General helices in the 3-dimensional Lorentzian space forms

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Rocky Mountain J. Math. 31 (2001), 373–388

(Partially supported by DGICYT grant PB97-0784 and Fundación Séneca PB/5/FS/97)

Abstract

We present Lancret-type theorems for general helices in the 3-dimensional Lorentzian space forms. We show outstanding and deep differences with regard to non flat Riemannian space forms as well as the classical Euclidean case. These will be pointed out when studying the problem of solving natural equations. Indeed we give a geometric approach to this problem and show that, for instance, general helices in the 3-dimensional Lorentz-Minkowski space correspond with geodesics either of right general cylinders or of flat B-scrolls. In this sense, the anti De Sitter and De Sitter worlds behave as the spherical and hyperbolic space forms, respectively.

Keywords: General helix, Lorentzian space form, Killing fields, solving natural equations, closed curves

MS classification: 53C50, 53A35

1. Introduction

A curve of constant slope or general helix in Euclidean space $\mathbb{R}^3$ is defined by the property that its tangent indicatrix is a planar curve. The straight line perpendicular to this plane is called the axis of the general helix. A classical result stated by M.A. Lancret in 1802 and first proved by B. de Saint Venant in 1845 (see [**11**] for details) is: "A necessary and sufficient condition in order to a curve be a general helix is that the ratio of curvature to torsion be constant".

For a given couple of one variable functions (eventually curvature and torsion parametrized by arclength) one might like to get an arclength parametrized curve for which the couple works as the curvature and torsion functions. This problem is usually referred as "the solving natural equations problem". The natural equations for general helices can be integrated, not only in $\mathbb{R}^3$, but also in the 3-sphere $\mathbb{S}^3$ (the hyperbolic space is poor in this kind of curves and only helices are general helices). Indeed one uses the fact that general helices are geodesics either of right general cylinders or of Hopf cylinders, according to the curve lies in $\mathbb{R}^3$ or $\mathbb{S}^3$, respectively (see [**3**] for further details).

In this note we deal with general helices in the 3-dimensional Lorentzian space forms. A non-null curve γ immersed in $\mathbb{L}^3$ is called a general helix if its tangent indicatrix is contained in some plane, say π, of $\mathbb{L}^3$. Since π can be either degenerate or non-degenerate, then both cases are distinguished by calling degenerate and non-degenerate general helices, respectively. Then we give a sort of Lancret theorem for general helices in $\mathbb{L}^3$ which formally agrees with the classical one. In fact we prove that "general helices in $\mathbb{L}^3$ correspond with non-null curves in $\mathbb{L}^3$ for which the ratio of curvature and torsion is constant".

In spite of this, we will point out a remarquable and deep difference between the behaviour of general helices in Euclidean and Lorentzian geometries. While in $\mathbb{R}^3$ general helices are geodesics in right general cylinders, as classically is shown, we will prove that general helices in $\mathbb{L}^3$ are geodesics in either right general cylinders or flat B-scrolls, according to the general helix is non-degenerate or degenerate (see Theorems 4.2 and 4.3), respectively. This nice difference between Euclidean and Lorentzian geometries (from the point of view of the behaviour of general helices) confirms once more the important role of the notion of B-scroll (see [**5**], [**1**], [**2**]) in Lorentzian geometries.

To extend the concept of general helix to 3-dimensional De Sitter space $\mathbb{S}^3_1$ and anti De Sitter space $\mathbb{H}^3_1$, we use the concept of Killing vector field along a curve in a 3-dimensional real space form, first introduced in [**8**]. The Lancret theorem in $\mathbb{S}^3_1$ and $\mathbb{H}^3_1$ underlines deep differences between the pseudospherical and pseudohyperbolic spaces. The pseudohyperbolic case is nicely analogous to the Lorentz-Minkowskian case, whereas in the pseudospherical case there are no nontrivial general helices. From this point of view, the roles played by the non flat Lorentzian space forms $\mathbb{H}^3_1$ and $\mathbb{S}^3_1$ correspond with those played by the non flat Riemannian space forms $\mathbb{S}^3$ and $\mathbb{H}^3$, respectively (see [**3**]).

To point out the interest of general helices it should be mentioned that they arise in the context of the interplay between geometry and integrable Hamiltonian systems (see [**7**] and [**9**]). In fact, the Betchov-Da Rios equation, also known as the localized induction equation (LIE), $\frac{\partial\gamma}{\partial t} = \frac{\partial\gamma}{\partial s} \times \frac{\partial^2\gamma}{\partial s^2}$, is a soliton equation for space curves $\gamma(s,t)$. It is a model for the behaviour of thin vortex tubes in an incompressible, inviscid, three-dimensional fluid. When $s \to \gamma(s,t)$ is arclength parametrized, that equation becomes the so-called vortex filament equation $\frac{\partial\gamma}{\partial t} = \kappa B$. If γ evolves according to this equation, then the complex wave function Ψ, given by the Hasimoto transformation (see [**7**] and [**9**], again), evolves according to the cubic nonlinear Schrödinger equation. The filament equation is even more directly related to still another model, the continuum limit of the classical Heisenberg chain, $\frac{\partial T}{\partial t} = T \times \frac{\partial^2 T}{\partial s^2}$, via the tangent indicatrix $\gamma \to T = \frac{\partial\gamma}{\partial s}$. By asuming that γ is an elastica, with Frenet frame T, N and B, Langer and Singer [**8**] have shown that $X = \kappa B$ and $Y = (\kappa^2 - \lambda)T + 2\kappa' N + 2\kappa\tau B$ are Killing fields along γ. The simplest soliton solutions for the flow $X = \frac{\partial\gamma}{\partial t}$ were elastic curves, which evolve by rigid motions. Elastic curves are also simple soliton solutions for the flow $Y = \frac{\partial\gamma}{\partial t}$. If γ evolves according to $Y = \frac{\partial\gamma}{\partial t}$, then Ψ evolves according to the modified Korteweg-de Vries equation (mKdV). In [**2**] we have found parametrized solutions of the LIE in the 3-dimensional anti De Sitter space, so that the soliton solutions are the null geodesics of the Lorentzian Hopf cylinders. Therefore there is a natural geometric evolution on general helices inducing a mKdV curvature evolution equation coming from the LIE. The role of general helices here is probably similar to that of curves of constant torsion or constant natural curvature (see [**7**]).

2. Setup

Let $\mathbb{R}^{n+2}_t$ be the $(n+2)$-dimensional pseudo-Euclidean space with index t endowed with the indefinite inner product given by

$$\langle x, y\rangle = -\sum_{i=1}^{t} x_i y_i + \sum_{j=t+1}^{n+2} x_j y_j,$$

where $(x_1, \dots, x_{n+2})$ is the usual coordinate system. Let $\mathbb{S}^{n+1}_\nu = \{x \in \mathbb{R}^{n+2}_\nu : \langle x, x\rangle = 1\}$ and $\mathbb{H}^{n+1}_\nu = \{x \in \mathbb{R}^{n+2}_{\nu+1} : \langle x, x\rangle = -1\}$ be the unit pseudo-sphere and the unit pseudo-hyperbolic space, respectively. They are pseudo-Riemannian hypersurfaces of index ν in $\mathbb{R}^{n+2}_\nu$ and $\mathbb{R}^{n+2}_{\nu+1}$, respectively, with constant sectional curvature $c = +1$ and $c = -1$, respectively. Throughout this paper, M will denote $\mathbb{S}^3_1$, $\mathbb{H}^3_1$ or $\mathbb{L}^3$ according to $c = +1$, $c = -1$ or $c = 0$, respectively, and E will stand for the pseudo-Euclidean space where M is lying.

Let p be a point in M and $v \in T_pM$ a tangent vector. Then v is said to be spacelike, timelike or null according to $\langle v, v\rangle > 0$, $\langle v, v\rangle < 0$, or $\langle v, v\rangle = 0$ and $v \neq 0$, respectively. Notice that the vector $v = 0$ is spacelike. The category into which a given tangent vector falls is called its causal character. These definitions can be generalized for curves as follows. A curve γ in M is said to be spacelike if all of its velocity vectors γ' are spacelike; similarly for timelike and null.

For a better understanding of the next construction we will bring back the notion of cross product in the tangent space T_pM at any point p in M. There is a natural orientation in T_pM defined as follows: an ordered basis $\{X, Y, Z\}$ in T_pM is positively oriented if det$[pXYZ] > 0$, where $[pXYZ]$ is the matrix with p, X, Y, Z as row vectors. Now let ω be the volumen element on M defined by $\omega(X, Y, Z) = \det[pXYZ]$. Then given X, $Y \in T_pM$, the cross product $X \times Y$ is the unique vector in T_pM such that $\langle X \times Y, Z\rangle = \omega(X, Y, Z)$, for any $Z \in T_pM$.

A non-null curve $\gamma(s)$ in M is said to be a unit speed curve if $\langle \gamma'(s), \gamma'(s)\rangle = \varepsilon$ (ε being $+1$ or -1 according to γ is spacelike or timelike, respectively). A unit speed curve $\gamma(s)$ in M, s being the arclength parameter, is called a Frenet curve if it admits a Frenet frame field $\{T = \gamma', N, B\}$, where $B = T \times N$, satisfying the Frenet equations

$$\begin{aligned}
\overline{\nabla}_T T &= \varepsilon_2 \kappa N, \\
\overline{\nabla}_T N &= -\varepsilon_1 \kappa T - \varepsilon_3 \tau B, \\
\overline{\nabla}_T B &= \varepsilon_2 \tau N,
\end{aligned}$$

where ε_1, ε_2 and ε_3 denote the causal characters of T, N and B, respectively (in particular, $\varepsilon_i = \pm 1$ and $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$), $\overline{\nabla}$ is the semi-Riemannian connection on M and $\kappa = \kappa(s)$ and $\tau = \tau(s)$ are the curvature and the torsion functions of γ, respectively.

The unit tangent vector field $T = \gamma'$ defines a mapping from γ to $Q = \{q \in E : \langle q, q\rangle = \pm 1\}$, which is usually called the *tangent indicatrix* of γ and, from now on, it will be also denoted by T.

Now let $\alpha(s)$ be a null curve in M with Cartan frame $\{A, B, C\}$, i.e., A, B, C are vector fields tangent to M along $\alpha(s)$ satisfying the following conditions:

$$\langle A, A\rangle = \langle B, B\rangle = 0, \quad \langle A, B\rangle = -1,$$

$$\langle A, C\rangle = \langle B, C\rangle = 0, \quad \langle C, C\rangle = 1,$$

and

$$\begin{aligned}
\dot{\alpha} &= A, \\
\dot{A} &= \rho C, \qquad \rho = \rho(s) \neq 0, \\
\dot{B} &= c\alpha + w_0 C, \qquad w_0 \text{ being a constant}, \\
\dot{C} &= w_0 A + \rho B.
\end{aligned}$$

If we consider the immersion $X : (s, t) \to \alpha(s) + tB(s)$, then X defines a Lorentz surface, with constant Gaussian curvature $c + w_0^2$, that L.K. Graves [**6**] called a *B-scroll*. An easy computation shows that the unit normal vector is given, up to the sign, by $\xi(s, t) = w_0 t B(s) + C(s)$.

3. Killing fields

This section is taken from [**8**]. Let $\gamma(t)$ be a non-null immersed curve in a 3-dimensional Lorentzian space form M with sectional curvature c and let $v(t) = |\gamma'(t)|$ be the speed of γ. Let us consider a variation of γ, $\Gamma = \Gamma(t,z) : I \times (-\varepsilon, \varepsilon) \to M$ with $\Gamma(t,0) = \gamma(t)$. In particular one can choose $\varepsilon > 0$ in such a way that all t-curves of the variation have the same causal character as that of γ. Associated with Γ there are two vector fields along Γ, $V(t,z) = \frac{\partial \Gamma}{\partial z}(t,z)$ and $W(t,z) = \frac{\partial \Gamma}{\partial t}(t,z)$. In particular $V(t) = V(t,0)$ is the variational vector field along γ and $W(t,z)$ is the tangent vector fields of the t-curves. We will use the notation $V = V(t,z)$, $v = v(t,z)$, $\kappa = \kappa(t,z)$, etc. with the obvious meanings. Also, if s denotes the arclength parameter of the t-curves, we will write $v(s,z)$, $V(s,z)$, $\kappa(s,z)$, etc. for the corresponding reparametrizations.

A straightforward but long computation allows us to obtain formulas for $\frac{\partial v}{\partial z}(t,0)$, $\frac{\partial \kappa^2}{\partial z}(t,0)$ and $\frac{\partial \tau^2}{\partial z}(t,0)$ which we collect, along with another standard identity, in the following lemma.

Lemma 3.1 *With the above notation, the following assertions hold:*

(1) $[V,W] = 0$;

(2) $\dfrac{\partial v}{\partial z}(t,0) = -\varepsilon_1 g v$, *with* $g = \langle \overline{\nabla}_T V, T\rangle$;

(3) $\dfrac{\partial \kappa^2}{\partial z}(t,0) = 2\varepsilon_2 \langle \overline{\nabla}^2_T V, \overline{\nabla}_T T\rangle + 4\varepsilon_1 g\kappa^2 + 2\varepsilon_2 \langle R(V,T)T, \overline{\nabla}_T T\rangle$;

(4) $\dfrac{\partial \tau^2}{\partial z}(t,0) = -2\varepsilon_2 \langle \dfrac{1}{\kappa}\overline{\nabla}^3_T V - \dfrac{\kappa'}{\kappa^2}\overline{\nabla}^2_T V + \varepsilon_1(\varepsilon_2\kappa + \dfrac{c}{\kappa})\overline{\nabla}_T V - \varepsilon_1 \dfrac{c\kappa'}{\kappa^2} V, \tau B\rangle$,

where $\langle , \rangle$ *denotes the Lorentzian metric of* M *and* $\kappa' = \dfrac{\partial \kappa}{\partial t}(t,0)$.

Without loss of generality we can assume γ to be arclength parametrized. A vector field $V(s)$ along γ, which infinitesimally preserves unit speed parametrization (that means $\frac{\partial v}{\partial z}(t,0) = 0$ for a V-variation of γ) is said to be a Killing vector field along γ if this evolves in the direction of V whithout changing shape, only position. In other words, the curvature and torsion functions of γ remain unchanged as the curve evolves. Hence Killing vector fields along γ are characterized by the equations

$$\frac{\partial v}{\partial z}(t,0) = \frac{\partial \kappa^2}{\partial z}(t,0) = \frac{\partial \tau^2}{\partial z}(t,0) = 0, \tag{1}$$

and this is well defined in the sense that it does not depend on the V-variation of γ one chooses to compute the derivatives involved in equation (1). In fact, we use Lemma 3.1 and (1) to see that V is a Killing vector field along γ if and only if it satisfies the following conditions:

$$\begin{aligned}
&\text{a)} \quad \langle \overline{\nabla}_T V, T\rangle = 0,\\
&\text{b)} \quad \langle \overline{\nabla}^2_T V, N\rangle + \varepsilon_1 c\langle V, N\rangle = 0,\\
&\text{c)} \quad \langle \frac{1}{\kappa}\overline{\nabla}^3_T V - \frac{\kappa'}{\kappa^2}\overline{\nabla}^2_T V + \varepsilon_1(\varepsilon_2\kappa + \frac{c}{\kappa})\overline{\nabla}_T V - \varepsilon_1 c\frac{\kappa'}{\kappa^2} V, \tau B\rangle = 0.
\end{aligned} \tag{2}$$

In particular, the solutions of (2) constitute a 6-dimensional linear space.

Now when M is simply connected, since the restriction to γ of any Killing field $\widetilde{V}$ of M is a Killing vector field along γ, one concludes from a well known dimension argument, the following lemma.

Lemma 3.2 *Let M be a complete, simply connected, Lorentzian space form and γ a non-null immersed curve in M. A vector field V on γ is a Killing vector field along γ if and only if it extends to a Killing field $\widetilde{V}$ on M.*

4. General helices in the 3-dimensional Lorentz-Minkowski space

Following the classical terminology of the Euclidean geometry (see, for instance, [**10**]) we will say that γ is a *general helix* in $\mathbb{L}^3$ if its tangent indicatrix lies in a plane of $\mathbb{L}^3$. That means that there exists a vector $v \neq 0$ in $\mathbb{L}^3$ which is orthogonal to the acceleration vector field of γ. The straight line generated by v is uniquely determined and we will refer to it as the *axis* of γ. In particular, we will say that a general helix is *degenerate* or *non-degenerate* according to its axis is null or non-null, respectively.

It is obvious that non-null curves in $\mathbb{L}^3$ with zero torsion are examples of non-degenerate general helices. In fact, such a curve lies in a non-degenerate 2-plane in $\mathbb{L}^3$ and a unit vector in $\mathbb{L}^3$ orthogonal to this plane works as the axis of the general helix.

Now given a general helix γ in $\mathbb{L}^3$ with axis v, we can define a translation vector field $\widetilde{V}$ in $\mathbb{L}^3$ by $\widetilde{V} = v$, for any $p \in \mathbb{L}^3$. Let V be $\widetilde{V}$ restricted to γ. Then V defines a Killing vector field along γ with constant length, i.e., $\langle V, V\rangle$ is constant, and orthogonal to the acceleration vector field of γ.

Assume now that W is a Killing vector field along a non-null curve γ, with constant length and orthogonal to its normal vector field N. From (2a) we can write $W = aT + bB$, a and b being constants. Now use (2b) to get $\overline{\nabla}_T W = \lambda N$, where $\lambda = \varepsilon_2(a\kappa + b\tau)$ is constant. Finally, equation (2c) yields $\lambda\tau(\tau/k)' = 0$. From here we consider the following cases.

(i) $\tau \equiv 0$. Then γ is a a non-degenerate general helix. It is not difficult to see that $W = B$ (unless γ is a circle which will be considered next) and so it extends to a translation vector field $\widetilde{W}$ in $\mathbb{L}^3$.

(ii) *k and τ both are constant*. Then γ is a helix. Now the Killing vector field W is not uniquely determined. In fact, for any couple of contants a and b, in the rectifying plane we define the vector field $W(s) = aT + bB$, which works as a Killing vector field along γ. In spite of that, we can determine a Killing vector field along γ, say $V(s)$, being parallel along γ and thus it extends to a translation vector field $\widetilde{V}(s)$ such that $\widetilde{V}(s) = v \in \mathbb{L}^3$. Indeed just choose a and b such that $a\kappa + b\tau = 0$. Therefore γ is a non-degenerate general helix, unless $\varepsilon_2 = 1$ (which means that N is spacelike or the rectifying plane is Lorentzian anywhere) and $\tau = \pm\kappa$, and then γ is degenerate.

(iii) $\lambda = 0$. Then W is a uniquely determined Killing vector field along γ. Furthermore, it is parallel along γ and extends to a translation vector field $\widetilde{W}$ such that $\widetilde{W} = v \in \mathbb{L}^3$. Therefore, γ is a general helix whose axis is v (or W). And γ is degenerate when W is null, which yields $\varepsilon_2 = 1$ and $\tau = \pm\kappa$.

We will refer to curves in the first two classes as trivial general helices.

Summarizing we have the following.

Theorem 4.1 (The Lancret theorem in $\mathbb{L}^3$) *Let γ be a non-null immersed curve in $\mathbb{L}^3$ with curvature and torsion functions κ and τ, respectively. Then the following statements are equivalent:*

(a) *γ is a general helix in $\mathbb{L}^3$;*

(b) *There exists a constant length Killing vector field V along γ which is orthogonal to the acceleration vector field of γ;*

(c) *There exists a constant r such that $\tau = r\kappa$.*

Moreover a general helix γ is degenerate if and only if $r = \pm 1$ and its normal vector field is spacelike. The Killing vector field V in (b) is not uniquely determined if γ is a helix (κ and τ both are constant); however, in this case, V can be uniquely determined, up to constants, once it is chosen parallel along γ (say otherwise, its extended Killing vector field in $\mathbb{L}^3$ is a translation vector field).

Theorem 4.2 (Solving natural equation for non-degenerate general helices.) *Let β be a non-null immersed curve in $\mathbb{L}^3$. Then β is a non-degenerate general helix if and only if it is a geodesic in some right cylinder whose directrix and generatrix are both non-null.*

Proof. Let v be a unit vector in $\mathbb{L}^3$ and α a unit speed curve in a plane orthogonal to v. The Frenet equations of α are

$$\begin{aligned} \overline{\nabla}_{\overline{T}}\overline{T} &= \delta_2\overline{\kappa}\overline{N}, \\ \overline{\nabla}_{\overline{T}}\overline{N} &= -\delta_1\overline{\kappa}\overline{T}, \end{aligned} \tag{1}$$

where $\{\overline{T}, \overline{N}\}$ is the Frenet frame along α, $\overline{\kappa}$ its curvature function and δ_1, δ_2 the causal characters of $\overline{T}$ and $\overline{N}$, respectively. Notice that the causal character of v is $-\delta_1\delta_2$.

Let us consider the right cylinder $C_{\alpha,v}$ in $\mathbb{L}^3$ generated by α and v, which is naturally parametrized as $X(s,t) = \alpha(s) + tv$. It is well known that the geodesics of $C_{\alpha,v}$ are the images under X of straight lines in the (s,t)-plane. Choose such a geodesic $\gamma(s) = \alpha(s) + msv$, where m is a certain constant. Then the translation field $\widetilde{V}$ in $\mathbb{L}^3$ determined by v induces a Killing vector field along γ with constant length and orthogonal to the acceleration vector $\gamma''(s) = \alpha''(s)$. Since v is non-null, Theorem 4.1 implies that γ is a non-degenerate general helix.

Conversely, suppose β is a non-degenerate general helix. Then there exists a certain constant r such that $\tau = r\kappa$ (of course κ and τ denote the curvature and torsion functions of β). One can also choose a unit vector, say v, lying on the axis of β. We take a (non-degenerate) plane P in $\mathbb{L}^3$ which is orthogonal to v. Up to congruences in P, there exists a unique curve in P, say α, with curvature function $\overline{\kappa} = |\beta'|\kappa$ and $\delta_2 = \varepsilon_2$ (notice that the causal character δ_1 of α is determined by δ_2 and the causal character of v). Let $C_{\alpha,v}$ be the right cylinder generated by α and v. Then it is parametrized by $X(s,t) = \alpha(s) + tv$. Finally we choose the geodesic of $C_{\alpha,v}$ defined by $\gamma(s) = \alpha(s) + msv$, where $m = \delta_1\varepsilon_3 r$. Then γ is a non-null geodesic, because $\delta_2 = -1$ provided that $m^2 = 1$. Finally, it is easy to see that γ and β have the same curvature and torsion functions, as well as the same causal characters. This concludes the proof.

Theorem 4.3 (Solving natural equation for degenerate general helices.) *Let β be a non-null immersed curve in $\mathbb{L}^3$. Then β is a degenerate general helix if and only if it is a geodesic in some flat B-scroll in $\mathbb{L}^3$.*

Proof. Let $\alpha(s)$ be a null curve in $\mathbb{L}^3$ with Cartan frame $\{A, B, C\}$ and $S_{\alpha,B}$ the flat B-scroll (i.e., $w_0 = 0$) parametrized by $X(s,t) = \alpha(s) + tB$. We choose a non-null geodesic of $S_{\alpha,B}$, say $\gamma(u) = \alpha(s(u)) + t(u)B(s(u))$. Then the translation field $\widetilde{B}$ in $\mathbb{L}^3$ determined by B induces a Killing vector field along γ, also denoted by B, with constant length and such that $\langle\gamma'(u), B\rangle = -s'(u)$ is constant, because the geodesic γ is the image under X of a straight line. Therefore, from Theorem 4.1, γ is a degenerate general helix in $\mathbb{L}^3$.

To prove the converse, let β be a degenerate general helix that we parametrize with constant speed, say $\langle\beta', \beta'\rangle = p$ constant. From the theorem of Lancret we know that the curvature τ and torsion κ functions of β agree (we can change orientation if necessary) and the acceleration vector field of β is spacelike, i.e., $\varepsilon_2 = 1$. We define the following vector fields

$$\begin{aligned} A &= \frac{|\beta'|}{2}(T+B), \\ B &= -\frac{\varepsilon_1}{|\beta'|}(T-B), \\ C &= N, \end{aligned}$$

where $\{T, N, B\}$ is the Frenet frame along β, $|\beta'| = \sqrt{\varepsilon_1 p}$ and ε_1 denotes, as usual, the causal character of β.

Let α be a curve in $\mathbb{L}^3$ with tangent vector field A, then α is a null curve in $\mathbb{L}^3$. Furthermore, $\{A, B, C\}$ is a Cartan frame along α with $w_0 = 0$ and $\rho = \kappa|\beta'|$ (see Section 2). Let $S_{\alpha,B}$ be the corresponding flat B-scroll, which is parametrized by $X(s,t) = \alpha(s) + tB$. Finally, choose the geodesic in $S_{\alpha,B}$ given by $\gamma(s) = \alpha(s) + msB$, where $m = -p/2$. It is not difficult to see that γ and β have the same curvature and torsion functions, and also the same causal character, showing that they are congruent in $\mathbb{L}^3$.

5. General helices in non-flat 3-dimensional Lorentzian space forms

In order to generalize the notion of general helix to 3-dimensional Lorentzian spaces of nonzero constant curvature, we profit by Theorem 4.1. A curve γ in M is said to be a general helix if there exists a Killing vector field V along γ with constant length and orthogonal to the acceleration vector field of γ. We will say that V is an axis of the general helix γ. Obvious examples of general helices in M are the following. Curves with torsion vanishing anywhere, where the unit binormal works as an axis. Helices are also general helices, where any vector field chosen in the rectifying plane having constant coordinates relative to T and B runs as an axis.

We can follow notation and terminology used in $\mathbb{L}^3$ to say that zero torsion curves are non-degenerate general helices, because the axis B is obviously non-null. As for curves with both constant curvature and torsion we know that for any pair of constants a and b the vector field along γ given by $V(s) = aT + bB$ is always a Killing vector field. Of course, when $\varepsilon_2 = -1$, i.e., the rectifying plane is positive definite at any point, all Killing vector fields $V(s)$ are non-null and we will say that the general helix is non-degenerate. However, if $\varepsilon_2 = 1$, i.e., the rectifying plane is Lorentzian, we have Killing vector fields along γ being either spacelike, or timelike, or null. It does not allow us to decide if such a general helix is degenerate or not. However, we can determine a unique Killing vector field along the helix by forcing it to be parallel along γ. The helix is said to be *degenerate* or *non-degenerate* according to V is null or non-null, respectively.

Let $\gamma(s)$ be a general helix in M with curvature $\kappa > 0$. Let $V(s)$ be an axis and assume, without loss of generality, that $\langle V, V\rangle = \varepsilon$, where $\varepsilon = -1, 0, 1$. From equation (2a) we deduce that

$$V(s) = fT(s) + hB(s), \quad \text{and} \quad \varepsilon = \varepsilon_1 f^2 + \varepsilon_3 h^2, \tag{1}$$

for certain constants f and h. By using the Frenet equations of γ we get

$$\overline{\nabla}_T V = \varepsilon_2(f\kappa + h\tau)N,$$

and

$$\overline{\nabla}_T^2 V = -\varepsilon_1\varepsilon_2\kappa(f\kappa + h\tau)T + \varepsilon_2(f\kappa' + h\tau')N - \varepsilon_2\varepsilon_3\tau(f\kappa + h\tau)B.$$

Now from equations (2b), (1) and (3) we deduce that $f\kappa' + h\tau' = 0$ from which we get

$$\tau = b\kappa + a,$$

for certain constants a and b. On the other hand, from (3), jointly with the Frenet equations of γ, we obtain

$$\overline{\nabla}^3_T V = -\varepsilon_1\varepsilon_2\lambda\kappa' T - \lambda(\varepsilon_1\kappa^2 + \varepsilon_3\tau^2)N - \varepsilon_2\varepsilon_3\lambda\tau' B,$$

where λ stands for the constant $f\kappa + h\tau$. Now equation (2c), jointly with equations (1)–(5), yields

$$\tau(\lambda\tau'\kappa - \lambda\kappa'\tau - ch\kappa') = 0,$$

and then

$$h\kappa'\tau(a^2 + c) = 0.$$

In particular, the above equation shows that in the De Sitter space $\mathbb{S}^3_1$, ($c = +1$), the only general helices are the two classes described just before this discussion. So we have prove the following result.

Theorem 5.1 (The Lancret theorem in the De Sitter space) *A non-null immersed curve γ in $\mathbb{S}^3_1$ is a general helix if and only if either*
(1) $\tau \equiv 0$ *and γ is a curve in some totally geodesic surface of $\mathbb{S}^3_1$; or*
(2) γ *is a helix in $\mathbb{S}^3_1$ (i.e. curvature κ and torsion τ constants).*

Furthermore, general helices of the first type have only one axis (the binormal) which is parallel and so they are non-degenerate. In contrast, general helices of the second type have a plane (the rectifying plane) of axes. However they only have a parallel axis. This axis is null, and so the general helix is degenerate, if and only if $\varepsilon_2 = +1$ (the normal vector is spacelike) and $\tau = \pm\kappa$; otherwise the helix is non-degenerate.

In the anti De Sitter space, besides the two classes of trivial general helices, we have another class. This kind of general helices can be characterized from equations (5) and (6), where $c = -1$, as the curves in $\mathbb{H}^3_1$ whose curvature and torsion are related by

$$\tau = b\kappa \pm 1,$$

for a certain constant b. These general helices admit only one axis $V = fT + hB$, which is defined by

$$\frac{f}{h} = -b = \frac{1-\tau}{\kappa}.$$

The causal character of this axis is

$$\varepsilon = h^2\left(\varepsilon_1\frac{(\tau-1)^2}{\kappa^2} + \varepsilon_3\right).$$

In particular, a general helix of this type is degenerate if and only if $\varepsilon_2 = 1$ and $b = \pm 1$.

Summarizing we have shown the following.

Theorem 5.2 (The Lancret theorem in the anti De Sitter space) *A non-null immersed curve γ in $\mathbb{H}^3_1$ is a general helix if and only if either*

(1) $\tau \equiv 0$ *and γ is a curve in some totally geodesic surface of $\mathbb{H}^3_1$. The curve admits only one axis which agrees with its binormal, being parallel along the curve and non-null. The general helix is non-degenerate; or*

(2) *γ is a helix in $\mathbb{H}^3_1$. It admits a plane (the rectifying plane) of axes but only one is parallel along γ. This parallel axis is null, and so γ is degenerate, if and only if $\varepsilon_2 = +1$ and $\tau = \pm\kappa$. Otherwise γ is non-degenerate; or*

(3) *there exists a certain constant b such that the curvature κ and the torsion τ functions of γ are related by $\tau = b\kappa \pm 1$. The curve admits a unique axis which can not be parallel along γ. It is null, and so γ is degenerate, if and only if $b = \pm 1$ and γ has spacelike normal vector ($\varepsilon_2 = +1$).*

Remark 5.3 *Compare Theorem 5.1 and Theorem 5.2 with Theorems 1 and 3 in [**3**], respectively.*

Now we are going to solve the natural equations for general helices in M.

In [**4**] we have just constructed a new class of submanifolds in $\mathbb{H}^3_1(-1)$ defined by means of two semi-Riemannian submersions $\pi_s : \mathbb{H}^3_1(-1) \to \mathbb{H}^2_s(-4)$, $s = 0, 1$ (see details therein). By pulling back via π_s a non-null curve γ in $\mathbb{H}^2_s(-4)$ we get the total horizontal lift of γ, which is an immersed flat surface M_γ in $\mathbb{H}^3_1(-1)$, that will be called the *semi-Riemannian Hopf cylinder associated to* γ. Notice that if $s = 0$, M_γ is a Lorentzian surface, whereas if $s = 1$, M_γ is Riemannian or Lorentzian, according to γ be spacelike or timelike, respectively.

Let $\gamma : I \to \mathbb{H}^2_s(-4)$ be a unit speed curve with Frenet frame $\{\overline{T}, \overline{N}\}$ and curvature function $\overline{\kappa}$. Let $\overline{\gamma}$ be a horizontal lift of γ to $\mathbb{H}^3_1(-1)$ with Frenet frame $\{T, N, B\}$, curvature $\kappa = \overline{\kappa} \circ \pi_s$ and torsion $\tau = 1$. Recall that B is nothing but the unit tangent vector field to the fibers along $\overline{\gamma}$. Then the Hopf Cylinder M_γ can be orthogonally parametrized by

$$X(t,z) = \begin{cases} \cos(z)\overline{\gamma}(t) + \sin(z)B(t) \text{ when } s = 0, \\ \cosh(z)\overline{\gamma}(t) + \sinh(z)B(t) \text{ when } s = 1. \end{cases}$$

Notice that a unit normal vector field to M_γ into $\mathbb{H}^3_1(-1)$ is obtained from the complete horizontal lift of $\overline{N}$ and it is, of course, N along each horizontal lift of γ. As a consequence we have that M_γ is a flat surface with mean curvature function α is given by $\alpha = \frac{1}{2}\kappa$.

Theorem 5.4 (Solving natural equation for non-degenerate general helices in $\mathbb{H}^3_1(-1)$.) *Let β a non-null immersed curve in $\mathbb{H}^3_1$. Then β is a non-degenerate general helix if and only if it is a geodesic in some Hopf cylinder M_γ.*

Proof. Let $\beta(s)$ be an arclength parametrized geodesic in M_γ, then there exists two constants a and b such that

$$T(s) = \beta'(s) = aX_t + bX_z,$$

with $\varepsilon_1 a^2 + \varepsilon_3 b^2 = \delta_1$, δ_1 being the causal character of β. A direct computation shows that the curvature ρ and the torsion τ of β satisfy

$$\begin{aligned} \rho &= \varepsilon_2 a^2 \kappa + 2ab, \\ \tau^2 &= \varepsilon_2 \rho^2 - \varepsilon_1 \delta_1 \kappa \rho + 1. \end{aligned}$$

It is not difficult to see that $\tau = r\rho \pm 1$, $r = b/a$, showing that β is a general helix. Moreover, if the normal vector N is spacelike, then $r \neq 1$ and then β is non-degenerate.

To prove the converse, let β be a non-degenerate general helix in $\mathbb{H}^3_1(-1)$ with curvature ρ and torsion τ. Then there exists a constant r (with $r \neq \pm 1$ if the normal vector to β is spacelike) such that $\tau = r\rho \pm 1$. We choose $\varepsilon_1 = \pm 1$ and s in $\{0, 1\}$ in order to $\delta_1(\varepsilon_1 - (-1)^s r^2)$ be positive, δ_1 being the causal character of β. Let γ be the unique curve, up to motions, in $\mathbb{H}^2_1(-4)$ with curvature $\overline{\kappa} = \delta_1((-1)^s - \varepsilon_1 r^2)\rho - 2\varepsilon_1(-1)^s r$ and causal character defined by ε_1. Let α be the geodesic in the Hopf cylinder M_γ given by $\alpha(s) = X(as, bs)$ with

$$a^2 = \frac{\delta_1}{\varepsilon_1 - (-1)^s r^2} \qquad \text{and} \qquad b^2 = r^2 a^2.$$

It is easy to see that β and α have the same curvature and torsion, and also the same causal character, showing that they are congruent.

Theorem 5.5 (Solving natural equation for degenerate general helices in $\mathbb{H}^3_1(-1)$.) *Let β a non-null immersed curve in $\mathbb{H}^3_1$. Then β is a degenerate general helix if and only if it is a geodesic in some flat B-scroll over a null curve.*

Proof. Let $\beta(u)$ be a geodesic of some flat B-scroll $S_{\alpha,B}$ in $\mathbb{H}^3_1(-1)$ (i.e., $w_0 = \pm 1$) parametrized by $\beta(u) = \alpha(s(u)) + t(u)B(s(u))$. Then the normal vector to β in $\mathbb{H}^3_1(-1)$ is given by $N(u) = \beta(u) - \alpha(s(u)) + C(s(u))$. From here we obtain that $\overline{\nabla}_T N = T + s'(u)\rho B$. By using the Frenet equations for β we deduce that the vector $(1 + \varepsilon_1\kappa)T + \varepsilon_3\tau B$ is null, where κ y τ stand for the curvature and torsion of β, respectively. Therefore N is spacelike and $\tau = \pm\varepsilon_1\kappa \pm 1$, which proves that β is a degenerate general helix.

Conversely, let β be a curve in $\mathbb{H}^3_1(-1)$ with curvature κ and torsion τ satisfying that $\tau = \kappa + \varepsilon_1$ and the normal vector of β is spacelike (the other cases are similar). We define the null curve α in $\mathbb{H}^3_1(-1)$ by the equation

$$\alpha(s) = \beta(s) - \frac{1}{2}s(T(s) - B(s)),$$

and the following vector fields

$$\begin{aligned} A(s) &= -\frac{\varepsilon_1}{2}s\beta(s) + \frac{1}{2}(T(s) + \overline{B}(s)) + \frac{\varepsilon_1}{2}sN(s), \\ B(s) &= -\varepsilon_1(T(s) - B(s)), \\ C(s) &= -\frac{1}{2}s(T(s) - B(s)) + N(s). \end{aligned}$$

It is not difficult to see that $\{A, B, C\}$ is a Cartan frame along α with $w_0 = 1$ and $\rho = \tau$. Let $S_{\alpha,B}$ be the B-scroll in $\mathbb{H}^3_1(-1)$ parametrized by $X(s,t) = \alpha(s) + tB(s)$. Then it is clear that $\beta(s) = X(s, -\frac{\varepsilon_1}{2}s)$ and so β is a geodesic of that B-scroll.

Remark 5.6 *It is worth noting that in the "if" part in Theorems 8 and 9 we have only used the existence of an axis, not necessarily parallel. Then if γ is a helix in $\mathbb{H}^3_1(-1)$ with Lorentzian rectifying plane anywhere, it turns out have both null and non-null axes. Therefore γ is a geodesic in a Hopf cylinder as well as in a flat B-scroll over a null curve.*

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