Geometric integration of the Betchov-Da Rios equation in a Gravity-Electromagnetism unified model

Manuel Barros, Angel Ferrández and Pascual Lucas
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Abstract

In this paper we find out explicit solutions of the Betchov-Da Rios soliton equation in a principal circle bundle $\pi : P \to M$ on a surface M. If P is endowed with a generalized Kaluza-Klein metric $\bar{g}_u$, we show that the complete lift of any curve γ is a solution of the Betchov-Da Rios equation if and only if the function u restricted to γ is just the curvature of γ. Some interesting applications are given.

1. Introduction

The first model to get the natural unification of gauge fields and gravitation goes back to the classical model of Kaluza (1921) and Klein (1926). This is a 5-dimensional model to unificate gravitiy and electromagnetism. In this note the space time M is 2-dimensional with gravity determined by a pseudo-Riemannian metric g. Then, we consider a $U(1) \equiv \mathbb{S}^1$ principal fibre bundle P on M, endowed with a gauge potential (of electromagnetism) ω. The metrics that we consider on P are conformal to the so called Kaluza-Klein metrics, where the conformal factor is constant along the fibres. In some sense, these metrics are locally warped products with warping function defined by the conformal factor. The aim of this note is to obtain geometric solutions of the Betchov-Da Rios equation, also called localized induction equation (LIE), in these models. Some applications and examples are obtained. In particular, one of them relates the solution with the theory of elasticae. We also obtain an irrational one-parameter family of gauge potentials (all of them with the same holonomy) producing models which are foliated by solutions of LIE. The algorithms exhibited here can be applied to construct solutions of LIE in other models.

2. Setup

Let $\pi : P \to M$ be a principal bundle with structure group $\mathbb{S}^1$ on a surface M and let ω be the connection 1-form of a principal connection on P. For any metric g on M and any positive smooth function u on M, we define

$$\bar{g}_u = \pi^*(g) + \varepsilon_3 (u \circ \pi)^2 \omega^*(dt^2),$$

where $\varepsilon_3 = \pm 1$ stands for the causal character of the fibres. Then $\pi : (P, \bar{g}_u) \to (M, g)$ is a pseudo-Riemannian submersion and $\mathbb{S}^1$ acts by isometries of $(P, \bar{g}_u)$. Let γ be a curve in M and let $\bar{\gamma}$ be any horizontal lift of γ. Then $\bar{\gamma}$ is arclength parametrized just taking γ arclength parametrized. The tube $T_\gamma = \pi^{-1}(\gamma)$ is the complete lift of γ and can be parametrized by

$$\Psi(s,t) = e^{it}\bar{\gamma}(s).$$

In order to compute the $\bar{g}_u$-induced metric on T_γ we have

$$\begin{aligned}\Psi_s &= e^{it}\bar{\gamma}'(s),\\ \Psi_t &= ie^{it}\bar{\gamma}(s) \equiv V(s,t),\end{aligned}$$

V being the tangent vector field to the fibres. Then the induced metric is given by

$$\begin{pmatrix} \langle\Psi_s,\Psi_s\rangle & \langle\Psi_s,\Psi_t\rangle \\ \langle\Psi_t,\Psi_s\rangle & \langle\Psi_t,\Psi_t\rangle \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_3(u\circ\pi)^2 \end{pmatrix},$$

where we have used that V is the fundamental vector field 1^*, so that $\omega(V)=1$. For the sake of simplicity we will write u by $\bar{u}$.

The Betchov-Da Rios equation in 3-dimensional hydrodynamics,

$$\frac{\partial\Psi}{\partial s}\wedge\overline{D}_{\frac{\partial}{\partial s}}\frac{\partial\Psi}{\partial s}=\frac{\partial\Psi}{\partial t},$$

is a soliton equation for space curves $\Psi(s,t)$, $\overline{D}$ being the Levi-Civita connection of the space. This can be rewritten as $\Psi_t=\kappa B$ (the filament equation), where κ and B stand for the curvature and the unit binormal of Ψ, respectively. The evolution of Ψ governed by this equation of motion can be viewed as an idealization of the motion of a thin vortex cylinder (see [4] and [5] for details).

The first interesting result states as follows.

Lemma 2.1 *Let γ be a curve in (M,g). Then $\pi^{-1}(\gamma)=\Psi(s,t)$ is a solution of the Betchov-Da Rios equation in $(P,\bar{g}_u)$ if and only if $u^2(\gamma(s))=\kappa^2(s)$.*

Proof. It is easy to see that

$$\overline{D}_{\Psi_s}\Psi_s=\varepsilon_2 e^{it}\bar{\kappa}(s)\bar{N}(s),$$

ε_2 standing for the causal character of the unit normal. Then we have

$$\Psi_s\wedge\overline{D}_{\Psi_s}\Psi_s=\varepsilon_2\bar{\kappa}(s)e^{it}(\bar{\gamma}'(s)\wedge\bar{N}(s))=\varepsilon_2\bar{\kappa}(s)e^{it}\bar{B}(s),$$

which finishes the proof.

3. Main results and applications

The first applications arise by choosing cylindrical coordinates in $\mathbb{R}^3$. We consider the following subsets

$$\begin{aligned}P &= \{(w,z)=(x,y,z)\in\mathbb{R}^3 : x^2+y^2\neq 0\}\equiv\mathbb{C}^*\times\mathbb{R},\\ H &= \{(x,y,z)\in\mathbb{R}^3 : y=0 \quad\text{and}\quad x>0\}.\end{aligned}$$

Then $P=H\times\mathbb{S}^1$. Let $\pi: P\to H$ be the canonical projection, which can be viewed as a $\mathbb{S}^1$-bundle with the action

$$e^{it}(w,z)=(e^{it}w,z).$$

We have on this bundle an obvious connection associated to the horizontal distribution defined by H.

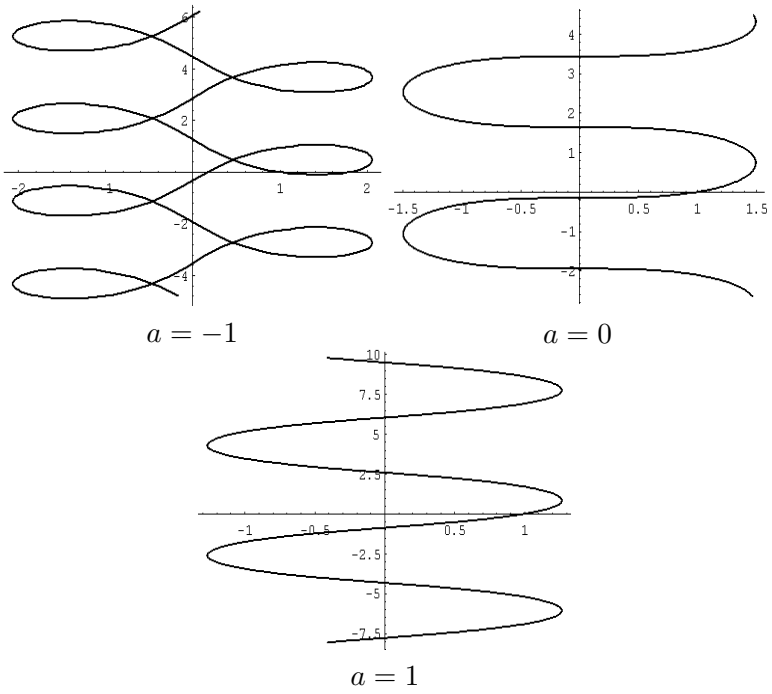

Figure 3: Elasticae in the Euclidean plane $\mathbb{R}^2$.

3.1. The Euclidean case

Let g be the Euclidean metric on H and let $u : H \to \mathbb{R}$ be the positive function defined by $u(x, 0, z) = x$, $x > 0$, which measures the distance to the z-axis. Then $\bar{g}_u$ is the Euclidean metric on P provided $\varepsilon_3 = +1$.

Our main theorem states as follows.

Theorem 3.1 *Let γ be a curve in (H, g). The tube $\pi^{-1}(\gamma) = \Psi(s, t)$ is a congruence solution of the Betchov-Da Rios soliton equation in $(P, \bar{g}_u)$ if and only if γ is an elastica in (H, g).*

Proof. Let $\gamma(s) = (x(s), 0, z(s))$, $x(s) > 0$, be an arclength parametrized curve in (H, g) . From Lemma 2.1 we see that $\Psi(s, t)$ is a solution of the Betchov-Da Rios equation in $(P, \bar{g}_u)$ if and only if $u(\gamma(s)) = \kappa(s) = x(s)$, where κ stands for the curvature function of γ. On the other hand, $\kappa(s) = (-x''z' + x'z'')(s)$, that jointly with $x'^2 + y'^2 = 1$ yield

$$x'' = -xz', \tag{1}$$
$$z'' = xx'. \tag{2}$$

A first integral of (2) gives

$$z' = a + \frac{1}{2}x^2, \tag{3}$$

for a certain constant a. We now combine the equations (1) and (3) to obtain

$$x'' + \frac{1}{2}x^3 + ax = 0.$$

This equation shows that $x(s) = \kappa(s)$ is a solution of the Euler-Lagrange equation of the elastica in the Euclidean plane (see [6]) with Lagrange multiplier $\lambda = -2a$. That means that γ is a critical point of the functional $\int_\gamma (\kappa^2 - 2a)ds$ acting on curves which satisfy certain first order boundary data in $\mathbb{R}^2$. The converse is a result due to H. Hasimoto [3] (see also [4]).

In Figure 3 we sketch some examples of elasticae γ in $\mathbb{R}^2$ giving congruence solutions T_γ of the Betchov-Da Rios equation in $\mathbb{R}^3$.

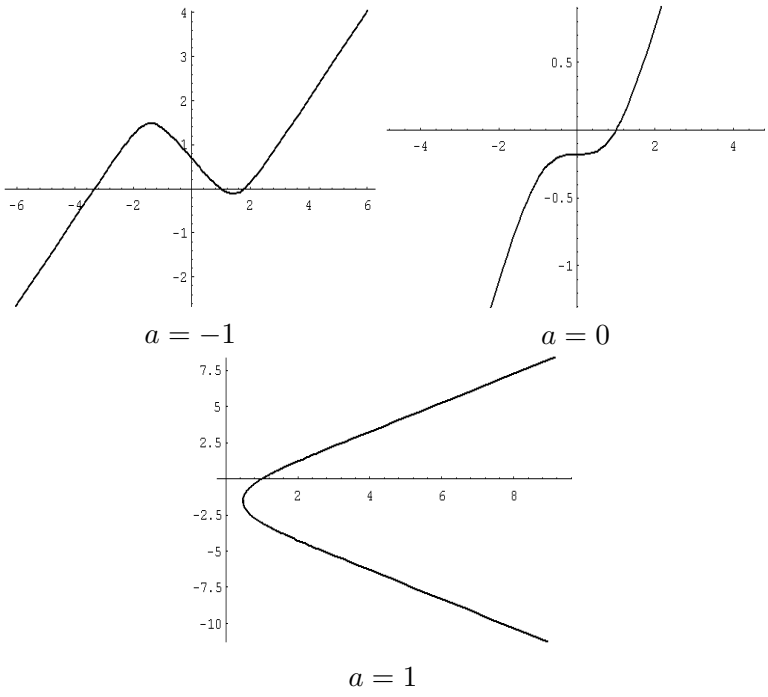

Figure 4: Spacelike elasticae in the Lorentzian plane $\mathbb{L}^2$.

3.2. The Lorentzian case

Let $g = dx^2 - dz^2$ be a Lorentz metric on H and choose $u \in C^\infty_+(H)$ as above. Then $\bar{g}_u$ gives a Lorentzian metric on P provided $\varepsilon_3 = +1$. Given any curve γ in (H, g) we observe that $\pi^{-1}(\gamma)$ is just the surface of revolution obtained by rotating γ around the z-axis. As $\varepsilon_3 = +1$, for the sake of simplicity we will write $\varepsilon = \varepsilon_1$ to denote the causal character of γ. If $\Psi(s,t) = (x(s)\cos t, x(s)\sin t, z(s))$ is a solution of the Betchov-Da Rios equation, the curvature function $\kappa(s) = (x''z' - x'z'')(s)$ of $\gamma(s)$ must be equal to $x(s)$. From here and the unit speed condition $(x')^2 - (z')^2 = \varepsilon$ on γ we get the following system of differential equations

$$x'' = \varepsilon x z', \tag{5}$$
$$z'' = \varepsilon x x'. \tag{6}$$

In order to find spacelike solutions of the Betchov-Da Rios equation we must take $\varepsilon = 1$.

Then a first integral of (6) gives

$$z' = a + \frac{1}{2}x^2, \tag{7}$$

a being a constant. By combining equations (5) and (7) we have

$$x'' - \frac{1}{2}x^3 - ax = 0.$$

This equation proves that γ is an elastica in $\mathbb{L}^2$ with Lagrange multiplier $\lambda = 2a$ (see [2]). Said otherwise, γ is a critical point of the functional $\int_\gamma (\kappa^2 + 2a)ds$ acting on curves which satisfy certain first order boundary data in $\mathbb{L}^2$ (see [2] for details).

In Figure 4 we exhibit some examples of spacelike elasticae γ in $\mathbb{L}^2$ which provide spacelike congruence solutions T_γ of the Betchov-Da Rios equation in $\mathbb{L}^3$.

Furthermore, we can also find interesting examples of Lorentzian congruence solutions of the Betchov-Da Rios equation in $\mathbb{L}^3$ shaped on timelike elasticae in $\mathbb{L}^2$. These are obtained by solving the system

$$\begin{aligned} x'' &= \frac{1}{2}x^2 - ax, \\ z' &= -\frac{1}{2}x^2 + a, \end{aligned}$$

where a is again a constant related to the constrained length of curves on which the elastic energy functional is defined. In Figures 5 and 6 we sketch some of these curves and their corresponding congruence solutions.

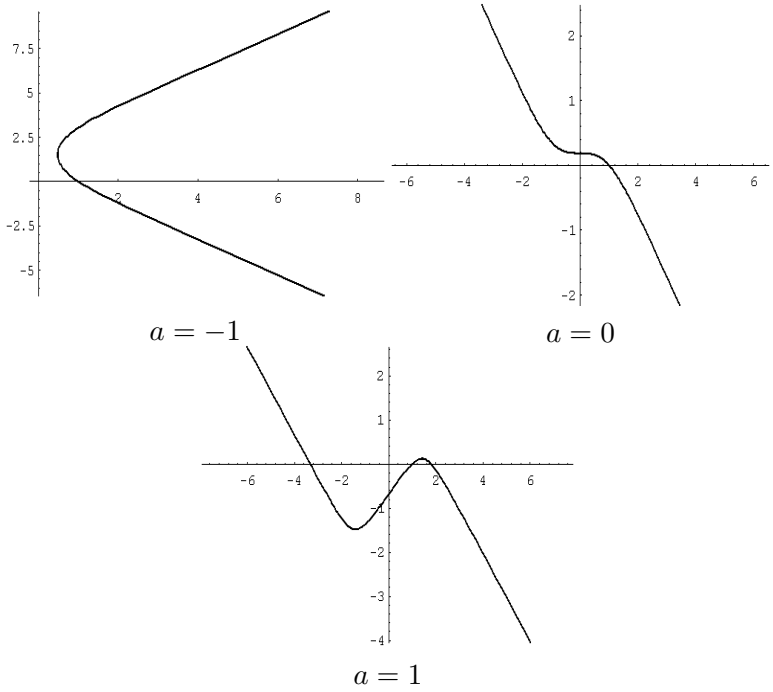

Figure 5: Timelike elasticae in the Lorentzian plane $\mathbb{L}^2$.

3.3. Non standard 3-spheres

Let $\pi : \mathbb{S}^3 \to \mathbb{S}^2$ be the usual Hopf fibration. Let $\bar{g}$ be the standard metric on $\mathbb{S}^3$ of constant curvature 1 and let g be the standard one on $\mathbb{S}^2$ of constant curvature 4. Thus $\pi : (\mathbb{S}^3, \bar{g}) \to (\mathbb{S}^2, g)$ be-

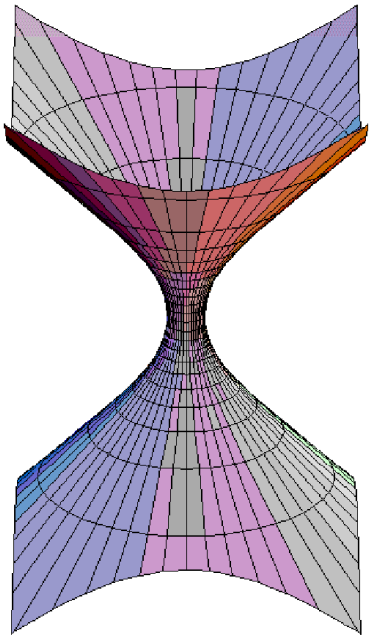

Figure 6: Revolution timelike surface in $\mathbb{L}^3$ with $a = -1$.

comes a totally geodesic pseudo-Riemannian submersion whose fibres are isometric to $\mathbb{S}^1$. Given any smooth function $u : \mathbb{S}^2 \to \mathbb{R}^+$, consider $\bar{g}_u = \pi^*(g) + \varepsilon(u \circ \pi)^2 \omega^*(dt^2)$, where ω stands for the natural connection associated to the horizontal ($\bar{g}_u$-orthogonal to the fibres) distribution. It is obvious that $(\mathbb{S}^3, \bar{g}_u)$ has Lorentzian causal character provided $\varepsilon = -1$. Then $\{\pi : (\mathbb{S}^3, \bar{g}_u) \to (\mathbb{S}^2, g)\}$ defines a class of pseudo-Riemannian submersions with the same horizontal distribution. Furthermore, such a submersion is totally geodesic if and only if u is a constant function. If this is the case, $(\mathbb{S}^3, \bar{g}_u)$ has constant scalar curvature.

Let γ be an immersed curve in $(\mathbb{S}^2, g)$ and assume that γ has positive curvature function. By Lemma 2.1 we take $u \in C^\infty_+(\mathbb{S}^2)$ such that $u(\gamma(s)) = \kappa(s)$. Then the tube $M_\gamma = \pi^{-1}(\gamma)$, naturally parametrized by fibres and horizontal lifts, provides a solution of the Betchov-Da Rios soliton equation in $(\mathbb{S}^3, \bar{g}_u)$.

To exhibit examples of this kind of solutions we propose the following algorithm.

First step (see [7]). Consider $\mathbb{S}^3$ as the set of unit quaternions $\{q \in \mathbb{H} \ : \ q \cdot \bar{q} = 1\}$ and $\mathbb{S}^2$ as the 2-sphere of radius 1/2 in the subspace $\mathrm{span}\{1, i, j\} \subset \mathbb{H}$. Let $q \to \tilde{q}$ be the skew-automorphism of $\mathbb{H}$ that fixes 1, j and k, but sends i to $-i$. Then $\pi : \mathbb{S}^3 \to \mathbb{S}^2$ is given by $\pi(q) = \frac{1}{2}\tilde{q} \cdot q$.

Second step. Given any point $p = (A, 0, B, C)$ in $\mathbb{S}^2$, the fibre $\pi^{-1}(p)$ is given by

$$\pi^{-1}(p) = \left\{ \left(D\cos\alpha, D\sin\alpha, \frac{1}{D}(B\cos\alpha - C\sin\alpha), \frac{1}{D}(B\cos\alpha + C\sin\alpha \right) \ : \ \alpha \in \mathbb{R} \right\},$$

where $D = \sqrt{\frac{1+2A}{2}}$.

Third step. Let $\gamma(s) = (A(s), 0, B(s), C(s))$ be an arc length parametrized curve in $\mathbb{S}^2$. A straightforward but long computation shows that any horizontal lift $\bar{\gamma}$ of γ to $(\mathbb{S}^3, \bar{g}_u)$ is given by

$$\bar{\gamma}(s) = (M(s)\cos\alpha, M(s)\sin\alpha, N(s)\cos\alpha - P(s)\sin\alpha, P(s)\cos\alpha + N(s)\sin\alpha),$$

where $M(s) = \sqrt{1+2A(s)}/\sqrt{2}$, $N(s) = \sqrt{2}B(s)/\sqrt{1+2A(s)}$, $P(s) = \sqrt{2}C(s)/\sqrt{1+2A(s)}$, and

$$\alpha(s) = 2\int \frac{C(s)B'(s) - B(s)C'(s)}{1+2A(s)}ds.$$

Fourth step. The natural parametrization of the Hopf tube $T_\gamma = \pi^{-1}(\gamma)$ is just given by

$$\Psi(s,t) = e^{it}\bar{\gamma}(s) = \cos t\bar{\gamma}(s) + i\ \sin t\bar{\gamma}(s),$$

where $\bar{\gamma}(s)$ stands for a fixed horizontal lift of γ. Notice that in the last formula the quaternions are identified with $\mathbb{C}^2$ so as $i = \sqrt{-1}$.

An application of the algorithm can be seen in the following example.

Example. *A rectangular torus as a Hopf surface*.

A rectangular torus in $(\mathbb{S}^3, \bar{g}_u)$ is associated with a small circle $\gamma(s) = (a, 0, r\ \cos\frac{r}{s}, r\ \sin\frac{s}{r})$ of radius r in $\mathbb{S}^2$, a being a suitable constant. We now apply the first three steps to obtain $M(s) = \sqrt{2(1+2a)}/2$, $N(s) = (\sqrt{2}r/\sqrt{1+2a})\cos s/r$, $P(s) = (\sqrt{2}r/\sqrt{1+2a})\sin s/r$, and $\alpha(s) = -(2r/1+2a)s$.

The fourth step yields to the parametrization of the rectangular torus considered as a Hopf tube given by

$$\begin{aligned}\Psi(s,t) \ = \ & \left(\sqrt{\frac{1+2a}{2}}\ \cos(t - \frac{2r}{1+2a}s), \sqrt{\frac{1+2a}{2}}\ \sin(t - \frac{2r}{1+2a}s),\right.\\ & \left.\sqrt{\frac{2}{1+2a}}\ r\cos(t + \frac{1+2a}{2r}s), \sqrt{\frac{2}{1+2a}}\ r\sin(t + \frac{1+2a}{2r}s\right).\end{aligned}$$

As a consequence of our method the parametrization $\Psi(s,t)$ gives a congruence solution of the Betchov-Da Rios soliton equation in $(\mathbb{S}^3, \bar{g}_u)$ for any positive smooth function u in $\mathbb{S}^2$, which is the constant $\sqrt{1-4r^2}/r$ along γ. Notice that $r < 1/2$ because $\mathbb{S}^2$ is of radius 1/2.

3.4. Metrics which admit a foliation with leaves being solutions of LIE

In this section we are going to show that any principal $\mathbb{S}^1$-bundle over a surface of revolution (in $\mathbb{R}^3$ or $\mathbb{L}^3$) admits a pseudo-Riemannian metric which is foliated and whose leaves are solutions of LIE. To do that, it is enough to find vector fields, say V, in any surface of revolution having neither zeroes nor inflexion points. We sketch the argument for a surface of revolution $S \subset \mathbb{R}^3$ parametrized by $X(s,u) = (f(s)\cos u, f(s)\sin u, g(s))$ (a similar argument holds in $\mathbb{L}^3$). The profile curve is assumed to be arc-length parametrized, so that $f(s) > 0$ anywhere and $(f')^2 + (g')^2 = 1$. A trivial case occurs provided that the parallels of S never are geodesics (for example a bugle surface), since in this case we choose V generating the parallel flow. Otherwise, we locally deformate this flow around any critical point of f. Therefore, writes $V(s) = a(s)X_s + b(s)X_u$, with $a^2 + f^2b^2 = 1$. Suppose $f'(s_0) = 0$ is a local minimum for f (other possibilities admit similar computations), so that there exists $\varepsilon > 0$ such that $f(s) > f(s_0)$ for $|s - s_0| < \varepsilon$. We set $a(s) = \cos\phi(s)$ and $f(s)b(s) = \sin\phi(s)$, then

$$D_V V = \frac{1}{f}(f\sin\phi)'(-\sin\phi X_s + \frac{\cos\phi}{f}X_u),$$

where D denotes the Levi-Civita connection on the surface. Now we choose ϕ to be an increasing differentiable function on $(s_0-\varepsilon, s_0+\varepsilon)$ satisfying

(1) $\phi(s) = -\frac{\pi}{2}$ if $s \leqslant s_0 - \varepsilon$,

(2) $\phi(s_0) = 0$,

(3) $\phi(s) = \frac{\pi}{2}$ if $s \geqslant s_0 + \varepsilon$.

To illustrate this idea we exhibit another algorithm to get a Betchov-Da Rios foliation over a principal $\mathbb{S}^1$-bundle endowed with a principal flat connection on a surface of revolution S (see [1] for details). For the sake of simplicity we assume that the profile curve is not closed (for example a catenary), in this case the fundamental group of S if free abelian with one generator.

First step. Let $\tilde{S}$ be the universal covering of S, so it is diffeomorphic to $\mathbb{R}^2$. The group $\mathbb{Z}$ works as structure group of the principal fibre bundle $\pi_0 : \tilde{S} \to S$ and admits a trivial flat connection, say ω_0.

Second step. Let η be a real number such $\eta/\pi \notin \mathbb{Q}$ (the set of rational numbers). The map $\phi_\eta : \mathbb{Z} \to \mathbb{S}^1$ given by $\phi_\eta(k) = e^{ik\eta}$ defines a monomorphism between $(\mathbb{Z}, +)$ and $\mathbb{S}^1 \subset \mathbb{C}$ regarded as a multiplicative group.

Third step. The transition functions of $\pi_0 : \tilde{S} \to S$ can be extended, via ϕ_η, to obtain $\mathbb{S}^1$-functions which can be chosen as transition functions of a principal $\mathbb{S}^1$-bundle $\pi : P \to S$. Furthermore, ϕ_η can be extended to a monomorphism from $\tilde{S}$ to P which maps ω_0 in a flat connection, say ω, on P.

Forth step. We choose, as above, a vector field V on S having neither zeroes nor inflexion points. Let $\{\gamma_s : s \in I\}$ be the flow of V, I being the domain of the profile curve of S. We define a positive smooth function u on S by $u(p) = \kappa(p)$, where κ is the curvature of the V-integral curve through p. Then $\{\pi^{-1}(\gamma_s) : s \in I\}$ defines a Betchov-Da Rios foliation on $(P, \bar{g}_u)$, where g is the metric of S.

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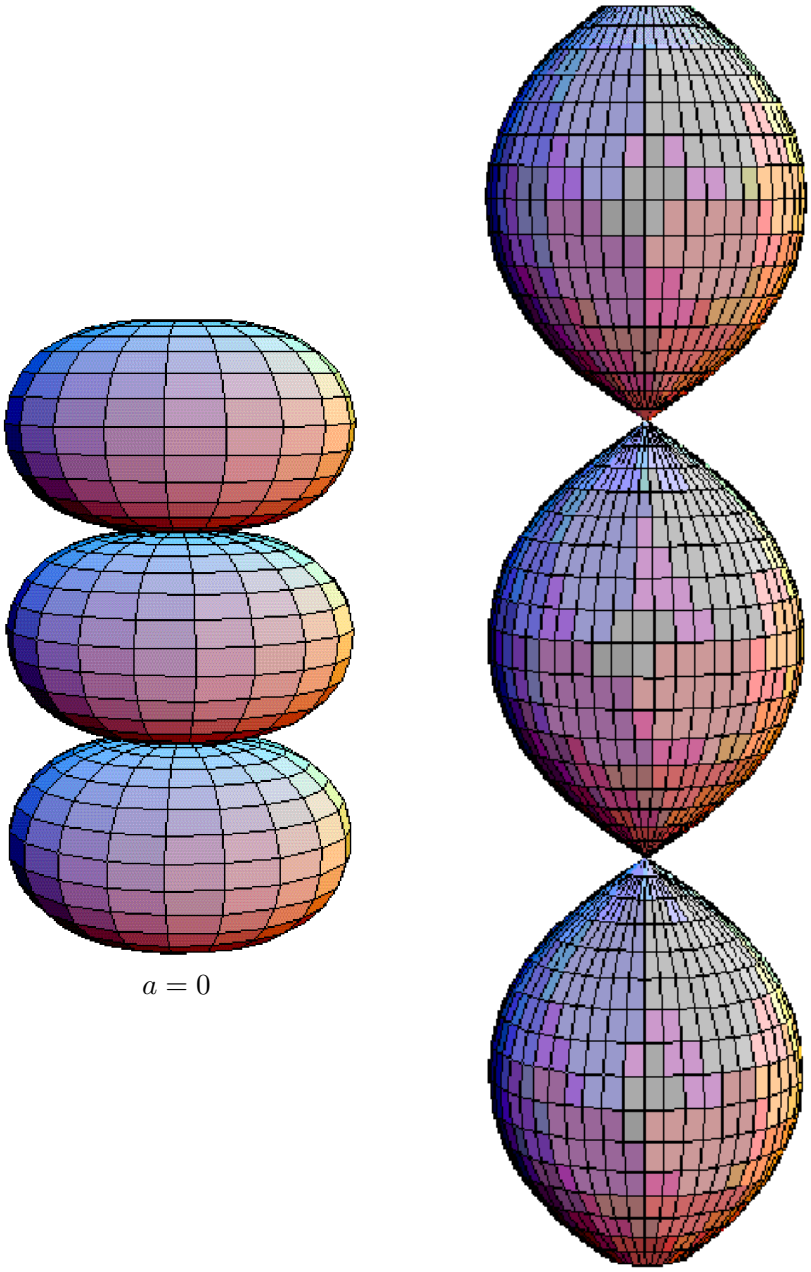
Manuel Barros: Departamento de Geometría y Topología, Universidad de Granada, 18071 Granada, Spain. e-mail: mbarros@ugr.es

Angel Ferrández: Departamento de Matemáticas, Universidad de Murcia, 30100 Espinardo, Murcia, Spain. e-mail: aferr@fcu.um.es

Pascual Lucas: Departamento de Matemáticas, Universidad de Murcia, 30100 Espinardo, Murcia, Spain. e-mail: plucas@fcu.um.es



$a = 0$ $a = 1$

Figure 7: Revolution surfaces in $\mathbb{R}^3$.