On the Gauss map of B-scrolls in 3-dimensional Lorentzian space forms

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Abstract

In this note we show that B-scrolls over null curves in a 3-dimensional Lorentzian space form $\bar{M}_1^3(c)$ are characterized as the only ruled surfaces with null rulings whose Gauss maps G satisfy the condition $\Delta G = \Lambda G$, $\Lambda : \mathfrak{X}(\bar{M}) \to \mathfrak{X}(\bar{M})$ being a parallel endomorphism of $\mathfrak{X}(\bar{M})$.

1. Introduction

In [4], C. Baikoussis and D.E. Blair studied ruled surfaces in $\mathbb{R}^3$ such that their Gauss maps satisfy $\Delta G = \Lambda G$, where Δ denotes the Laplace operator of the surface with respect to the induced metric and Λ stands for a fixed endomorphism of the ambient space $\mathbb{R}^3$. They showed that the only ones are planes and circular cylinders. Recently, S. M. Choi in [5] investigated the Lorentz version of the above result and she essentially obtained the same result.

It is worth pointing out that all surfaces obtained above have diagonalizable shape operator. However, it is possible that a self-adjoint linear operator on a Lorentzian plane is not diagonalizable (see, for example, [1], [2] and [7], where that chief difference with regard to the Riemannian case has been greatly exploited). To illustrate the current situation, let $\gamma(s)$ be a null curve in a 3-dimensional Lorentzian space form $\bar{M}_1^3(c)$ and $B(s)$ a null vector field along $\gamma(s)$. Under a certain hypothesis (see Example 1 for more details) the map $X : (s,t) \to \gamma(s) + tB(s)$ defines a "ruled surface" in $\bar{M}_1^3(c)$ whose shape operator has a minimal polynomial of degree two with a double real eigenvalue, so that it is not diagonalizable. That surface is called a B-*scroll* and was introduced by L.K. Graves (see [8] and also [1]). The main purpose of [3] was to complete Choi's classification of ruled surfaces in $\mathbb{L}^3$ whose Gauss maps satisfy the condition $\Delta G = \Lambda G$. Actually, it was shown that B-scrolls over null curves in $\mathbb{L}^3$ are the only ruled surfaces in $\mathbb{L}^3$ with null rulings satisfying the above condition.

In this note we extend the main result of [3] and show that B-scrolls over null curves in a 3-dimensional Lorentzian space form $\bar{M}_1^3(c)$ are characterized as the only ruled surfaces with null rulings whose Gauss maps G satisfy the condition $\Delta G = \Lambda G$, $\Lambda : \mathfrak{X}(\bar{M}) \to \mathfrak{X}(\bar{M})$ being a parallel endomorphism of $\mathfrak{X}(\bar{M})$. The important point to note here is the technique we have used. The advantage of using Jacobi vector fields is that the characterization of such surfaces has been obtained without viewing $\bar{M}_1^3(c)$ as a hypersurface into the corresponding pseudo-Euclidean space. Hence our proof provides a natural and intrinsic characterization of those surfaces.

2. Setup

Let $\bar{M}_1^3(c)$ be a 3-dimensional Lorentzian space form of constant curvature c. As usual, $\bar{M}_1^3(c)$ is either the pseudo-Euclidean space $\mathbb{R}_1^3$, or the pseudo-sphere $\mathbb{S}_1^3(c) \subset \mathbb{R}_1^4$, or the pseudo-hyperbolic space $\mathbb{H}_1^3(c) \subset \mathbb{R}_2^4$, according to $c = 0$, $c > 0$ or $c < 0$, respectively. For the sake of simplicity, and provided that we need explicitly mention neither curvature c nor index, we will simply write down $\bar{M}$ instead of $\bar{M}_1^3(c)$.

Let $\alpha : I \subset \mathbb{R} \to \bar{M}$ be an immersed curve and let $B \in \mathfrak{X}(\alpha)$ be a vector field along α in $\bar{M}$. Let us consider the ruled surface M in $\bar{M}$, generated by α and B, which is naturally parametrized by

$$\begin{aligned} X : I \times (-a, a) &\rightarrow \bar{M}, \\ (s,t) &\rightarrow X(s,t) = \exp_{\alpha(s)}(tB(s)). \end{aligned}$$

For each fixed s, the curve γ_s defined by $t \to \gamma_s(t) = X(s,t)$ is the geodesic of $\bar{M}$ uniquely determined by the initial conditions $\gamma_s(0) = \alpha(s)$ and $\gamma_s'(0) = B(s)$. Let $\{X_s, X_t\}$ be the frame defined by

$$X_s(s,t) = dX_{(s,t)}\left(\left.\frac{\partial}{\partial s}\right|_{(s,t)}\right) = (\mathrm{dexp}_{\alpha(s)})_{tB(s)}(\alpha'(s) + tB'(s))$$

and

$$X_t(s,t) = dX_{(s,t)}\left(\left.\frac{\partial}{\partial t}\right|_{(s,t)}\right) = (\mathrm{dexp}_{\alpha(s)})_{tB(s)}(B(s)),$$

where $B'(s)$ stands for the covariant derivative of $B(s)$ along α. Observe that, at $t = 0$, $X_s(s,0) = \alpha'(s)$ and $X_t(s,0) = B(s)$, so that $X(s,t)$ will define a regular pseudo-Riemannian surface into $\bar{M}$ whenever $\alpha'(s)$ and $B(s)$ are linearly independent and the plane $\Pi = \mathrm{span}\{\alpha'(s), B(s)\}$ is non degenerate in $\bar{M}$. According to the causal character of α' and B, there are four possibilities:

(1) α' and B are non-null and linearly independent.
(2) α' is null and B is non-null with $\langle \alpha', B \rangle \neq 0$.
(3) α' is non-null and B is null with $\langle \alpha', B \rangle \neq 0$.
(4) α' and B are null with $\langle \alpha', B \rangle \neq 0$.

It is easy to see that, with an appropiate change of the curve α, cases (2) and (3) reduce to (1) and (4), respectively (see [3] for details). We will pay attention to cases (3) and (4) which we aim to characterize in terms of the Laplacian of their Gauss maps. Therefore, let M be a ruled surface in $\bar{M}$ whose directrix $\alpha(s)$ and rulings $B(s)$ both are null, and assume without loss of generality that $\langle \alpha', B \rangle(s) = -1$.

To compute the metric induced on M, we apply the Gauss lemma to get that

$$\begin{aligned} \langle X_s, X_t \rangle(s,t) &= \langle \alpha' + tB', B \rangle(s) &= -1, \\ \langle X_t, X_t \rangle(s,t) &= \langle B, B \rangle(s) &= 0. \end{aligned}$$

Note that, for each fixed s, the vector field J_s defined by $J_s(t) = X_s(s,t)$ is a Jacobi vector field along γ_s with initial conditions $J_s(0) = \alpha'(s)$ and $J_s'(0) = B'(s)$. As $\bar{M}$ is a space of constant curvature, we can write

$$J_s(t) = P_s(t) + tQ_s(t),$$

$P_s(t)$ and $Q_s(t)$ being parallel translation vector fields along $\gamma_s(t)$ of vectors $\alpha'(s)$ and $B'(s)$, respectively. Then we have

$$\begin{aligned}\langle X_s, X_s\rangle(s,t) &= \langle P_s, P_s\rangle(t) + 2t\langle P_s, Q_s\rangle(t) + t^2\langle Q_s, Q_s\rangle(t) \\ &= \langle \alpha', \alpha'\rangle(s) + 2t\langle \alpha', B'\rangle(s) + t^2\langle B', B'\rangle(s).\end{aligned}$$

Hence the matrix (g_{ij}) of the induced metric on M reads as follows:

$$\begin{pmatrix} 2t\langle \alpha', B'\rangle(s) + t^2\langle B', B'\rangle(s) & -1 \\ -1 & 0 \end{pmatrix}.$$

Assume now that we have chosen an orientation on $\bar{M}$. Then a volume element ω is determined on $\bar{M}$ by the condition $\omega(X,Y,Z) = -1$, for any positively oriented orthonormal frame $\{X,Y,Z\}$. Therefore, for any couple X and Y of tangent vectors to $\bar{M}$, the vector product $X \wedge Y$ is the unique tangent vector to $\bar{M}$ such that $\langle X \wedge Y, Z\rangle = \omega(X,Y,Z)$ for any tangent vector Z. It is well known that the vector product of parallel vector fields also is a parallel vector field, so that the Gauss map G can be given in terms of $X_s \wedge X_t$ getting

$$\begin{aligned}G(s,t) &= P_s(t) \wedge X_t(s,t) + tQ_s(t) \wedge X_t(s,t) \\ &= \tilde{P}_s(t) + t\tilde{Q}_s(t),\end{aligned}$$

where $\tilde{P}_s(t)$ and $\tilde{Q}_s(t)$ are parallel translation vector fields along $\gamma_s(t)$ of $(\alpha' \wedge B)(s)$ and $(B' \wedge B)(s)$, respectively. Bearing in mind that $\langle X \wedge Y, X \wedge Y\rangle = \langle X, Y\rangle^2 - \langle X, X\rangle\langle Y, Y\rangle$, we see that $\langle G, G\rangle = 1$.

If we put $C(s) = (\alpha' \wedge B)(s)$, then $\{\alpha', B, C\}$ is a pseudo-orthonormal frame field of $\bar{M}$ along α. In this frame, we easily see that $B' \wedge B = -\beta B$, β being the function defined on I by $\beta(s) = \langle x', B' \wedge B\rangle(s)$. Hence $\tilde{Q}_s(t)$ is the parallel translation vector field of $-\beta(s)B(s)$, so that $\tilde{Q}_s(t) = -\beta(s)X_t(s,t)$. Thus

$$G(s,t) = \tilde{P}_s(t) - t\beta(s)X_t(s,t).$$

We are going to compute the shape operator S. To do that, a simple computation yields

$$\begin{aligned}\frac{\bar{D}G}{\partial t}(s,t) &= \frac{\bar{D}\tilde{P}_s}{dt}(t) - \beta(s)X_t(s,t) - t\beta(s)\frac{\bar{D}X_t}{\partial t}(s,t) \\ &= -\beta(s)X_t(s,t),\end{aligned}$$

where $\bar{D}/\partial t$ and $\bar{D}/dt$ stand for the covariant derivative in $\bar{M}$ along M and γ_s, respectively. Now observe that $W_s(t) = G(s,t)$ is a Jacobi vector field along $\gamma_s(t)$ with initial conditions $W_s(0) = C(s)$ and $W_s'(0) = -\beta(s)B(s)$, so that

$$W_s(t) = (d\exp_{\alpha(s)})_{tB(s)}(C(s) - t\beta(s)B(s)).$$

Then a straightforward computation leads to

$$\frac{\bar{D}G}{\partial s} = -\beta(s)X_s(s,t) - (\langle \alpha', \alpha'' \wedge B\rangle(s) + t\beta'(s))X_t(s,t),$$

where we have used that $C'(s) = -\beta(s)\alpha'(s) - \langle\alpha', \alpha'' \wedge B\rangle(s)$. So the shape operator writes down as

$$S = \begin{pmatrix} \beta(s) & 0 \\ t\alpha'(s) + \langle\alpha', \alpha'' \wedge B\rangle(s) & \beta(s) \end{pmatrix}.$$

On the other hand, the Laplacian of the Gauss map can be computed as follows:

$$\Delta G = -\frac{\bar{D}}{\partial s}\frac{\bar{D}G}{\partial t} - \frac{\bar{D}}{\partial t}\frac{\bar{D}G}{\partial s} - 2\{\langle\alpha', B'\rangle + t\langle B', B'\rangle\}\frac{\bar{D}G}{\partial t} - \{2t\langle\alpha', B'\rangle + t^2\langle B', B'\rangle\}\frac{\bar{D}^2G}{\partial t^2}.$$

A straightforward computation yields

$$\begin{aligned}
\frac{\bar{D}}{\partial s}\frac{\bar{D}G}{\partial t}(s,t) &= -\beta'(s)X_t(s,t) - \beta(s)Q_s(t), \\
\frac{\bar{D}}{\partial t}\frac{\bar{D}G}{\partial s}(s,t) &= \frac{\bar{D}}{\partial s}\frac{\bar{D}G}{\partial t}(s,t), \\
\frac{\bar{D}}{\partial t}\frac{\bar{D}G}{\partial t}(s,t) &= 0.
\end{aligned}$$

Hence the Laplacian of G is given by

$$\Delta G(s,t) = 2\{\beta'(s) + \beta(s)(\langle\alpha', B'\rangle(s) + t\langle B', B'\rangle(s))\}X_t(s,t) + 2\beta(s)Q_s(t).$$

3. Main result

We start this section with a typical example.

Example 1. Let $\gamma(s)$ be a null curve in $\bar{M}$ with an associated Cartan frame $\{A, B, C\}$, i.e., $\{A, B, C\}$ is a pseudo-orthonormal frame of vector fields along $\gamma(s)$,

$$\begin{aligned}
\langle A, A\rangle = \langle B, B\rangle = 0, &\qquad \langle A, B\rangle = -1, \\
\langle A, C\rangle = \langle B, C\rangle = 0, &\qquad \langle C, C\rangle = 1,
\end{aligned}$$

such that

$$\begin{aligned}
\dot{\gamma}(s) &= A(s), \\
\dot{C}(s) &= -hA(s) - k(s)B(s),
\end{aligned}$$

where h is a nonzero constant and $k(s) \neq 0$ for all s. Then the map $X : (s,t) \to \gamma(s) + tB(s)$ parametrizes a Lorentzian surface into $\bar{M}$ which is called a *B-scroll* (see [1] and [6]).

It is not difficult to see that the Gauss map G is given by

$$G(s,t) = -htB(s) + C(s),$$

and the shape operator, in the usual frame $\left\{\frac{\partial X}{\partial s}, \frac{\partial X}{\partial t}\right\}$, writes down as

$$S = \begin{pmatrix} h & 0 \\ k(s) & h \end{pmatrix}.$$

Thus the B-scroll has a non-diagonalizable shape operator with the minimal polynomial $P_S(u) = (u-h)^2$. It has constant mean and Gaussian curvatures $H = h$ and $K = c + a^2$, respectively, and satisfies $\Delta G = 2KG$.

Then it seems natural to pose the following question: *is a B-scroll the only ruled surface in $\bar{M}$ with null rulings satisfying the equation $\Delta G = \Lambda G$?* The answer is affirmative and can be stated as follows.

Theorem 3.1 *B-scrolls over null curves are the only ruled surfaces in $\bar{M}$ with null rulings satisfying the equation $\Delta G = \Lambda G$, Λ being a parallel endomorphism on $\mathfrak{X}(\bar{M})$.*

Proof. Let M be a ruled surface in $\bar{M}$ with null rulings satisfying $\Delta G = \Lambda G$. Without loss of generality, we can assume that the directrix curve $\alpha(s)$ is null and according to [1] we only have to prove that β is constant. Consider the set $\mathcal{U} = \{s \in I : \beta(s)\beta'(s) \neq 0\}$ and study the equation $\Delta G = \Lambda G$ on the set $U \times (-a, a)$. Differentiating with respect to t we have

$$2\beta(s)\langle B', B'\rangle(s)X_t(s,t) = -\beta(s)\Lambda X_t(s,t),$$

so we get that $\lambda = -2\langle B', B'\rangle(s)$ is an eigenvalue of Λ. It is not difficult to show that $B'(s) = -\langle \alpha', B'\rangle(s)B(s) - \beta(s)C(s)$ and $C'(s) = -\beta(s)\alpha'(s) - \langle \alpha', \alpha'' \wedge B\rangle(s)B(s)$ and so $\lambda = -2\beta(s)^2$. At $t = 0$, a long and messy computation yields $\mathrm{tr}(\Lambda)(s,0) = -\langle \Lambda\alpha', B\rangle(s) - \langle \Lambda B, \alpha'\rangle(s) + \langle \Lambda C, C\rangle(s) = 3\lambda$ and hence the gradient of λ is $\nabla\lambda = (1/3)\mathrm{tr}(\nabla\Lambda) = 0$. Therefore λ and $\beta(s)$ both are constant and the proof is complete.

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