtained in the last theorem. Notice that these sets give the complete moduli space (up to motions in $AdS_3$) of generalized elasticae with constant curvature in $AdS_3$.

**Remark 3.3** It is not difficult to deduce from $F(\kappa, q) = 0$ that the relationship between $q$ and $\kappa$ gives the following property. Every $q \neq 0$ occurs for exactly one $\kappa$, while each $\kappa$ determines exactly two values of $q$, except when $\kappa^2 = \frac{4r-2}{r}$ (which corresponds to $q = \frac{1}{2}$ or $q = -\frac{1}{2}$). The product of these two values of $q$ is always $\frac{1}{4}$, therefore when one of the them is rational the other one must also be rational. Thus the corresponding Hopf tori in $AdS_3$ has transverse foliations by closed generalized elastic helices.

Now we can combine Corollary 2.9 with Theorem 3.2 to obtain a more general existence result for critical points of the two-parameter family of functionals $\{\mathcal{W}_{ar}\}$. However, we will change a little bit the notation (namely the order in products) to agree with the classical one used in Physics. The setting can be described as follows. Let $(M, g)$ be a compact homogeneous space with group of isometries $H$. On the product space $AdS \times M$, we consider the Lorentzian metric $f^2 g_o + g$, where $g_o$ is the canonical metric on $AdS$, say for instance with constant curvature $-1$, and $f$ is a positive smooth function on $M$, which works as a warping function on the above product. We consider the action $\mathcal{W}_{ar}$ acting on $I\left(Q, (AdS \times M, f^2 g_o + g)\right)$, here dim $Q$ =dim $M + 1$, and use the above mentioned results to get the complete classification of $H$-invariant critical points of $\mathcal{W}_{ar}$ which have constant mean curvature according to the following statement.

**Corollary 3.4** *The class of $H$-invariant critical points of $\mathcal{W}_{ar}$ which have constant mean curvature in $(AdS \times M, f^2 g_o + g)$ is $\{\beta_{qr} \times M : \beta_{qr} \in \Lambda_{qr}\}$. In particular, if dim $M = r - 1$, the above family gives all the $H$-invariant, Willmore-Chen submanifolds with dimension $r$ in $(AdS \times M, [f^2 g_o + g])$ that have constant mean curvature relative to $f^2 g_o + g$.*

As a first conclusion, we get a rational one-parameter family of solutions of the motion equations associated with the actions $\mathcal{W}_{ar}$, at any stage of the Hopf T-duality chain. These solutions emerge from closed generalized elastic helices in $AdS_3$ and can be regarded as solitons corresponding to extended dynamical objects obtained when the compactification space propagates in the target space by describing closed helicoidal orbits. We can also study, of course, non-closed helicoidal motions. In this case a real one-parameter class of non congruent solutions may be obtained and the above closeness property is obtained in terms of the rationality nature of the parameter. However we are interested in closed solutions which are narrowly related with the $AdS$-geometry.

**Corollary 3.5** *For any couple of real numbers $a, r$, and any non-zero rational number $q$ we have*

1. $\beta_{qr} \times \mathbb{S}^5$ *belongs to* $\Sigma^{ar}$ *in the Type IIB string on* $AdS_5 \times \mathbb{S}^5$.

2. $\beta_{qr} \times \mathbb{CP}^2 \times \mathbb{S}^1$ *belongs to* $\Sigma^{ar}$ *in the Type IIA string on* $AdS_5 \times \mathbb{CP}^2 \times \mathbb{S}^1$.

3. $\beta_{qr} \times \mathbb{CP}^2 \times T^2$ *belongs to* $\Sigma^{ar}$ *in the M-theory on* $AdS_5 \times \mathbb{CP}^2 \times T^2$.

*Furthermore, these configurations always lie in a codimension two, totally geodesic submanifold of the corresponding background. Also they are either $SO(6)$-invariant, $(SU(3) \times U(1))$-invariant or $(SU(3) \times U(1) \times U(1))$-invariant depending on whether we are in Type IIB, Type IIA strings or M-theory, respectively.*

## 4. Willmore-Chen submanifolds in Type IIA string theories and M-theory

The Hopf T-duality has the effect of untwisting $\mathbb{S}^5$ to $\mathbb{CP}^2 \times \mathbb{S}^1$. This corresponds with Type IIB configurations carrying strictly R-R electric and magnetic 5-form charges [14]. In our case, the compactification spaces are $\mathbb{CP}^2 \times \mathbb{S}^1$ for Type IIA string and $\mathbb{CP}^2 \times T^2$ for M-theory, and they can be treated according to the settings we have considered in section 2. In both cases we naturally break symmetry to study generalized elasticae in the complex projective plane $\mathbb{CP}^2$ endowed with its usual Fubini-Study metric $g_o$, with holomorphic sectional curvature $4$.

For a curve $\gamma$ in $\mathbb{CP}^2$, one can consider the angle $\phi$ between the complex tangent plane span$\{\gamma'(s), J\gamma'(s)\}$ and the osculating plane of $\gamma$, $J$ standing for the usual complex structure of $\mathbb{CP}^2$. A curve is said to be of constant slant if the angle $\phi$ is constant along $\gamma$. In [1] the first author gave the complete classification of curves with constant slant in $\mathbb{CP}^2$ which are critical points of the elastic energy functional $\int(\kappa^2 + \lambda)^2 ds$, where $\lambda$ is a certain constant. The argument used there can be adapted now to get the complete classification of critical points with constant slant for elastic energy functionals of the form $\int(\kappa^2 + \lambda)^{r/2} ds$, where $r$ is any real number, in particular generalized elasticae with constant slant in $\mathbb{CP}^2$ (see also [5] for another related problem). In the next discussion, the term generalized elastica will be used to name a critical point of the above types of functionals. Before to explain the main points in that classification, where the parameters $\lambda$ and $r$ are referred as the potentials, we will exhibit the argument for an arbitrary $\lambda$ because it will be used later.

1. First of all, notice that the standard Frenet equations of curves in $\mathbb{CP}^2$ are useful, for example, in defining the concept of helix. However, to study generalized elasticae in $\mathbb{CP}^2$ one needs a different reference frame along curves in $\mathbb{CP}^2$ which involves the complex structure $J$ of $\mathbb{CP}^2$. One way to describe this frame is to begin by lifting horizontally the curve $\gamma(s)$ in $\mathbb{CP}^2$, via the usual Hopf mapping, to a curve $Y(s)$ in $\mathbb{S}^5$. The unit tangent vector field $T(s) = \gamma'(s)$ lifts to $\bar{T}(s) = Y'(s)$. Now, we may choose a vector field $U(s)$ along $\gamma(s)$ such that its horizontal lifting $\bar{U}(s)$ gives the third component in a special unitary frame $\sigma(s) = \{Y(s), \bar{T}(s), \bar{U}(s)\}$ in $\mathbb{C}^3$. In other words, $\sigma(s)$ is a lift of the curve $\gamma(s)$ to a curve in $SU(3)$. This curve satisfies a natural differential equation which projects down to $\mathbb{CP}^2$ and gives the natural equations of the new frame $\{T(s), JT(s), U(s), JU(s)\}$ along $\gamma(s)$ (see [1, 5] for more details).

2. By computing the Euler-Lagrange equations for generalized elasticae in terms of the new frame, one can see that each generalized elastica with constant slant in $\mathbb{CP}^2$ is a helix.

3. The main point in the geometrical integration of the motion equations for helices is the following: *Every generalized elastic helix in $\mathbb{CP}^2$ is the image, under the natural projection, of a one-parameter subgroup of $SU(3)$.*

4. In this framework one can obtain the moduli space, up to congruences in $\mathbb{CP}^2$, of generalized elasticae with constant slant in $\mathbb{CP}^2$. This space consists of a real three-parameter family of helices in $\mathbb{CP}^2$ where two parameters in this family can be chosen to be the potentials. Now the closedness characterization for these curves can be obtained in terms of a rationality condition of the third parameter. Therefore, for any couple of real numbers $r$ and $\lambda$, the potentials, we obtain a rational one-parameter family $\Gamma_r^\lambda = \{\beta_{rq}^\lambda : q \in E_r^\lambda \subset \mathbb{Q}\}$ of generalized elastic closed helices in $\mathbb{CP}^2$, here $E_r^\lambda$ is a certain subset of rational numbers determined in terms of the potentials.

The above argument, which shows the existence (in particular) of generalized elasticae in $\mathbb{CP}^2$, can be combined with the methods of breaking symmetry in the motion equations associated with the action $\mathcal{W}_{ar}$ that was given in section 2. It allows us to obtain the following result of existence of solutions in the Type IIA string on $AdS_5 \times \mathbb{CP}^2 \times \mathbb{S}^1$.
*For any couple of real numbers $a$, $r$ the class $\{\beta_{rq}^0 \times \mathbb{S}^1 : q \in E_r^0\}$ is a rational one-parameter family of $U(1)$-invariant tori which belong to $\Sigma^{ar}$ in the Type IIA string on $AdS_5 \times \mathbb{CP}^2 \times \mathbb{S}^1$.*

In particular, if $r = 2$, we obtain Willmore tori as solutions of the conformal motion equations associated with the corresponding action. Of course we can exploit, once more, the breaking symmetry process to obtain solutions relative to metrics on $AdS_5 \times \mathbb{CP}^2 \times \mathbb{S}^1$ which are given as double warping metrics coming from a couple of warping functions on the circle.

The above solutions can be Hopf T-dualized to obtain solutions not only in the M-theory on $AdS_5 \times \mathbb{CP}^2 \times T^2$, but also in the M-theory on any $AdS_5 \times P$, where $P$ is a circle bundle on $\mathbb{CP}^2 \times \mathbb{S}^1$ endowed with a Kaluza-Klein ansatz associated with a principal flat connection. To be precise, let $\eta$ be a real number such that $\eta/\pi$ is not a rational number, the map $\phi_\eta : \mathbb{Z} \to \mathbb{S}^1$ given by $\phi_\eta(k) = e^{ik\eta}$ defines a monomorphism between $(\mathbb{Z}, +)$ and $\mathbb{S}^1 \subset \mathbb{C}$ regarded as a multiplicative group. Let $U = \mathbb{CP}^2 \times \mathbb{R}$ be the universal covering of $\mathbb{CP}^2 \times \mathbb{S}^1$. Certainly $U$ can be regarded as a principal $\mathbb{Z}$-bundle on $\mathbb{CP}^2 \times \mathbb{S}^1$, which admits an obvious principal flat connection, say $\omega$. The transition functions of this bundle can be extended, via $\phi_\eta$, to $\mathbb{S}^1$-valued functions and they can be used as transition functions to define a principal $\mathbb{S}^1$-bundle, say $P_\eta$, on $\mathbb{CP}^2 \times \mathbb{S}^1$. Moreover, one can extend $\omega$ to a principal flat connection, also called $\omega$, on the whole $\mathbb{S}^1$-bundle whose holonomy subbundle is isomorphic to $U(\mathbb{CP}^2 \times \mathbb{S}^1, \mathbb{Z})$. Notice that when $\eta$ is chosen to be one, then $P_1$ is nothing but the direct product $\mathbb{CP}^2 \times T^2$ on which the usual M-theory is Hopf T-dualized. If we call $\pi : P_\eta \to \mathbb{CP}^2 \times \mathbb{S}^1$ the projection of the above fibration, then we have
$\{\pi^{-1}(\beta_{rq}^0 \times \mathbb{S}^1) : \beta_{rq}^0 \in \Gamma_r^0\}$ *is a rational one-parameter family of $(U(1) \times U(1))$-invariant submanifolds which are solutions of the motion equations associated with the action $\mathcal{W}_{ar}$ in the M-theory on $AdS_5 \times P$.*

## 5. A zoo of solutions in the Type IIB string theory

Throughout this section $M$ will be a principal fibre $H$-bundle, $H$ being a $d$-dimensional compact Lie group, over a certain pseudo-Riemannian manifold $(M', g')$. Let $p : M \to M'$ be the projection and let $\theta$ be a principal connection on this principal fibre $H$-bundle. We denote by $da^2$ the bi-invariant metric on $H$, so that the Kaluza-Klein metric writes down

$$g = p^*(g') + \varepsilon\theta^*(da^2).$$

As above, we can determine the manifold made up of $(d+1)$-dimensional, $H$-invariant, compact submanifolds in $(M,g)$. It can be identified with the set of complete lifts of closed curves immersed in $M'$. Therefore, to compute the critical points of $\mathcal{W}_{br}$ in $(M,g)$ which are $H$-invariant, we use again the principle of symmetric criticality and compute the restriction of $\mathcal{W}_{br}$ to the above submanifold. To do that, let $\alpha$ and $\tau_e$ be the mean curvature and the extrinsic scalar curvature functions of $p^{-1}(\gamma)$ in $(M,g)$, respectively. We also denote by $S$ and $S'$ the Ricci curvatures of $(M,g)$ and $(M',g')$, respectively. Finally, assume that $\gamma(s)$ is arclength parametrized, denote by $\kappa$ its curvature function in $(M',g')$ and let $\tilde{\gamma}'(s)$ be the horizontal lift to $(M,g)$ of its unit tangent $\gamma'(s)$. In this setting, we have (see [1])

$$\begin{aligned} \alpha^2 &= \frac{1}{(d+1)^2}(\kappa^2 \circ p), \\ \tau_e &= -\frac{1}{d(d+1)}(S'(\gamma') \circ p - S(\tilde{\gamma}')). \end{aligned}$$

On the unit tangent bundle $UM'$ of $(M',g')$, let $\psi : UM' \to \mathbb{R}$ be defined by

$$\psi(\gamma') \circ p = \frac{b(d+1)}{d}(S'(\gamma') \circ p - S(\tilde{\gamma}')).$$

Then

$$\mathcal{W}_{br}(p^{-1}(\gamma)) = \frac{\mathrm{vol}(H, da^2)}{(d+1)^r} \int_\gamma (\kappa^2 + \psi(\gamma'))^{r/2} ds.$$

Consequently we have

**Theorem 5.1** *$p^{-1}(\gamma)$ is a critical point of $\mathcal{W}_{br}$ in $(M,g)$ if and only if $\gamma$ is a critical point of the elastic energy functional $\mathcal{F}$ defined by*

$$\mathcal{F}(\gamma) = \int_\gamma (\kappa^2 + \psi(\gamma'))^{r/2} ds,$$

*on the space of closed curves in $(M',g')$.*

The most interesting situation in Theorem 5.1 occurs when the potential $\psi$ is constant. In this case, recall that we used the term generalized elastica, with potentials $(\psi, r/2)$, to refer to critical points of $\mathcal{F}$.

A sufficient condition to guarantee the constancy of $\psi$ is to assume that both $(M,g)$ and $(M',g')$ are Einstein, and then $\theta$ must be a Yang-Mills connection. Therefore, if $\lambda$ and $\lambda'$ denote the cosmological constants of $(M,g)$ and $(M',g')$, respectively, we find

$$\psi = \frac{b(d+1)}{d}(\lambda' - \lambda).$$

In particular $\psi \geqslant 0$ (see [1]).

However that condition is not necessary. In fact, suppose that both $(M,g)$ and $(M',g')$ are Einstein giving a constant potential $\psi$. Now, we deformate the metric $g$ by changing the relative scales of the base and the fibres. To be precise, we define a one-parameter family of metrics on $M$ by putting $\{g_t = p^*(g') + t\theta^*(da^2), t \geqslant 0\}$. This gives a one-parameter family of Riemannian submersions with totally geodesic fibres, all of them having the same horizontal distribution associated with $\theta$. This is nothing but the canonical variation of the starting Riemannian submersion. Since we are assuming that $g_1 = g$ is Einstein, then there is at most one more Einstein metric in $\{g_t\}$ (see [6]). However, if $\psi_t$ denotes the corresponding potential, it is not difficult to see that $\psi_t = t\psi$ and so it is constant for any $t$.

The existence of one-parameter families of generalized elastic closed helices, for arbitrary potentials, in $\mathbb{S}^3$ and $\mathbb{CP}^2$, respectively, has been established in [1] and in the last section, respectively.

**Example 5.2** Let $\pi$ be the usual Hopf map from the 5-sphere $\mathbb{S}^5$ over the complex projective plane $\mathbb{CP}^2$. Then for any $t > 0$ and any pair of real numbers $b$, $r$, there exists a rational one parameter family of $U(1)$-invariant flat tori with constant mean curvature in $(\mathbb{S}^5, g_t)$ which are critical points of $\mathcal{W}_{br}$. Here $g_1 = g$ is the canonical metric of curvature one on $\mathbb{S}^5$.

**Example 5.3** Let $\pi$ be the usual quaternion Hopf map from the 7-sphere $\mathbb{S}^7$ over the quaternion projective line $\mathbb{S}^4$. Then for any $t > 0$ and any pair of real numbers $b$, $r$, there exists a rational one parameter family of $SO(4)$-invariant 4-dimensional submanifolds with constant mean curvature in $(\mathbb{S}^7, g_t)$ which are critical points of $\mathcal{W}_{br}$. Here $g_1 = g$ is the canonical metric of curvature one on $\mathbb{S}^7$.

We have just obtained a rational one-parameter family of flat tori with constant mean curvature in the round 5-sphere, which are critical points of $\mathcal{W}_{br}$ for arbitrary $b$ and $r$. These tori appeared connected, via the Hopf map, with the existence of generalized elastic closed helices in the complex projective plane. The chief point to obtain this variational reduction of variables was provided by the Palais principle of symmetric criticality.

Now, we are going to exhibit a new method to get flat tori in $\mathbb{S}^{2n+1}$ which are not obtained as lifts of closed curves in the complex projective space $\mathbb{CP}^n$, but still are critical points for $\mathcal{W}_{br}$.

Let $T$ be a compact genus one surface and let $(\mathbb{S}^{2n+1}, g_0)$ be the round sphere of radius one. For any pair of real numbers $b$ and $r$ we have $\mathcal{W}_{br} : I(T, \mathbb{S}^{2n+1}) \to \mathbb{R}$ given now by

$$\mathcal{W}_{br}(\varphi) = \int_T (\alpha^2 + b - bG)^{r/2} dv,$$

where $\alpha$ is the mean curvature of $\varphi$, $G$ and $dv$ being the Gaussian curvature and the volume element of $(T, \varphi^*(g_0))$, respectively.

The computation of the first variation of $\mathcal{W}_{br}$ is not easy in general. Certainly, the case $r = 2$ (the Willmore functional corresponds with $r = 2$ and $b = 1$) is the simplest one. In fact, using the Gauss-Bonnet formula, the functional is reduced to $\int_T (\alpha^2 + b) dv$. In this case, the first variation formula was computed in [45], and from there one sees that the minimal immersions are automatically solutions of the corresponding Euler-Lagrange equations. Moreover, it is not clear, in general, that the minimal immersions are critical points of $\mathcal{W}_{br}$. We will overcome these obstacles by using again the Palais principle of symmetric criticality. Contrarily to those occasions where we used the principle for reduction of variables, now we will use it to improve rigidity. Notice that the highest rigidity for a metric on a compact genus one surface means flatness.

To clarify that idea, let $K$ be a 2-dimensional compact subgroup of $SO(2n+2)$ regarded as the isometry group of $(\mathbb{S}^{2n+1}, g_0)$. It is obvious that $K$ acts naturally on $I(T, \mathbb{S}^{2n+1})$. Let $\varphi$ be a $K$-invariant immersion, then $(T, \varphi^*(g_0))$ has a subgroup of isometries of dimension two.

We apply a well known classical argument to see that $K$ acts transitively on $(T, \varphi^*(g_0))$. This homogeneity implies constant Gaussian curvature and so $(T, \varphi^*(g_0))$ is a flat torus. Consequently, the submanifold of $K$-invariant immersions $I_K(T, \mathbb{S}^{2n+1})$ is made up of flat tori. It is not difficult to see that each isometric immersion from a flat torus in $(\mathbb{S}^{2n+1}, g_0)$ can be viewed as an orbit associated with a 2-dimensional, compact subgroup of $SO(2n+2)$ (see [26, 27] for minimal flat tori in the 5-sphere). Then we apply the principle of symmetric criticality to obtain the following.

**Proposition 5.4** *$\varphi \in I_K(T, \mathbb{S}^{2n+1})$ is a critical point of $\mathcal{W}_{br}$ if and only if it is a critical point of $\mathcal{W}_{br}$, but restricted to the submanifold $I_K(T, \mathbb{S}^{2n+1})$.*

Now we can compute the first variation of this restriction, that is, we take variations of $\varphi \in I_K(T, \mathbb{S}^{2n+1})$ in $I_K(T, \mathbb{S}^{2n+1})$. We use a standard argument, which involves some integrations by parts, to obtain the following Euler-Lagrange equation

$$r\Delta H - r\tilde{A}(H) + 4\alpha^2 H - 2(r-2)bH = 0,$$

where $H$ is the mean curvature vector field, $\Delta$ denotes the Laplacian associated with the normal connection and $\tilde{A}$ is the Simons operator [40].

A first consequence makes mention to the minimality.

**Corollary 5.5** *Every minimal flat tori in $(\mathbb{S}^{2n+1}, g_0)$ is a critical point of $\mathcal{W}_{br}$, for arbitrary $b$ and $r$.*

To illustrate that, we consider the following one-parameter family of minimal flat tori in $(\mathbb{S}^5, g_0)$ (see [26]). Let $q \in (0,1]$ be a rational number and consider, in the Euclidean plane $\mathbb{R}^2$, the lattice $\Gamma$ generated by

$$\left(\frac{\sqrt{2}}{q\sqrt{4-q^2}}, 0\right) \quad \text{and} \quad \left(\frac{2-\mathrm{q}^2}{\mathrm{q}\sqrt{2(4-\mathrm{q}^2)}}, \frac{\sqrt{2}}{2}\right).$$

We define $y : \mathbb{R}^2 \to \mathbb{C}^3$ by

$$y(s,t) = \frac{1}{\sqrt{4-q^2}}\left(e^{is}, e^{it}, \sqrt{2-q^2}e^{\frac{i(s+t)}{q}}\right).$$

It is not difficult to see that this gives an isometric immersion. Furthermore, it induces an isometric immersion $x$ from the flat torus $T = \mathbb{R}^2/\Gamma$ in $(\mathbb{S}^5, g_0)$. Since the coordinate functions of $x$ in $\mathbb{C}^3$ are eigenfunctions of the Laplacian of $T$, associated with the eigenvalue 2, $x$ is minimal in $(\mathbb{S}^5, g_0)$. Notice that the case $q = 1$ gives the so called equilateral flat torus, because it comes from an equilateral lattice.

Next we give some explicit examples of non-minimal flat tori in $(\mathbb{S}^5, g_0)$ which are critical points of the functional $\mathcal{W}_{br}$. Given three real numbers $c$, $d$ and $e$, with $c, d > 0$, we consider the lattice $\Gamma$ in $\mathbb{R}^2$ generated by $(2\pi c, 2\pi e)$ and $(0, 2\pi d)$. Choose $n$, $m$, $\bar{n} \in \mathbb{Z} - \{0\}$ such that $\omega = \frac{n}{c} - \frac{me}{cd}$ and $\bar{\omega} = \bar{n}c$ satisfy $\omega \neq \bar{\omega}$ and $\omega \neq 0$. We can also assume that $\frac{m}{d} > 1$. Then we define $y : \mathbb{R}^2 \to \mathbb{C}^3$ by

$$y(s,t) = \left(p\cos\left(\frac{t}{p}\right)e^{i\omega s}, p\sin\left(\frac{t}{p}\right)e^{i\omega s}, \sqrt{1-p^2}e^{i\bar{\omega}s}\right),$$

where $p = d/m$. It is easy to see that $y$ defines an isometric immersion if and only if $p^2\omega^2 + (1-p^2)\bar{\omega}^2 = 1$. Furthermore, in this case it induces an isometric immersion $x$ from the flat torus $T = \mathbb{R}^2/\Gamma$ in $(\mathbb{S}^5, g_0)$. Some interesting properties of these immersions are collected in the following.

1. The center of mass of $x$ in $\mathbb{C}^3$ coincides with the center of $\mathbb{S}^5$, in this sense we say that it is of mass symmetric. Notice that minimal submanifolds in a round sphere are always mass symmetric.

2. The immersion $x$ is not minimal in $(\mathbb{S}^5, g_0)$. In fact, it is constructed in $\mathbb{C}^3$ by using eigenfunctions of the Laplacian of $T$ associated with two different eigenvalues, namely $\omega^2 + \frac{1}{p^2}$ and $\bar{\omega}^2$. Therefore, we say that it is of 2-type in the sense of B.Y. Chen (see [11]).

3. These immersions have non-zero constant mean curvature.

4. They are not Hopf tori. That is, they are not invariant under the natural $U(1)$-action on $\mathbb{S}^5$ to obtain $\mathbb{CP}^2$ as space of orbits.

The following result shows the existence of an ample family of non-minimal flat tori in $(\mathbb{S}^5, g_0)$ which are critical points of $\mathcal{W}_{br}$ for arbitrary $b$ and $r > 0$.

**Proposition 5.6** *For any pair of real numbers $b$ and $r$, there exist infinitely many non-minimal flat tori in the family* (3) *which are critical points of $\mathcal{W}_{br}$ in $(\mathbb{S}^5, g_0)$.*

**Proof.** Let $A$ be the shape operator of $x : T \to (\mathbb{S}^5, g_0)$ in the unit direction defined by $H$. Let $\lambda = \omega^2 + \frac{1}{p^2}$ and $\mu = \bar{\omega}^2$ be the two eigenvalues of the Laplacian of $T$ which are involved in the 2-type nature of $x$ in $\mathbb{C}^3$. Now, a straightforward but long computation allows us to see that $x$ is a critical point of $\mathcal{W}_{br}$ in $(\mathbb{S}^5, g_0)$, that is, a solution of (1), if and only if the following equation holds

$$2r|A|^2 - 4\alpha^2 = r(\lambda + \mu - 2) + 2(2-r)b.$$

A messy computation shows that (4) is equivalent to

$$(2r-1)p^4\omega^4 + (4p^2 - 2rp^2 - r - 2)p^2\omega^2 + M(b,r,p) = 0,$$

where

$$M(b,r,p) = r - 1 - 4p(r - p - rp^2 + p^3) + p^2(p^2-1)(2(2-r)b - 2r).$$

The equation (5) can be regarded as a biquadratic one in $\omega$, so if $D$ denotes its discriminant, it is easy to see that $\lim_{p\to 1} D = r^2$. Then given $(b,r)$ there exists an open subset $I$ in $(0,1)$ such that for all $p \in I$ one can get solutions $\omega$ of (5). Now it is clear that we can determine flat tori in the family (3) with these parameters, and the proof finishes.

Along this section, we have obtained three explicit families of flat tori in $(\mathbb{S}^5, g_0)$ that are critical points of $\mathcal{W}_{br}$ for arbitrary $b$ and $r$. Let us recall them.

1. The family $\mathcal{C}_1$ was obtained by lifting, via the usual Hopf map $\pi : \mathbb{S}^5 \to \mathbb{CP}^2$, a rational one-parameter family of generalized elastic closed helices in $\mathbb{CP}^2$ with suitable potentials. Therefore, the flat tori in $\mathcal{C}_1$ have non-zero constant mean curvature in $(\mathbb{S}^5, g_0)$.

2. The family $\mathcal{C}_2$ is made up of minimal flat tori in $(\mathbb{S}^5, g_0)$. Explicit parametrizations for tori in $\mathcal{C}_2$ were given by K. Kenmotsu [26]. This is also a rational one-parameter family of tori.

3. The family $\mathcal{C}_3$ (see (3) and Proposition 5.6) consists of a multi-indexed family of non-zero constant mean curvature, flat tori in $(\mathbb{S}^5, g_0)$ which are constructed in $\mathbb{C}^3$ using eigenfunctions of their Laplacians associated with exactly two different eigenvalues.

# 6. Further backgrounds

The above considered $AdS_5 \times \mathbb{S}^5$ solution can be extended to any other configuration of the form $AdS_5 \times M_5$, where $M_5$ is any 5-dimensional compact Einstein space with positive scalar curvature. An interesting class of such solutions is provided by choosing $M_5$ to be a principal $U(1)$-bundle over a compact 4-dimensional manifold $N$. Now, the $U(1)$-bundles over such a $N$ are classified by the cohomology group $H^2(N, \mathbb{Z})$.

For example, let $N$ be the Grassmannian of oriented 2-planes in $\mathbb{R}^4$, viewed as the complex quadric $Q_2 = \mathbb{S}^2 \times \mathbb{S}^2$ with its natural Einstein metric $g$. By choosing $\frac{1}{3}c_1(N) \in H^2(N, \mathbb{Z})$, where $c_1(N)$ denotes the first Chern class of $N$, we obtain a principal $U(1)$-bundle $M = T_1\mathbb{S}^3$ on $N$ which coincides with the unit tangent bundle of the round 3-sphere and admits a natural Einstein metric (Stiefel manifold) $\bar{g}$. This example can be regarded as a special one of an integer two-parameter family of $U(1)$-bundles, $\{M(p,q) : p, q \in \mathbb{Z}\}$, over $\mathbb{S}^2 \times \mathbb{S}^2$, where the integers $p$ and $q$ are the winding numbers of the fibres over both $\mathbb{S}^2$ factors in the complex quadric. Natural Einstein metrics, with positive scalar curvature, can be obtained on each $M(p,q)$ [14]. Observe that one can consider $p$ and $q$ to be relatively prime, otherwise if $a = \gcd(p,q)$, then $M(p,q) = M(p/a, q/a)/\mathbb{Z}_p$ is a lens space. Notice also that $T_1\mathbb{S}^3 = M(1,1)$ is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^3$, however it is not a product either as a homogeneous space or as an Einstein manifold.

Finally remark that the situation of greatest interest comes out when the Einstein space $M(p,q)$ admits Killing spinors, which automatically implies that the $AdS_5 \times M(p,q)$ solution preserves some supersymmetries. However it only occurs when $p = q = 1$, i.e., for the Stiefel manifold [14]. We may follow steps analogous to those described for the five sphere [14], to reduce the $AdS_5 \times T_1\mathbb{S}^3$ solution of the Type IIB theory to $D = 9$, and perform a T-duality transformation. Upon oxidation back to $D = 10$ Type IIA theory, we have a solution on $AdS_5 \times Q_2 \times \mathbb{S}^1$. This can be oxidised further to $D = 11$ M-theory, giving a solution not only on $AdS_5 \times Q_2 \times T^2$, but also on any background of the form $AdS_5 \times P$, where $P$ is any principal $U(1)$-bundle over $Q_2 \times \mathbb{S}^1$, which admits a principal flat connection and it is endowed with the corresponding Kaluza-Klein antsaz, $I(P)$ being the corresponding group of isometries. Consequently, we obtain the following duality chain [$n = 4$ Yang-Mills]$\longrightarrow$[Type IIB string on $AdS_5 \times T_1\mathbb{S}^3$]$\longrightarrow$[Type IIA string on $AdS_5 \times Q_2 \times \mathbb{S}^1$]$\longrightarrow$[M-theory on $AdS_5 \times P$].

According to the reduction of symmetry program, we can obtain examples of solutions, for the motion equations associated with the $\mathcal{W}_{ar}$-dynamics, in any step of the above constructed duality chain which reduces to generalized elasticae in the complex quadric, as well as in a $AdS_3$ totally geodesic in $AdS_5$. In the latter case, we can use those rational one-parameter families of generalized elastic helices in $AdS_3$ which were obtained in section 3. Therefore we have a result analogous to Corollary 3.5.

1. $\{\beta_{qr} \times T_1\mathbb{S}^3 : q \in \mathbb{Q} - \{0\}\}$ *is a rational one-parameter family of* $(SO(4) \times U(1))$*-invariant solutions of the* $\mathcal{W}_{ar}$*-dynamic in the Type IIB theory on* $AdS_5 \times T_1\mathbb{S}^3$.

2. $\{\beta_{qr} \times Q_2 \times \mathbb{S}^1 : q \in \mathbb{Q} - \{0\}\}$ *is a rational one-parameter family of* $(SO(4) \times U(1))$*-invariant solutions of the* $\mathcal{W}_{ar}$*-dynamic in the Type IIA theory on* $AdS_5 \times Q_2 \times \mathbb{S}^1$.
3. $\{\beta_{qr} \times P : q \in \mathbb{Q} - \{0\}\}$ *is a rational one-parameter family of* $I(P)$*-invariant solutions of the* $\mathcal{W}_{ar}$*-dynamic in the M-theory on* $AdS_5 \times P$.

In the former case, we can emulate an argument given in [1] to construct real one-parameter families of generalized elastic closed helices in the complex quadric for any pair of given potentials. These helices appear as geodesic in certain flat tori embedded in $Q_2$, so that a simple argument allows us to get a closeness condition in terms of a rationality condition for the involved parameter (see the appendix of [1]). Using now the breaking of symmetry program, we can obtain holographic solutions for the $\mathcal{W}_{ar}$-motion equations at any stage of the above described T-duality chain. These solutions are similar to those obtained in sections 4 and 5 where $\mathbb{CP}^2$ played the part of $Q_2$ here.

Certainly, spherical compactifications of supergravity are maximally supersymmetric and therefore the boundary superconformal field theory, SCFT, has sixteen supercharges. However, the conjecture of Maldacena is believed to be true for any supersymmetry. This is a good reason to investigate SCFT with less than sixteen supercharges [22, 25, 29, 17, 18, 20, 34]. This reduction of supersymmetry is obtained by orbifolding the space transverse to the boundary. Hence, the $AdS$ part of the geometry remains intact while the $\mathbb{S}^n$ part of the geometry is orbifolded and depending on the orbifold one obtains distinct CFT with different amounts of supersymmetry. An interesting way to do it is by considering odd dimensional transverse spheres and then to regard them as Hopf fibrations, i.e., principal $U(1)$-bundles over complex projective spaces. Now, we can break supersymmetry either by reducing over $U(1)$-fibre (this Hopf reduction has been already widely used along this paper) or by considering multiple windings of the $U(1)$-fibre over the base space. In this second setting we do not reduce dimension and arrive to the lens spaces.

Let $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$ be the $(2n-1)$-dimensional sphere of radius one, i.e.,

$$\mathbb{S}^{2n-1} = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n \ : \ |z|^2 = \sum_{j=1}^{n} |z_j|^2 = 1\}.$$

For any natural number $r$, let $\varepsilon = e^{2\pi i/r}$ be a primitive $r$-th root of unity and $\{s_1, \dots, s_n\}$ integers which are relatively prime to $r$. We define an action of $\mathbb{Z}_r = \{1, \varepsilon, \varepsilon^2, \dots, \varepsilon^{r-1}\}$ on $\mathbb{S}^{2n-1}$ by

$$\varepsilon \cdot (z_1, \dots, z_n) = (\varepsilon^{s_1} z_1, \dots, \varepsilon^{s_n} z_n).$$

The orbit space is denoted by $L(r, s_1, \dots, s_n)$ and it will be called a lens space. The natural projection $p : \mathbb{S}^{2n-1} \to L(r, s_1, \dots, s_n)$ gives the universal covering of this space. Hence $\mathbb{Z}_r$ is not only the fundamental group of $L(r, s_1, \dots, s_n)$, but also the deck transformation group of this covering space. The classical case appears when $n = 2$ and $L(r, 1, s)$ is usually denoted by $L(r, s)$. In particular, $L(2, 1)$ is just the real projective space $\mathbb{R}P^3$.

Let $\{\psi_{k\ell}\}$ and $\omega_0$ be the transition functions of $\mathbb{S}^3(L(r, s), \mathbb{Z}_r)$ and the connection 1-form of its canonical flat principal connection, respectively. For any compact Lie group $G$ endowed with a bi-invariant metric $d\sigma^2$, we choose an arbitrary closed geodesic through the identity of $G$, say $\beta(t) = \exp(tA)$, where $A \in g$. We define a monomorphism $\phi_\beta : \mathbb{Z}_r \to G$ by identifying $\mathbb{Z}_r$ with the group of primitive $r$-th roots of unity and then using that the exponential mapping defines an isomorphism between $\mathbb{S}^1$ and $\beta$. We may then extend $\{\psi_{k\ell}\}$ via $\phi_\beta$ to obtain a set of $G$-valuated functions which can be used to construct a principal fibre bundle $P(L(r,s),G)$. Furthermore, $\phi_\beta$ is extended to get a monomorphism $\bar{\phi}_\beta : \mathbb{S}^3 \to P$ which maps $\omega_0$ into a flat connection on $P$. Summing up, we have obtained the following result.

**Proposition 6.1** *Let $G$ be a compact Lie group. Then there exists a $G$-principal fibre bundle $P(L(r,s),G)$ over the lens space $L(r,s)$ which admits a principal flat connection with holonomy subbundle isomorphic to the 3-sphere, that is, $\mathbb{S}^3(L(r,s),\mathbb{Z}_r)$.*

This proposition is also true if $G$ is not compact and the construction can also be generalized to lens spaces of higher dimensions.

Let $(L^5, g_0)$ be a five dimensional lens space. Then we can use the natural covering mapping from $\mathbb{S}^5$ over $L^5$ to project $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ (the three families of solutions obtained in the above section on the round five sphere). We also denote by $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ the projected families in $(L^5, g_0)$. Then we have,

**Corollary 6.2** *Let $G$ be an $m$-dimensional compact Lie group endowed with a bi-invariant metric $d\sigma^2$. Let $\pi : P \to L^5$ be a principal fibre $G$-bundle, endowed with a principal flat connection $\omega$. Let $[\bar{g}_0]$ be the Kaluza-Klein conformal class on $P$ associated with $\bar{g}_0 = \pi^*(g_0) + \omega^*(d\sigma^2)$. Then there exist infinitely many $(m+2)$-dimensional, $G$-invariant, Willmore-Chen submanifolds in $(P, [\bar{g}_0])$, which have constant mean curvature in $(P, \bar{g}_0)$. This family includes the three subfamilies obtained by lifting, via $p$, the families of flat tori $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$.*

The above result can be obviously extended to any action $\mathcal{W}_{ar}$, not necessarily to that giving the Willmore-Chen functional. Furthermore it can be applied to a wide variety of contexts. For example, suppose the Lie group $G$ is chosen to be $U(1) = \mathbb{S}^1$, so that $P$ is a principal $U(1)$-bundle on $L^5$ and then we have:
*The $\mathcal{W}_{ar}$-dynamic in the M-theory on $AdS_5 \times P$ has infinitely many $U(1)$-invariant compact solutions with dimension three. This class includes the three subfamilies obtained when lifting, via $p : P \to L^5$, the families of flat tori $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ on $L^5$.*

It should be noticed that these solutions have constant mean curvature (tension) in the original Kaluza-Klein metric on $AdS_5 \times P$.

**Remark 6.3** We can construct examples of 2-type (in the sense of B.Y.Chen) flat tori in $(\mathbb{S}^7, g_0)$ which are critical points of $\mathcal{W}_{br}$ for arbitrary $b$ and $r$. Moreover, we can use an argument similar to that used in [3] to show that 2-type compact surfaces which are solutions of the $\mathcal{W}_{br}$-dynamic in any round sphere $(\mathbb{S}^m, g_0)$ are actually flat tori lying fully in either $(\mathbb{S}^5, g_0)$ or $(\mathbb{S}^7, g_0)$.

## 7. Dyonic strings and non-dilatonic black holes

General spaces of the type $AdS \times N$, where $N$ are Einstein spaces, necessarily spheres, emerge naturally in supergravity as the near horizon geometries of supermembranes with fewer Killing spinors and whose boundary conformal field theories have so less supersymmetry. In particular, the space $AdS_3 \times \mathbb{S}^3$ appears as the near horizon geometry of the self-dual string or, more generally, the dyonic string (see [15] and references therein for details). In that paper, truncated six-dimensional type IIB and type IIA Lagrangians are obtained and their $T$-duality transformations were explicitly computed. This construction is necessary if one wishes to consider solutions carrying both NS-NS and R-R electric and magnetic 3-form charges and whose near horizon geometry contains $AdS_3 \times \mathbb{S}^3$.

All the non-dilatonic black holes in $D = 5$ and $D = 4$ were listed there and their near horizon limits, when they are oxidised to $D = 6$, were obtained. Since all these near horizon limits can be obtained by Hopf $T$-duality on $AdS_3 \times \mathbb{S}^3$ (actually, Hopf $T$-duality relates not only near horizon limits but also their associated full solutions), we can combine Corollary 2.9 with Theorem 3.2 to obtain an existence result of Willmore-Chen submanifolds which can be regarded as a Hopf $T$-duality invariant. In fact, it is known that Hopf dualities preserve the area of the horizons and hence they also preserve the black hole entropies. Now we have

**Corollary 7.1** *The conformal structure associated with the string-frame in the near horizon limit of any non dilatonic black hole in $D = 5$ and $D = 4$ admits a rational one parameter family of four dimensional Willmore-Chen submanifolds which have constant mean curvature in the original string metric. Moreover, in the case that there are only NS-NS charges, the invariance of these Willmore-Chen submanifolds is preserved, while this is decreased when R-R charges are carried.*

**Remark 7.2** Dilatons and axions are constant in the solutions and for simplicity we have considered them to be zero. Otherwise we could have started from original solutions $AdS_3 \times \mathbb{S}^3/\mathbb{Z}_n$, for the type IIB low-energy effective action. It should be pointed out that the string metric of any near horizon limit is always homogeneous and have constant scalar curvature. For example, by considering that dilatons and axions are zero and applying Hopf $T$-duality on the $U(1)$-fibres of $\mathbb{S}^3$, the following possibilities could appear:

1. There are only R-R charges, then $\mathbb{S}^3$ is untwisted to $\mathbb{S}^2 \times \mathbb{S}^1$.

2. There are only NS-NS charges, then $\mathbb{S}^3$ becomes a cyclic lens space $\mathbb{S}^3/\mathbb{Z}_p$ with its round metric and $p$ being the magnetic NS-NS charge.

3. In the generic case, with both NS-NS and R-R charges, $\mathbb{S}^3$ not only becomes $\mathbb{S}^3/\mathbb{Z}_p$, but it is also squashed, with a squashing parameter that is related to the values of the charges. In other words, the metric on $\mathbb{S}^3/\mathbb{Z}_p$ is covered by a metric on $\mathbb{S}^3$ which may be realized as a *distance sphere* in the complex projective plane or its symmetric dual (the complex hyperbolic plane) according to the squashing parameter is less than or greater than 1, respectively. The squashing parameter equal to 1 corresponds with the round metric.

Most of the near horizon limits are not simply connected. Only those solutions with dilatons and axions being zero of the type IIB Lagrangian are simply connected. Since Corollary 2.9 holds for any $\mathcal{W}_{br}$, one can obtain an analogous to Corollary 7.1 for any functional $\mathcal{W}_{br}$. Consequently, we combine this argument with the first reduction of symmetry phase to obtain the following result.

**Corollary 7.3** *Let $AdS_3 \times N$ be any near horizon limit and choose a monomorphism from the fundamental group $\pi_1(N)$ in a compact Lie group endowed with a bi-invariant metric $d\sigma^2$. Then*

1. *There exists a principal fibre $G$-bundle $\pi : P \to AdS_3 \times N$ which admits a principal flat connection $\omega$.*

2. *There exists a rational one parameter family of Willmore-Chen submanifolds in $(P, [\bar{g}])$, where $\bar{g} = \pi^*(g) + \omega^*(d\sigma^2)$, $g$ being the string metric, which have non zero constant mean curvature in $(P, \bar{g})$, and are $(G \otimes H)$-invariant, where $H$ is the group of isometries of $N$.*

This result works, in particular, along the oxidation process of the involved metrics to $D = 10$ and $D = 11$. Consequently, we have ample families of solutions of $\mathcal{W}_{ar}$-dynamics (in particular equivariant Willmore-Chen submanifolds) in the oxidised $D = 10$ metrics $AdS_3 \times \mathbb{S}^3 \times T^4$ and $AdS_3 \times \mathbb{S}^3 \times K3$ of type IIB string theory. Also in the duals $AdS_3 \times \mathbb{S}^2 \times \mathbb{S}^1 \times T^4$, $AdS_3 \times \mathbb{S}^2 \times \mathbb{S}^1 \times K3$, $AdS_3 \times \mathbb{S}^3 \times T^4$, $AdS_3 \times \mathbb{S}^3 \times K3$, $AdS_3 \times (\mathbb{S}^3/\mathbb{Z}_p) \times T^4$ and $AdS_3 \times (\mathbb{S}^3/\mathbb{Z}_p) \times K3$ Type IIA theories. Finally the result applies to the oxidised $D = 11$ metrics $AdS_3 \times \mathbb{S}^3 \times T^5$ and $AdS_3 \times \mathbb{S}^3 \times K3 \times \mathbb{S}^1$ of M-theory.

## 8. Higher-dimensional theories

The Type IIA string can be obtained by compactifying the $D = 11$ supermembrane on a circle. An obvious question is whether Type IIB string also admits a higher-dimensional explanation. The appearance of Majorana-Weyl spinors and self-dual tensors in both the twelve-dimensional and Type IIB theories supplied evidence in favour of a corresponding and natural conjecture posed in [7].

Despite of all the objections one might raise to a world with two time dimensions, associated with the idea of a $(2, 2)$ object moving in a $(10, 2)$ spacetime, it has been revived in the context of the F-theory [42]. This involves Type IIB compactification where the axion and dilaton from R-R sector are allowed to vary on the internal manifold. In general, given a manifold $M$ that has the structure of a fiber bundle , with fiber $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$, on some manifold $L$, then [F-theory on $M$]≡[Type IIB theory on $L$]. Of course, the most conservative point of view is that the twelfth dimension is merely a mathematical artifact and so the F-theory should be considered as a clever way of compactifying the Type IIB string [39]. However, time will tell [13].

Consequently in this section we will investigate the following general framework. Let $M$ be an $F$-bundle associated with a certain principal $H$-bundle $Q(L, H)$, where $H$ is a compact Lie group and the fibre $F$ is a homogeneous space with dimension $d$. Let $p : M \to L$ be the projection map, $\theta$ a gauge potential on $Q(L, H)$ and $da^2$ an $H$-invariant metric on $F$. In this setting, given any metric $g'$ on $L$, one can find a unique metric $g$ on $M$ such that $p : (M, g) \to (L, g')$ is a Riemannian submersion with totally geodesic fibres isometric to $(F, da^2)$ and horizontal distribution associated with $\theta$ [44]. As above, the manifold of the $(d+1)$-dimensional, $H$-invariant compact submanifolds in $(M, g)$ can be identified with the set of complete lifts of closed curves in $L$. Once more, the search for symmetric configurations which are solutions of the $\mathcal{W}_{br}$-dynamics on $(M, g)$ is reduced to that of solutions of the dynamics associated with the restriction of the action $\mathcal{W}_{br}$ to the space of symmetric submanifolds. This restriction has been computed in [2] and is formally similar to that obtained in section 5, that is

$$\mathcal{W}_{br}(p^{-1}(\gamma)) = \frac{vol(F, da^2)}{(d+1)^r} \int_\gamma (\kappa^2 + \psi(\gamma'))^{r/2} ds,$$

where $\kappa$ denotes de curvature of $\gamma$ in $(L, g')$ and $\psi$ is defined on the unit tangent bundle of $(L, g')$ to measure, up to a constant involving $b$, the difference between the Ricci curvature of $(L, g')$ in a direction and the Ricci curvature of $(M, g)$ in the corresponding horizontal direction. Therefore we formally have the same Theorem 5.1, though in a more general context

**Theorem 8.1** *$p^{-1}(\gamma)$ belongs to $\Sigma^{br}$ in $(M,g)$ if and only if $\gamma$ is a critical point of the elastic energy functional*

$$\mathcal{F}(\gamma)=\int_\gamma (\kappa^2+\psi(\gamma'))^{r/2}ds,$$

*defined on closed curves in $(L,g')$.*

As above, most of the important applications of this result occur when the potential $\psi$ is constant. In this case we will use again the term of generalized elastica to name a critical point of $\mathcal{F}$. An obvious sufficient condition to guarantee this constancy of $\psi$ is to assume that both $(M,g)$ and $(L,g')$ are Einstein. In this case, if $\lambda$ and $\lambda'$ denote the corresponding cosmological constants then $\psi=\frac{b(d+1)}{d}(\lambda'-\lambda)$. This condition is not sufficient as the squashing method shows.

Although the list of examples satisfying that condition is too large, we have chosen some of them as an illustration.

## First example

It is known that the field equations of $d=11$ supergravity describe a 4-dimensional space-time with negative cosmological constant and a compact 7-dimensional Einstein space $(M,g)$ with positive scalar curvature. Many of these spaces can be regarded as principal fibre $H$-bundles over certain spaces $(L,g')$ and the Einstein metric $g$ is obtained from the Kaluza-Klein mechanism, i.e., $g=p^{-1}(g')+\theta^*(da^2)$. Consequently, Theorem 8.1 can be directly applied to most of the stable vacuum states in the Freund-Rubin spontaneous compactification of $d=11$ supergravity.

On the other hand, if we squash an Einstein metric $(M,g)$ by scaling the size of fibres, we still obtain Riemannian submersions providing constant potentials. Therefore, if we consider the canonical variation of the quaternion Hopf fibration $\{p:(\mathbb{S}^7,g_t)\to(\mathbb{S}^4,g');t>0\}$, we can find exactly two values of $t$, namely $t=1$ (the round metric) and $t=1/5$ (the squashed metric) providing Einstein metrics. It is known that generalized elasticae in a round 4-sphere yield in a totally geodesic 3-sphere, as well as the existence, for any pair of potentials $r$, $\psi$, of a rational one parameter family of closed helices, $\Omega_{r,\psi}=\{\gamma_j\}$, in a round 3-sphere, which are generalized elasticae [2]. Therefore, for any $t>0$, we obtain a rational one parameter family $\{p^{-1}(\gamma);\gamma\in\Omega_{r,\psi}\}$ of critical points of $\mathcal{W}_{br}$ in $(\mathbb{S}^7,g_t)$. This family of critical points can be projected down to the lens space $\mathbb{S}^7/\mathbb{Z}_\ell=L^7(\ell,1,1,1)$, which is also obtained in the list of solutions for $d=11$ supergravity given in [10], as $\frac{SU(4)\times U(1)}{SU(3)\times U(1)}$ and $\ell$ is the number of times that a simple loop in the $U(1)$ of the denominator winds around the $U(1)$ in the numerator. We will also denote by $g$ the round metric on $\mathbb{S}^7/\mathbb{Z}_\ell$ and use the already recalled way to generate the principal fibre bundles admitting a flat connection to obtain

**Corollary 8.2** *Let $G$ be any compact Lie group endowed with a bi-invariant metric $d\sigma^2$ and $\phi$ a monomorphism from $\mathbb{Z}_\ell$ in $G$. Then*

1. *There exists a principal fibre $G$-bundle $\pi:P\to\mathbb{S}^7/\mathbb{Z}_\ell$ which admits a principal flat connection $\omega$.*

2. *There exists a rational one parameter family of Willmore-Chen submanifolds in the conformal Kaluza-Klein structure $[\bar g]$, $\bar g=\pi^*(g)+\omega^*(d\sigma^2)$ on $P$ which are $(G\otimes SU(2))$-invariant and have non zero constant mean curvature in $(P,\bar g)$.*

### Second example

In [9], the spaces $N^{abc} = \frac{SU(3)\times U(1)}{U(1)\times U(1)}$ were studied, where $a$, $b$ and $c \in \mathbb{Z}$ characterize the embedding of $U(1) \times U(1)$ in $SU(3) \times U(1)$. These spaces can be viewed as principal fibre $SU(2)$-bundles on $\mathbb{CP}^2$. Using the Kaluza-Klein inverse mechanism and the squashing method, one can see that these spaces, except when $3a = b$, admit exactly two different Einstein metrics which make the above fibration a Riemannian submersion with totally geodesic fibres and the base $\mathbb{CP}^2$ endowed with the Fubini-Study metric (see also [35]). We denote by $(N^{abc}, g)$ the above Castellani-Romans Einstein Riemannian manifolds. On the other hand, for any couple of constants $r$, $\psi$ one can find a rational one parameter family of closed helices in $\mathbb{CP}^2$ which are generalized elasticae [2]. Therefore, for arbitrary $b$, $r$ one can get rational one parameter families of critical points for $\mathcal{W}_{br}$ in each $(N^{abc}, g)$ which are $SU(2)$-invariant and have non-zero constant mean curvature. To avoid Riemannian product in the next result, we will consider $r \neq 0$ (otherwise, $N^{ab0}$ is simply connected) and the embedding of $U(1) \times U(1)$ in $SU(3) \times U(1)$ is carried out by mapping a simple loop of the $U(1)$ in the denominator to wind $\ell$ times around the $U(1)$ in the numerator. In this case the fundamental group of $N^{abc}$ is $\mathbb{Z}_\ell$ and we have

**Corollary 8.3** *Let $G$ be any compact Lie group with a bi-invariant metric $d\sigma^2$ and $\phi$ a monomorphism from $\mathbb{Z}_\ell$ in $G$. Then*

1. *There exists a principal fibre $G$-bundle $\pi : P \to N^{abc}$ which admits a principal flat connection $\omega$.*
2. *There exists a rational one parameter family of Willmore-Chen submanifolds in $(P, [\bar{g}])$, $\bar{g} = \pi^*)g) + \omega^*(d\sigma^2)$, which are $(G \otimes SU(2))$-invariant and have non zero constant mean curvature.*

**Remark 8.4** We can obtain similar results in other spaces giving solutions for $d = 11$ supergravity, such as the spaces of Witten $M^{abc} = \frac{\mathbb{S}^5\times\mathbb{S}^3\times U(1)}{U(1)\times U(1)}$, which can be viewed as circle bundles on $\mathbb{CP}^2 \times \mathbb{S}^2$.

Since our result works for associated bundles (not necessarily principal), it can be applied, for instance, to certain Penrose twistor spaces. Namely, to the twistor spaces of $\mathbb{S}^4$ and $\mathbb{CP}^2$ to obtain examples of immersions in $\Sigma^{br}$ [2].

The method that we have developed in this paper can be applied to other backgrounds different from those considered here. It is also open to be extended to other contexts.

## 9. Conclusions

In this paper we have considered a two-parameter class of actions, $\{\mathcal{W}_{ar} : a, r \in \mathbb{R}\}$, defined on the space of immersions of a given smooth manifold $Q$ in a pseudo-Riemannian manifold $(L, ds^2)$. This constitutes a natural variation of the conformal total tension action, also known as the Willmore-Chen functional, whose importance is due in part to its invariance under conformal changes in the surrounding metric $ds^2$. This class also includes the popular Willmore action, so as the Canham-Helfrich bending energy of fluid membranes and lipid vesicles, as well as the Polyakov extrinsic action in the bosonic string theory. Roughly speaking, given a universe $(L, ds^2)$ and a Lie group $G$, which acts on $L$ through isometries of $(L, ds^2)$, the "best" worlds to live in this universe are those submanifolds which satisfy the following properties

1. They are $G$-invariant configurations. That means that they have a natural degree of *a priori* stated $G$-symmetry.

2. They are solutions of the isoperimetric area-volume problem, in particular they must have constant mean curvature (tension) in $(L, ds^2)$.

3. They are extremes for some tension action $\mathcal{W}_{ar}$ and so solutions of the corresponding motion equations.

Along this section, we will use the term $G_{ar}$-configuration to name those submanifolds in $(L, ds^2)$ which are $G$-invariant, critical points of some $\mathcal{W}_{ar}$ and have constant mean curvature in $(L, ds^2)$.

The existence of $G_{ar}$-configurations is investigated in string theories, M-theory and F-theory (even in higher-dimensional theories) on backgrounds of the form $AdS \times M$, where $M$ is some principal $U(1)$-bundle.

Recall that string theory, emerging as a candidate for the unification of the fundamental forces in the nature, has a main objection. In fact, there are five different, but consistent, ten dimensional string theories which are all distinct in their perturbative spectra. The understanding of the problems derivated from this ambiguity has undergone a great improvement with the appearance of dualities. String dualities are, in some sense, the statements one has to relate all five different perturbative superstrings. It is tempting then to imagine that they are the expansion of a single and more powerful eleven dimensional, non-perturbative, unified theory (known as M-theory and that contains the $D = 11$ supergravity as low energy limit) around five different sets of perturbative variables. A very important ingredient to account for these dualities are those properties (encoded in the Physics or in the Geometry of the theory) which remain invariant along the duality chains. When the compactifying space $M$ is a principal $U(1)$-bundle, one can define a natural kind of T-duality [14, 15]. These Hopf T-dualities relate different black holes preserving entropies [15]. Therefore, the black holes entropies provide a nice invariant along duality chains.
In this paper we have shown that
*The existence of wide families of $G_{ar}$-configurations, for arbitrary $a$, $r$ and suitable choices of $G$, is also an invariant along any duality chain.*

However, this general statement is actually a series of results that were obtained as conclusions from a general program of breaking symmetry which was developed in section 2. The method exhibited there is mainly based on a formulation due to R.S.Palais [36] of the so-called principle of symmetric criticality. This method allows one to break the $G$-symmetry by reducing the number of variables in the study of $\mathcal{W}_{ar}$-variational problems. Furthermore, the constancy of the mean curvature function remains invariant through this process.

To conclude, we summarize the main points in the above series of invariants. The first natural problem, that we solved, corroborates once more the important role that $AdS$-geometry plays in these theories and specially the group manifold $AdS_3$. It corresponds with the case where the transverse space $(M, g)$ is a $G$-homogeneous one. In $AdS_p \times M_{D-p}$, we consider the metric $f^2 g_o + g$, where $f$ is any smooth positive function on $M$. Given a curve $\gamma$ in $AdS$, one can evolve the transverse space $M$ through $\gamma$ to generate the tube $T_\gamma = \gamma \times M$ with the metric $f^2 dt^2 + g$. It is obvious that $T_\gamma$ is $G$-invariant and so it seems natural to ask whether $T_\gamma$ is a $G_{ar}$-configuration. The method we exhibited here allows us to reduce this problem to one for $\gamma$ in the $AdS$ part. Using the nice geometry of $AdS$, we are able to get not only the characterization of $T_\gamma$ to be a $G_{ar}$-configuration, but also the complete classification of these solutions. This provides us the moduli space of $G_{ar}$-configurations with dimension $D-p+1$. These moduli spaces must be understood, up to isometries of $f^2 g_o + g$, except in the case of the Willmore-Chen action, where we can relax to conformal transformations of $f^2 g_o + g$. In this context, we can also obtain $G_{ar}$-configurations with dimension greater than $D-p+1$, however we do not know the moduli spaces and this can be regarded as an open problem.

Other solutions in this paper are directly obtained in the transverse space. For example, if we start from Type IIB theory on $AdS_5 \times M_5$, where $M_5$ is a principal $U(1)$-bundle over, say $B$, then we have the duality chain [Type IIB theory on $AdS_5 \times M_5$]→[Type IIA theory on $AdS_5 \times B \times \mathbb{S}^1$]→[M-theory on $AdS_5 \times B \times T^2$]. Then we use our method of breaking symmetry to reduce the search for $G_{ar}$-configurations, where $G$ is now an internal gauge group, in these backgrounds to that for certain elastic curves in the base space $B$. Giving explicit examples of elasticae, for example, in $\mathbb{CP}^2$ or in the complex quadric $Q_2 = \mathbb{S}^2 \times \mathbb{S}^2$, we get wide families of $G_{ar}$-configurations along the above duality chain.

The method also works on theories that carry both NS-NS and R-R electric and magnetic charges. Therefore, we showed the existence of ample families of solutions in the near horizon limit of any non-dilatonic black hole in $D=5$ and $D=4$. The degree of symmetry of these configurations is preserved only when NS-NS charges appear, while that is decreased if R-R charges are carried. Higher-dimensional theories, such as F-theory, are also investigated in relation with the existence of $G_{ar}$-configurations

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