Null helices in Lorentzian space forms

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Abstract

In this paper we introduce a reference along a null curve in an n-dimensional Lorentzian space with the minimun number of curvatures. That reference generalizes the reference of Bonnor for null curves in Minkowski space-time and it is called the Cartan frame of the curve. The associated curvature functions are called the Cartan curvatures of the curve. We characterize the null helices (that is, null curves with constant Cartan curvatures) in n-dimensional Lorentzian space forms and we obtain a complete classification of them in low dimensions.

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1. Introduction

The general theory of curves in a Riemannian manifold M have been developed a long time ago and now we have a deep knowledge of its local geometry as well as its global geometry. When M is a proper semi-Riemannian manifold (that is, the index ν of the metric of M satisfies $1 \leqslant \nu \leqslant \dim(M)-1$) there exist three families of curves (spacelike, timelike, and null or lightlike curves) depending on their causal characters. It is well-known[1] that the study of timelike curves has many analogies and similarities with that of spacelike curves. However, the fact that the induced metric on a null curve is degenerate leads to a much more complicated study and also different from the non-degenerate case, even sometimes the geometry of a lightlike hypersurface in a semi-Euclidean space can be investigated by using the geometry of the hypersurface as a Riemannian hypersurface in a Euclidean space[2].

In the geometry of null curves difficulties arise because the arc length vanishes, so that it is not possible to normalize the tangent vector in the usual way. A method of proceeding is to introduce a new parameter called the pseudo-arc (already used by Vessiot[3]) which normalizes the derivative of the tangent vector. This was the point of view followed by W.B. Bonnor[4] which defined two curvatures K_2 and K_3 in terms of the pseudo-arc and a third curvature K_1 which takes only two values, 0 and 1, according as the null curve is a straight line or otherwise without any points of inflexion (see also the work by M. Castagnino[5]). J.L. Synge[6] follows a different procedure by supposing that in addition to the equation of the null curve $\gamma(w)$, a null vector $\mathbf{p}$ parallel to $\gamma'(w)$ is also given. Then there exists a unique parameter u such that $\mathbf{p} = \gamma'(u)$, and this new parameter allows us to study the geometry of the curve. From a physical point of view, this corresponds to specify both the world-line and the momentum of a photon.

The importance of the study of null curves and its presence in the physic theories is clear from the fact that the classical relativistic string is a surface or world-sheet in Minkowski space which satisfies the Lorentzian analogue of the minimal surface equations[7]. The string equations simplify to the wave equation, plus a couple of extra simple equations, and by solving the 2-dimensional wave equation it turns out that strings are equivalent to pairs of null curves, or a single null curve in the case of an open string (see also [8, 9, 10, 11, 12]). Recently, A. Nersessian and E. Ramos[13] show that there exists a geometrical particle model based enterely on the geometry of the null curves in Minkowskian 4-dimensional spacetime which under quantization yields the wave equations corresponding to massive spinning particles of arbitrary spin. The same authors[14] consider the simplest geometrical particle model associated with null curves in 3-dimensional Lorentz-Minkowski space and show that under quantization the action, which is proportional to the pseudo-arc length, yields the $(2+1)$-dimensional anyonic field equation supplemented with a Majorana-like relation on mass and spin, i.e. $\text{mass} \times \text{spin} = \alpha^2$, with α the coupling constant in front of the action.

Motivated by the growing importance of null curves in mathematical physics, A. Bejancu[15] initiated an ambitious program for the general study of the differential geometry of null curves in Lorentzian manifolds and, more generally, in semi-Riemannian manifolds. From a complementary vector subbundle to the tangent bundle of a null curve, he obtains the Frenet equations (with respect to a general Frenet frame) and proves certain theorems of existence and uniqueness for null curves in Lorentzian manifolds (see also his book[16]).

In this paper we generalize the results of Bonnor in a double sense. First, for a null curve in an n-dimensional Lorentzian space form we introduce a Frenet frame with the minimun number of curvature functions (which we call the Cartan frame), and then we study the null helices in those spaces, that is, null curves with every constant curvatures. Secondly, we find a complete classification of these curves in the Lorentzian space forms of low dimensions: the 5-dimensional Lorentz-Minkowski space $\mathbb{R}^5_1$, the 4-dimensional De Sitter space-time $\mathbb{S}^4_1$ and the 4-dimensional anti-De Sitter space-time $\mathbb{H}^4_1$. The main theorems of this paper (Theorems 4.8, 5.4 and 6.12) state that in $\mathbb{R}^5_1$ there are 3 different families of helices, in $\mathbb{S}^4_1$ there is only one type of helices and, surprisingly, in $\mathbb{H}^4_1$ we can find up to nine distinct types of helices. The importance of helices and generalized helices in physical theories is also well known, as one can see, for example, in the previous work of the authors[17] and references therein.

2. Preliminaries

Let $(\bar{M}, g)$ be a proper $(m+2)$-dimensional semi-Riemannian manifold of index q and let us consider C a smooth curve in $\bar{M}$ locally parametrized by $\gamma : I \subset \mathbb{R} \to \bar{M}$. The curve C is said to be *null or lightlike* if the tangent vector $\gamma'(t)$ to C at any point is a null vector.

The following concepts are taken from [16]. Let TC denote the tangent bundle of C and define, as in the non-generate case, the bundle $TC^\perp$ by

$$TC^\perp = \bigcup_{p \in C} T_pC^\perp; \quad T_pC^\perp = \{\xi_p \in T_pM : g(\xi_p, V_p) = 0\},$$

where V_p is a null vector tangent to C at p. It is well known that $TC^\perp$ is of rank $m+1$. Since V_p is a null vector, it easily follows that TC is a vector subbundle of $TC^\perp$ of rank 1. Then we may consider a complementary vector subbundle $S(TC^\perp)$ to TC in $TC^\perp$ such that

$$TC^\perp = TC \perp S(TC^\perp),$$

where $\perp$ means orthogonal direct sum. It is known that the subbundle $S(TC^{\perp})$, called the *screen vector bundle* of C, is non-degenerate and of dimension m. Note that, in contrast with the non-degenerate case, the tangent bundle is contained in the normal bundle, and the screen bundle is not unique. These two properties leads to a much more difficult and also different geometry of null curves with respect to non-degenerate (spacelike or timelike) curves.

Since $S(TC^{\perp})$ is non-degenerate, we have the decomposition

$$T\bar{M}|_C = S(TC^{\perp}) \perp S(TC^{\perp})^{\perp},$$

where $S(TC^{\perp})^{\perp}$ is the complementary orthogonal vector bundle to $S(TC^{\perp})$ in $T\bar{M}|_C$. The following result is well-known[15].

Theorem 2.1 *Let C be a null curve of a semi-Riemannian manifold $(\bar{M}, g)$ and consider $S(TC^{\perp})$ a screen vector bundle of C. Then there exists a unique vector bundle E over C, of rank 1, such that on each coordinate neighbourhood $U \subset C$ there is a unique section $N \in \Gamma(E|_C)$ satisfying*

$$\langle \gamma'(t), N \rangle = 1$$

and

$$\langle N, N \rangle = \langle N, X \rangle = 0, \quad \textit{for all } X \in \Gamma(S(TC^{\perp})|_U).$$

The above vector bundle E will be denoted by $ntr(C)$ and it is called the *null transversal bundle* of C with respect to $S(TC^{\perp})$. The vector field N is called the *null transversal vector field* of C with respect to $\gamma'(t)$. We define the transversal vector bundle of C, $tr(C)$, as the vector bundle

$$tr(C) = ntr(C) \perp S(TC^{\perp}),$$

and then we have

$$T\bar{M}|_C = TC \oplus tr(C) = (TC \oplus ntr(C)) \perp S(TC^{\perp}),$$

from which the following result easily follows[16].

Proposition 2.2 *Let C be a null curve of a semi-Riemannian manifold $(\bar{M}, g)$ of index q. Then any screen vector bundle of C is semi-Riemannian of index $q-1$. Hence, if M is a Lorentzian manifold, then any screen vector bundle is Riemannian.*

Let M_1^n be an orientable Lorentzian manifold and consider C a null curve locally parametrized by $\gamma : I \subset \mathbb{R} \to M_1^n$. Assume that $\{\gamma', \gamma'', \ldots, \gamma^{(n)}\}$ is a linearly independent family and define $E_i = \text{span}\{\gamma', \gamma'', \ldots, \gamma^{(i)}\}$, $i = 1, \ldots, n$. We denote by $E_i^{\perp}$ the complementary vector bundle to E_i in E_{i+1}, that is, $E_{i+1} = E_i \perp E_i^{\perp}$. Now we are going to construct a Frenet frame on $\bar{M}$ along C.

To do that, let $L \in E_1$, so that $\gamma' = \bar{k}_1 L$, for a certain function $\bar{k}_1$. Since $E_2 = E_1 \oplus \text{span}\{\gamma''\}$, we have

$$\dim E_1^{\perp} = \dim E_2 - \dim E_1 + \dim(\text{Rad}(E_2) \cap E_1) = 1 + d_{21},$$

where we write $d_{ij} = \dim(\text{Rad}(E_i) \cap E_j)$. It is easy to see that $d_{21} = 1$ and then $E_1^{\perp} = E_2$, so that we can choose a unit spacelike vector W_1 satisfying $E_2 = \text{span}\{L, W_1\}$. Now, since $E_3 = E_2 \oplus \text{span}\{\gamma^{(3)}\}$ we obtain that $d_{32} = 0$ and then $E_2^{\perp} = E_1$. Since E_3 is a Lorentzian subspace of E_n, then there exists only one null vector N such that $\langle L, N \rangle = 1$, $\langle W_1, N \rangle = 0$ and

$E_3 = \text{span}\{L, N, W_1\}$. In general, for $i = 2, \ldots, n-3$, we can find orthonormal spacelike vectors $\{W_1, \ldots, W_i\}$ such that $E_{i+2} = \text{span}\{L, N, W_1, \ldots, W_i\}$ and the basis $\{\gamma', \gamma'', \ldots, \gamma^{(i+2)}\}$ and $\{L, N, W_1, \ldots, W_i\}$ have the same orientation. Finally, the vector W_m, $m = n-2$, is chosen in order that the basis $\{L, N, W_1, \ldots, W_m\}$ is positively oriented. The vector bundle $\text{span}\{W_1, \ldots, W_m\}$ is a screen vector bundle of C. An easy computation shows that there exist functions $\{\bar{k}_1, \ldots, \bar{k}_{m+3}\}$ such that the following equations hold (compare with [15])

$$\begin{aligned}
\gamma' &= \bar{k}_1 L, \\
L' &= \bar{k}_2 L + \bar{k}_3 W_1, \\
N' &= -\bar{k}_2 N + \bar{k}_4 W_1 + \bar{k}_5 W_2, \\
W_1' &= -\bar{k}_4 L - \bar{k}_3 N, \\
W_2' &= -\bar{k}_5 L + \bar{k}_6 W_3, \\
&\vdots \\
W_i' &= -\bar{k}_{i+3} W_{i-1} + \bar{k}_{i+4} W_{i+1}, \\
&\vdots \\
W_{m-1}' &= -\bar{k}_{m+2} W_{m-2} + \bar{k}_{m+3} W_m, \\
W_m' &= -\bar{k}_{m+3} W_{m-1}.
\end{aligned} \tag{3}$$

The set $F = \{L, N, W_1, \ldots, W_m\}$ satisfying the above equations is called the *Frenet frame* on $\bar{M}$ along C with respect to the screen vector bundle $\text{span}\{W_i\}$. The functions $\{\bar{k}_1, \ldots, \bar{k}_{m+3}\}$ are called the *curvature functions* of C with respect to F. Those equations are called the *Frenet equations* of C with respect to F.

3. The Cartan frame of a null curve

The goal of this section is to find a Frenet frame with the minimal number of curvatures and such that they are invariant under Lorentzian transformations. Without loss of generality we may assume that γ is parametrized by the pseudo-arc parameter, that is, $\langle \gamma'', \gamma'' \rangle = 1$. Now choose $L = \gamma'$ and $W_1 = \gamma''$, so that $\bar{k}_1 = 1$, $\bar{k}_2 = 0$ and $\bar{k}_3 = 1$.

It is not difficult to show that the null transversal bundle is generated by the section

$$N = -\gamma^{(3)} - \frac{1}{2} \left\langle \gamma^{(3)}, \gamma^{(3)} \right\rangle \gamma',$$

and then the forth curvature is given by

$$\bar{k}_4 = \langle N', W_1 \rangle = \frac{1}{2} \left\langle \gamma^{(3)}, \gamma^{(3)} \right\rangle.$$

From the equation

$$\bar{k}_5 W_2 = N' - \bar{k}_4 W_1 = -\gamma^{(4)} - \left\langle \gamma^{(4)}, \gamma^{(3)} \right\rangle \gamma' - \left\langle \gamma^{(3)}, \gamma^{(3)} \right\rangle \gamma'',$$

we easily deduce

$$\begin{aligned}
\bar{k}_5 &= \pm \sqrt{\langle \gamma^{(4)}, \gamma^{(4)} \rangle - \langle \gamma^{(3)}, \gamma^{(3)} \rangle^2}, \\
W_2 &= \mp \frac{1}{\bar{k}_5} \left(\gamma^{(4)} + \left\langle \gamma^{(3)}, \gamma^{(3)} \right\rangle \gamma'' + \left\langle \gamma^{(4)}, \gamma^{(3)} \right\rangle \gamma' \right).
\end{aligned}$$

A direct computation shows that $W_1' = -\bar{k}_4 L - \bar{k}_3 N$. After rename the curvature functions ($k_1 = \bar{k}_4$, $k_2 = \bar{k}_5$ and $k_3 = \bar{k}_6$), the first four Frenet equations write down as follows:

$$\begin{aligned} L' &= W_1, \\ N' &= k_1 W_1 + k_2 W_2, \\ W_1' &= -k_1 L - N, \\ W_2' &= -k_2 L + k_3 W_3. \end{aligned}$$

If the dimension is greater than 5, then by a similar reasoning we have the following Frenet equations

$$\begin{aligned} W_i' &= -k_i W_{i-1} + k_{i+1} W_{i+1}, \qquad i \in \{3, \ldots, m-1\}, \\ W_m' &= -k_m W_{m-1}, \quad m = n-2. \end{aligned}$$

Then we have shown the following theorem.

Theorem 3.1 *Let $\gamma : I \longrightarrow M_1^n$, $n = m+2$, be a null curve parametrized by the pseudo-arc such that $\left\{\gamma'(t), \gamma''(t), \ldots, \gamma^{(n)}(t)\right\}$ is a basis of $T_{\gamma(t)} M_1^n$ for all t. Then there exists only one Frenet frame satisfying the equations*

$$\begin{aligned} L' &= W_1, \\ N' &= k_1 W_1 + k_2 W_2, \\ W_1' &= -k_1 L - N, \\ W_2' &= -k_2 L + k_3 W_3, \\ W_i' &= -k_i W_{i-1} + k_{i+1} W_{i+1} \quad i \in \{3, \ldots, m-1\}, \\ W_m' &= -k_m W_{m-1}, \end{aligned} \tag{3}$$

and verifying

i) For $2 \leqslant i \leqslant m-1$, $\left\{\gamma', \gamma'', \ldots, \gamma^{(i+2)}\right\}$ and $\{L, N, W_1, \ldots, W_i\}$ have the same orientation.

ii) $\{L, N, W_1, \ldots, W_m\}$ is positively oriented.

Definition 3.2 *A null curve in M_1^n satisfying the conditions of the above theorem is called a* Cartan curve. *The above Frenet frame and curvatures $\{k_1, k_2, \ldots, k_m\}$ are called the* Cartan reference *and the* Cartan curvatures, *respectively, of the curve γ.*

Let $\mathcal{A}$ and $\mathcal{B}$ denote the references $\left\{\gamma', \gamma'', \ldots, \gamma^{(n)}\right\}$ and $\{L, N, W_1, W_2, \ldots, W_m\}$, respectively, and write A and B for the matrices of the metric with respect to $\mathcal{A}$ and $\mathcal{B}$, respectively. Let D_i denote the i-th order main determinant of the matrix A, that is,

$$D_i = \begin{vmatrix} \langle\gamma', \gamma'\rangle & \langle\gamma', \gamma''\rangle & \cdots & \left\langle\gamma', \gamma^{(i)}\right\rangle \\ \langle\gamma'', \gamma'\rangle & \langle\gamma'', \gamma''\rangle & \cdots & \left\langle\gamma'', \gamma^{(i)}\right\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \left\langle\gamma^{(i)}, \gamma'\right\rangle & \left\langle\gamma^{(i)}, \gamma''\right\rangle & \cdots & \left\langle\gamma^{(i)}, \gamma^{(i)}\right\rangle \end{vmatrix}$$

Proposition 3.3 *Let $\gamma : I \longrightarrow M_1^n$ be a Cartan curve. Then the Cartan curvatures $\{k_1, k_2, k_3, \dots, k_m\}$ are given by*

$$k_1 = \frac{1}{2}\left\langle \gamma^{(3)}, \gamma^{(3)} \right\rangle, \quad k_2^2 = -D_4, \quad k_i^2 = \frac{D_i D_{i+2}}{D_{i+1}^2}.$$

Moreover, $k_2 < 0$, $k_i > 0$ for all $i \in \{3, \dots, m-1\}$, and $k_m > 0$ or $k_m < 0$ according to $\mathcal{A}$ is positively or negatively oriented, respectively.

Proof. The formula for k_1 (old $\bar{k}_4$) was computed in (1). After a straightforward computation, it is not difficult to see that

$$\begin{pmatrix} \gamma' \\ \gamma'' \\ \gamma^{(3)} \\ \gamma^{(4)} \\ \gamma^{(5)} \\ \gamma^{(6)} \\ \vdots \\ \gamma^{(n)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ -k_1 & -1 & 0 & 0 & 0 & 0 & \dots & 0 \\ -k_1' & 0 & -2k_1 & -k_2 & 0 & 0 & \dots & 0 \\ * & * & * & * & -k_2k_3 & 0 & \dots & 0 \\ * & * & * & * & * & -k_2k_3k_4 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & * & * & * & \dots & -k_2k_3\dots k_m \end{pmatrix} \begin{pmatrix} L \\ N \\ W_1 \\ W_2 \\ W_3 \\ W_4 \\ \vdots \\ W_m \end{pmatrix}.$$

If we denote by P the above matrix, then from $A = P^t B P$, P^t standing for the transpose matrix, we get

$$\det A = (\det P)^2 \det B = -k_m^2 k_{m-1}^4 \dots k_2^{2m-2}.$$

On the other hand, a direct computation leads to

$$\begin{aligned} D_3 &= -1, \\ D_4 &= -k_2^2, \\ D_5 &= -k_3^2 k_2^4, \\ D_6 &= -k_4^2 k_3^4 k_2^6, \\ &\vdots \\ D_i &= -k_{i-2}^2 k_{i-3}^4 \dots k_2^{2(i-3)}, \\ &\vdots \\ D_{m+2} &= -k_m^2 k_{m-1}^4 \dots k_2^{2(m-1)}. \end{aligned}$$

Now the result is clear by using Theorem 3.1. □

Corollary 3.4 *The Cartan curvatures of a curve C in M_1^n are invariant under Lorentzian transformations.*

Definition 3.5 *A curve is said to be a* helix *if it has constant Cartan curvatures.*

A long and messy computation shows that if γ is a helix then it satisfies the following differential equation

$$\gamma^{(n+1)} = a_1\gamma' + a_2\gamma^{(3)} + \dots + a_s\gamma^{(n-1)}, \quad \text{if } n \text{ is even, } n = 2s,$$

and

$$\gamma^{(n+1)} = a_1\gamma'' + a_2\gamma^{(4)} + \cdots + a_s\gamma^{(n-1)}, \quad \text{if } n \text{ is odd, } n = 2s+1,$$

where the coefficients are given by

$$\begin{aligned}
a_i &= k_2^2 \sum_{\substack{4\leqslant j_1<\cdots<j_{s-i-1}\leqslant n-2\\ j_r-j_{r-1}\geqslant 2 \quad \forall r}} k_{j_1}^2\cdots k_{j_{s-i-1}}^2 - 2k_1 \sum_{\substack{3\leqslant j_1<\cdots<j_{s-i}\leqslant n-2\\ j_r-j_{r-1}\geqslant 2 \quad \forall r}} k_{j_1}^2\cdots k_{j_{s-i}}^2\\
&\quad - \sum_{\substack{3\leqslant j_1<\cdots<j_{s-i+1}\leqslant n-2\\ j_r-j_{r-1}\geqslant 2 \quad \forall r}} k_{j_1}^2\cdots k_{j_{s-i+1}}^2, \qquad i = 1,\ldots,s-2,\\
a_{s-1} &= k_2^2 - 2k_1\sum_{j=3}^{n-2} k_j^2 - \sum_{\substack{3\leqslant j_1<j_2\leqslant n-2\\ j_2-j_1\geqslant 2}} k_{j_1}^2 k_{j_2}^2,\\
a_s &= -2k_1 - \sum_{j=3}^{n-2} k_j^2.
\end{aligned}$$

Bonnor[4] study and classify helices in $\mathbb{R}_1^4$. In this paper we are going to extend his results to $\mathbb{R}_1^5$, $\mathbb{S}_1^4$ and $\mathbb{H}_1^4$.

4. Null curves in $\mathbb{R}_1^n$

Definition 4.1 *A basis $\mathcal{B} = \{L_1, N_1, \ldots, L_r, N_r, W_1, \ldots, W_m\}$ of $\mathbb{R}_q^n$, with $2r \leqslant 2q \leqslant n$ and $m = n-2r$, is said to be* pseudo-orthonormal *if it satisfies the following equations:*

$$\begin{aligned}
\langle L_i, L_j\rangle = \langle N_i, N_j\rangle = 0, &\quad \langle L_i, N_j\rangle = \delta_{ij},\\
\langle L_i, W_\alpha\rangle = \langle N_i, W_\alpha\rangle = 0, &\quad \langle W_\alpha, W_\beta\rangle = \varepsilon_\alpha\delta_{\alpha\beta},
\end{aligned}$$

where $i,j \in \{1,\ldots,r\}$, $\alpha,\beta \in \{1,\ldots,m\}$, $\varepsilon_\alpha = -1$ if $1 \leqslant \alpha \leqslant q-r$ and $\varepsilon_\alpha = 1$ if $q-r+1 \leqslant \alpha \leqslant m$.

Lemma 4.2 *Let $\mathcal{B} = \{L_1, N_1, \ldots, L_r, N_r, W_1, \ldots, W_m\}$ be a basis of $\mathbb{R}_q^n$, with $2r \leqslant 2q \leqslant n$ and $m = n-2r$. Consider $\mathcal{B}' = \{V_1, \ldots, V_q, V_{q+1}, \ldots, V_n\}$ where*

$$V_i = \begin{cases}
\dfrac{1}{\sqrt{2}}(L_i - N_i) & i = 1,\ldots,r\\
W_{i-r} & i = r+1,\ldots,q\\
\dfrac{1}{\sqrt{2}}(L_{i-q} + N_{i-q}) & i = q+1,\ldots,q+r\\
W_{i-2r} & i = q+r+1,\ldots,n
\end{cases}$$

The following conditions are equivalent:

(i) $\mathcal{B}$ is a pseudo-orthonormal basis.

(ii) $\mathcal{B}'$ is an orthonormal basis.

(iii) $\mathcal{B}'$ *satisfies*

$$-\sum_{\alpha=1}^{q} V_{\alpha i} V_{\alpha j} + \sum_{\beta=q+1}^{n} V_{\beta i} V_{\beta j} = \eta_{ij}.$$

(iv) $\mathcal{B}$ *satisfies*

$$\sum_{\alpha=1}^{r} \left(L_{\alpha i} N_{\alpha j} + L_{\alpha j} N_{\alpha i} \right) - \sum_{\beta=1}^{q-r} W_{\beta i} W_{\beta j} + \sum_{\theta=q-r+1}^{m} W_{\theta i} W_{\theta j} = \eta_{ij}.$$

Here $V_{\rho k}, L_{\rho k}, N_{\rho k}$ *and* $W_{\rho k}$ *stand for the components of vectors* V_ρ, L_ρ, N_ρ *and* W_ρ*, respectively, and* (η_{ij}) *denotes the matrix of the canonical metric in the standard coordinates.*

Proof. (i) $\Leftrightarrow$ (ii) It is obvious.
(ii) $\Leftrightarrow$ (iii) Consider the matrices $V = (V_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ in $\mathcal{M}_{n\times n}(\mathbb{R})$ given by

$$\begin{aligned} b_{ij} &= \langle V_i, V_j \rangle \,, \\ c_{ij} &= -\sum_{\alpha=1}^{q} V_{\alpha i} V_{\alpha j} + \sum_{\beta=q+1}^{n} V_{\beta i} V_{\beta j}. \end{aligned}$$

Put

$$V = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix},$$

where A_1, B_1 and C_1 are matrices in $\mathcal{M}_{q\times q}(\mathbb{R})$. Consider the complex matrix

$$A = \begin{pmatrix} A_1 & iA_2 \\ -iA_3 & A_4 \end{pmatrix} \in \mathcal{M}_{n\times n}(\mathbb{C}).$$

Then a straightforward computation shows that

$$AA^T = \begin{pmatrix} -B_1 & iB_2 \\ iB_3 & B_4 \end{pmatrix} \quad \text{and} \quad A^T A = \begin{pmatrix} -C_1 & -iC_2 \\ -iC_3 & C_4 \end{pmatrix}.$$

Then $\mathcal{B}'$ is orthonormal if and only if $C_1 = -I$, $C_4 = I$ and $C_2 = C_3 = 0$, and that concludes the proof.
(iii) $\Leftrightarrow$ (iv) From (1) we have

$$L_\alpha = \frac{1}{\sqrt{2}} \left(V_{\alpha+q} + V_\alpha \right) \quad \text{and} \quad N_\alpha = \frac{1}{\sqrt{2}} \left(V_{\alpha+q} - V_\alpha \right), \quad \alpha \in \{1, \ldots, r\},$$

and then

$$L_{\alpha i} N_{\alpha j} + N_{\alpha i} L_{\alpha j} = -V_{\alpha i} V_{\alpha j} + V_{(\alpha+q)i} V_{(\alpha+q)j}, \quad \alpha \in \{1, \ldots, r\}, \quad i, j \in \{1, \ldots, n\},$$

which finishes the proof. $\square$

Theorem 4.3 *Let $k_1, k_2, \dots, k_m : [-\varepsilon, \varepsilon] \longrightarrow \mathbb{R}$ be differentiable functions with $k_2 < 0$ and $k_i > 0$ for $i \in \{3, \dots, m-1\}$. Let p be a point in $\mathbb{R}^n_1$, $n = m+2$, and consider $\{L^0, N^0, W^0_1, \dots, W^0_m\}$ a positively oriented pseudo-orthonormal basis of $\mathbb{R}^n_1$. Then there exists a unique Cartan curve γ in $\mathbb{R}^n_1$, with $\gamma(0) = p$, whose Cartan reference $\{L, N, W_1, \dots, W_m\}$ satisfies*

$$L(0) = L^0, N(0) = N^0, W_i(0) = W^0_i, \quad i \in \{1, \dots, m\}.$$

Proof. According to the general theory of differential equations, there exists a unique solution $\{L, N, W_1, \dots, W_m\}$ of (2), defined on an interval $[-\varepsilon, \varepsilon]$, and satisfying the initial conditions of the theorem. A straightforward computation, bearing in mind (2), leads to

$$\frac{d}{dt}\left(L_i N_j + L_j N_i + \sum_{\alpha=1}^{m} W_{\alpha i} W_{\alpha j}\right) = 0, \quad i, j \in \{1, \dots, n\}.$$

Since $\{L(0), N(0), W_1(0), \dots, W_m(0)\}$ is a pseudo-orthonormal basis, then the above equation jointly with Lemma 4.2 implies

$$L_i(t)N_j(t) + L_j(t)N_i(t) + \sum_{\alpha=1}^{m} W_{\alpha i}(t) W_{\alpha j}(t) = \eta_{ij}, \quad \forall t \in [-\varepsilon, \varepsilon].$$

Then by using again Lemma 4.2 we deduce that $\{L, N, W_1, \dots, W_m\}$ is a pseudo-orthonormal basis for all t, and this concludes the proof. $\square$

The following result shows that the Cartan curvatures determine curves satisfying the nondegeneracy conditions stated in Theorem 3.1.

Theorem 4.4 *If two Cartan curves C and $\bar{C}$ in $\mathbb{R}^n_1$ have Cartan curvatures $\{k_1, \dots, k_m\}$, where $k_i : [-\varepsilon, \varepsilon] \longrightarrow \mathbb{R}$ are differentiable functions, then there exists a Lorentzian transformation of $\mathbb{R}^n_1$ which maps C into $\bar{C}$.*

4.1. Null helices in $\mathbb{R}^5_1$

The goal of this section is to classify the family of null helices in $\mathbb{R}^5_1$. Before to do that, we present some examples. From the general equation, we know that a null helix in $\mathbb{R}^5_1$ satisfies the following differential equation

$$\gamma^{(6)} + (2k_1 + k_3^2)\gamma^{(4)} - (k_2^2 - 2k_1 k_3^2)\gamma'' = 0,$$

which will help us to find the examples.

Example 4.5 [Helices of Type 1] Let ω, σ and h be three non-zero constants such that $\frac{1}{\sigma^2} < h^2 < \frac{1}{\omega^2}$ and let $\gamma_{\omega,\sigma,h} : \mathbb{R} \to \mathbb{R}^5_1$ be the curve defined by

$$\gamma_{\omega,\sigma,h}(t) = \left(ht, \frac{1}{\omega}\sqrt{\frac{h^2\sigma^2 - 1}{\sigma^2 - \omega^2}}\sin \omega t, \frac{1}{\omega}\sqrt{\frac{h^2\sigma^2 - 1}{\sigma^2 - \omega^2}}\cos \omega t, \frac{1}{\sigma}\sqrt{\frac{1 - h^2\omega^2}{\sigma^2 - \omega^2}}\sin \sigma t, \right.$$
$$\left. \frac{1}{\sigma}\sqrt{\frac{1 - h^2\omega^2}{\sigma^2 - \omega^2}}\cos \sigma t \right).$$

Then it is easy to see that $\gamma_{\omega,\sigma,h}$ is a helix with curvatures given by

$$\begin{aligned} k_1 &= \frac{1}{2}\left(\sigma^2 + \omega^2(1-\sigma^2 h^2)\right), \\ k_2^2 &= -\omega^2\sigma^2\left(\omega^2 h^2 - 1\right)\left(\sigma^2 h^2 - 1\right), \\ k_3^2 &= \omega^2\sigma^2 h^2. \end{aligned}$$

Example 4.6 [Helices of Type 2] Let ω, σ and h be three non-zero constants such that $0 < h^2\omega^2 < 1$ and let $\gamma_{\omega,\sigma,h} : \mathbb{R} \to \mathbb{R}^5_1$ be the curve defined by

$$\begin{aligned} \gamma_{\omega,\sigma,h}(t) = \Bigg(& \frac{1}{\omega}\sqrt{\frac{1+h^2\sigma^2}{\omega^2+\sigma^2}}\sinh\omega t, \frac{1}{\omega}\sqrt{\frac{1+h^2\sigma^2}{\omega^2+\sigma^2}}\cosh\omega t, \frac{1}{\sigma}\sqrt{\frac{1-h^2\omega^2}{\omega^2+\sigma^2}}\sin\sigma t, \\ & \frac{1}{\sigma}\sqrt{\frac{1-h^2\omega^2}{\omega^2+\sigma^2}}\cos\sigma t, ht\Bigg). \end{aligned}$$

Then $\gamma_{\omega,\sigma,h}$ is a helix with curvatures given by

$$\begin{aligned} k_1 &= \frac{1}{2}\left(\sigma^2 - \omega^2(1+\sigma^2 h^2)\right), \\ k_2^2 &= -\omega^2\sigma^2\left(\omega^2 h^2 - 1\right)\left(\sigma^2 h^2 + 1\right), \\ k_3^2 &= \omega^2\sigma^2 h^2. \end{aligned}$$

Example 4.7 [Helices of Type 3] Let σ and h be two non-zero constants such that $0 < 2h^2 < 1$ and let $\gamma_{\sigma,h} : \mathbb{R} \to \mathbb{R}^5_1$ be the curve defined by

$$\begin{aligned} \gamma_{\sigma,h}(t) = \Bigg(& \frac{3}{2}\left(\frac{1-2h^2}{\sigma^2 h^2}\right)t + \frac{1}{6}h^2 t + t^3, \frac{h}{\sqrt{2}}t^2, \frac{3}{2}\left(\frac{1-2h^2}{\sigma^2 h^2}\right)t - \frac{1}{6}h^2 t + t^3, \\ & \frac{\sqrt{1-2h^2}}{\sigma^2}\sin\sigma t, \frac{\sqrt{1-2h^2}}{\sigma^2}\cos\sigma t\Bigg). \end{aligned}$$

Then $\gamma_{\sigma,h}$ is a helix with curvatures given by

$$\begin{aligned} k_1 &= \frac{1}{2}\sigma^2\left(1-2h^2\right), \\ k_2^2 &= 2\sigma^4 h^2\left(1-2h^2\right), \\ k_3^2 &= 2\sigma^2 h^2. \end{aligned}$$

Theorem 4.8 *Let γ be a null curve fully immersed in $\mathbb{R}^5_1$. Then γ is a helix if and only if it is congruent to a helix either of type 1, or type 2 or type 3.*

Proof. Let k_1, k_2 and k_3 be the constant curvatures of γ such that $k_2 \neq 0 \neq k_3$. By the Congruence Theorem 4.4 it suffices to find a helix (of one of the above types) with these curvatures.

Case 1: Assume that $(2k_1 + k_3^2) > \sqrt{(2k_1 - k_3^2)^2 + 4k_2^2}$. Take the helix $\gamma_{\omega,\sigma,h}$ of type 1 determined by

$$\begin{aligned} \omega^2 &= \frac{1}{2}\left(2k_1 + k_3^2\right) - \frac{1}{2}\sqrt{(2k_1 - k_3^2)^2 + 4k_2^2}, \\ \sigma^2 &= \frac{1}{2}\left(2k_1 + k_3^2\right) + \frac{1}{2}\sqrt{(2k_1 - k_3^2)^2 + 4k_2^2}, \\ h^2 &= \frac{k_3^2}{2k_1 k_3^2 - k_2^2}. \end{aligned}$$

A straightforward computation shows that $\dfrac{1}{\sigma^2} < h^2 < \dfrac{1}{\omega^2}$ and that the curvatures of $\gamma_{\omega,\sigma,h}$ are k_1, k_2 and k_3.

Case 2: Suppose that $(2k_1 + k_3^2) < \sqrt{(2k_1 - k_3^2)^2 + 4k_2^2}$. Take the helix $\gamma_{\omega,\sigma,h}$ of type 2 determined by

$$\begin{aligned} \omega^2 &= -\frac{1}{2}\left(2k_1 + k_3^2\right) + \frac{1}{2}\sqrt{(2k_1 - k_3^2)^2 + 4k_2^2}, \\ \sigma^2 &= \frac{1}{2}\left(2k_1 + k_3^2\right) + \frac{1}{2}\sqrt{(2k_1 - k_3^2)^2 + 4k_2^2}, \\ h^2 &= \frac{k_3^2}{2k_1k_3^2 - k_2^2}. \end{aligned}$$

As before we have that the curvatures of $\gamma_{\omega,\sigma,h}$ are k_1, k_2 and k_3.

Case 3: Finally, if $(2k_1 + k_3^2) = \sqrt{(2k_1 - k_3^2)^2 + 4k_2^2}$, then we can take the helix $\gamma_{\sigma,h}$ determined by

$$\sigma^2 = \frac{k_2^2 + k_3^4}{k_3^2} \quad \text{and} \quad h^2 = \frac{k_3^4}{2\left(k_2^2 + k_3^4\right)}.$$

It is easy to see that $h^2 < 1/2$ and the curvatures of $\gamma_{\sigma,h}$ are k_1, k_2 and k_3. That concludes the proof. □

5. Null curves in $\mathbb{S}_1^n$

Let $\gamma : I \longrightarrow \mathbb{S}_1^n \subset \mathbb{R}_1^{n+1}$ be a null curve and denote by D_t the covariant derivative in $\mathbb{S}_1^n$ along γ. Then for any vector field V along γ we have

$$D_t V = V' + \left\langle V, \gamma' \right\rangle \gamma,$$

where $\langle , \rangle$ stands for the canonical metric in $\mathbb{R}_1^{n+1}$. If $\{L, N, W_1, \ldots, W_m\}$ denotes the Cartan reference, then equations (2) write down as follows:

$$\begin{aligned} \gamma' &= L, \\ L' &= W_1, \\ N' &= -\gamma + k_1 W_1 + k_2 W_2, \\ W_1' &= -k_1 L - N, \\ W_2' &= -k_2 L + k_3 W_3, \\ W_i' &= -k_i W_{i-1} + k_{i+1} W_{i+1}, \quad i \in \{3, \ldots, m-1\}, \\ W_m' &= -k_m W_{m-1}. \end{aligned} \tag{2}$$

Theorem 5.1 *Let $k_1, k_2, \ldots, k_m : [-\varepsilon, \varepsilon] \longrightarrow \mathbb{R}$ be differentiable functions with $k_2 < 0$ and $k_i > 0$ for $i \in \{3, \ldots, m-1\}$. Let p be a point in $\mathbb{S}_1^n$, $n = m+2$, and let $\left\{L^0, N^0, W_1^0, \ldots, W_m^0\right\}$ be a positively oriented pseudo-orthonormal basis of $T_p\mathbb{S}_1^n$. Then there exists a unique Cartan curve γ in $\mathbb{S}_1^n$, with $\gamma(0) = p$, whose Cartan reference $\{L, N, W_1, \ldots, W_m\}$ satisfies*

$$L(0) = L^0, N(0) = N^0, W_i(0) = W_i^0, \quad i \in \{1, \ldots, m\}.$$

Proof. We proceed as in the proof of Theorem 4.3. Let $\{L, N, \gamma, W_1, \ldots, W_m\}$ be a solution of (1) with initial conditions $\left\{L^0, N^0, p, W_1^0, \ldots, W_m^0\right\}$. A straightforward computation yields

$$\frac{d}{dt}\left(L_i(t)N_j(t) + L_j(t)N_i(t) + \gamma_i(t)\gamma_j(t) + \sum_{\beta=1}^{m} W_{\beta i}(t)W_{\beta j}(t)\right) = 0.$$

Since $\left\{L^0, N^0, p, W_1^0, \ldots, W_m^0\right\}$ is a pseudo-orthonormal basis, from Lemma 4.2, with $q = r = 1$, we obtain

$$L_i(t)N_j(t) + L_j(t)N_i(t) + \gamma_i(t)\gamma_j(t) + \sum_{\beta=1}^{m} W_{\beta i}(t)W_{\beta j}(t) = \eta_{ij}, \quad \forall t \in [-\varepsilon, \varepsilon],$$

and then $\{L(t), N(t), \gamma(t), W_1(t), \ldots, W_m(t)\}$ is a local pseudo-orthonormal frame. This concludes the proof. $\square$

The following result is clear.

Theorem 5.2 *If two Cartan curves C and $\bar{C}$ in $\mathbb{S}_1^n$ have Cartan curvatures $\{k_1, \ldots, k_m\}$, where $k_i : [-\varepsilon, \varepsilon] \longrightarrow \mathbb{R}$ are differentiable functions, then there exists a Lorentzian transformation of $\mathbb{S}_1^n$ which maps C into $\bar{C}$.*

5.1. Null helices in $\mathbb{S}_1^4$

In this section we are going to classify the null helices in the 4-dimensional De Sitter space. The Cartan frame of a null curve γ in $\mathbb{S}_1^4$ satisfies the following equations

$$\begin{aligned}
\gamma' &= L, \\
L' &= W_1, \\
N' &= -\gamma + k_1 W_1 + k_2 W_2, \\
W_1' &= -k_1 L - N, \\
W_2' &= -k_2 L.
\end{aligned} \tag{4}$$

An easy computation shows that if γ is a helix, then it satisfies the following differential equation

$$\gamma^{(5)} + 2k_1\gamma^{(3)} - (1 + k_2^2)\gamma' = 0,$$

whose general solution is

$$\gamma(t) = A_1 \sinh \omega t + A_2 \cosh \omega t + A_3 \sin \sigma t + A_4 \cos \sigma t + A_5,$$

where A_1, A_2, A_3, A_4 and A_5 are constant vectors in $\mathbb{R}_1^5$.

Example 5.3 Let $\gamma_{\omega,\sigma} : \mathbb{R} \to \mathbb{S}_1^4 \subset \mathbb{R}_1^5$ be the null curve defined by

$$\gamma_{\omega,\sigma}(t) = \sqrt{\frac{1}{\omega^2 + \sigma^2}}\left(\frac{1}{\omega}\sinh \omega t, \frac{1}{\omega}\cosh \omega t, \frac{1}{\sigma}\sin \sigma t, \frac{1}{\sigma}\cos \sigma t, \sqrt{\frac{\omega^4 - 1}{\omega^2} + \frac{\sigma^4 - 1}{\sigma^2}}\right),$$

where ω and σ are non-zero constants such that $\omega^2\sigma^2 > 1$.

A direct computation shows that γ is a null helix with curvatures

$$k_1 = \frac{1}{2}\left(\sigma^2 - \omega^2\right) \quad \text{and} \quad k_2^2 = \omega^2\sigma^2 - 1.$$

Theorem 5.4 *Let γ be a null curve fully immersed in $\mathbb{S}_1^4$. Then γ is a helix if and only if it is congruent to one of the family described in Example 5.3.*

Proof. Let γ be a null helix in $\mathbb{S}_1^4$ with curvatures k_1 and k_2. Consider the constants ω and σ given by

$$\omega = \sqrt{-k_1 + \sqrt{k_1^2 + k_2^2 + 1}} \quad \text{and} \quad \sigma = \sqrt{k_1 + \sqrt{k_1^2 + k_2^2 + 1}}.$$

Then from (5) we deduce that $\gamma_{\omega,\sigma}$ has curvatures k_1 and k_2, so that we get the result by using the congruence Theorem 5.2. □

6. Null curves in $\mathbb{H}_1^n$

Let $\gamma : I \longrightarrow \mathbb{H}_1^n \subset \mathbb{R}_2^{n+1}$ be a null curve and denote by D_t the covariant derivative in $\mathbb{H}_1^n$ along γ. Then for any vector field V along γ we have

$$D_t V = V' - \langle V, \gamma' \rangle \gamma,$$

where $\langle , \rangle$ stands for the canonical metric in $\mathbb{R}_2^{n+1}$. If $\{L, N, W_1, \ldots, W_m\}$ denotes the Cartan reference, then we have the following equations

$$\begin{aligned} \gamma' &= L, \\ L' &= W_1, \\ N' &= \gamma + k_1 W_1 + k_2 W_2, \\ W_1' &= -k_1 L - N, \\ W_2' &= -k_2 L + k_3 W_3, \\ W_i' &= -k_i W_{i-1} + k_{i+1} W_{i+1}, \quad i \in \{3, \ldots, m-1\}, \\ W_m' &= -k_m W_{m-1}. \end{aligned} \tag{2}$$

Theorem 6.1 *Let $k_1, k_2, \ldots, k_m : [-\varepsilon, \varepsilon] \longrightarrow \mathbb{R}$ be differentiable functions with $k_2 < 0$ and $k_i > 0$ for $i \in \{3, \ldots, m-1\}$. Let p be a point in $\mathbb{H}_1^n$, $n = m+2$, and let $\{L^0, N^0, W_1^0, \ldots, W_m^0\}$ be a positively oriented pseudo-orthonormal basis of $T_p\mathbb{H}_1^n$. Then there exists a unique Cartan curve γ in $\mathbb{H}_1^n$, with $\gamma(0) = p$, whose Cartan reference $\{L, N, W_1, \ldots, W_m\}$ satisfies*

$$L(0) = L^0, N(0) = N^0, W_i(0) = W_i^0, \quad i \in \{1, \ldots, m\}.$$

Proof. We proceed as in the proof of Theorem 5.1. Let $\{L, N, \gamma, W_1, \ldots, W_m\}$ be a solution of (1) with initial conditions $\{L^0, N^0, p, W_1^0, \ldots, W_m^0\}$. A straightforward computation yields

$$\frac{d}{dt}\left(L_i(t)N_j(t) + L_j(t)N_i(t) - \gamma_i(t)\gamma_j(t) + \sum_{\beta=1}^{m} W_{\beta i}(t) W_{\beta j}(t) \right) = 0.$$

Since $\{L^0, N^0, p, W_1^0, \ldots, W_m^0\}$ is a pseudo-orthonormal basis, from Lemma 4.2, with $q = 2$ and $r = 1$, we obtain

$$L_i(t)N_j(t) + L_j(t)N_i(t) - \gamma_i(t)\gamma_j(t) + \sum_{\beta=1}^{m} W_{\beta i}(t) W_{\beta j}(t) = \eta_{ij}, \quad \forall t \in [-\varepsilon, \varepsilon],$$

and then $\{L(t), N(t), \gamma(t), W_1(t), \ldots, W_m(t)\}$ is a local pseudo-orthonormal frame with $\langle\gamma,\gamma\rangle = -1$. That concludes the proof. $\square$

The following result is clear.

Theorem 6.2 *If two Cartan curves C and $\bar{C}$ in $\mathbb{H}_1^n$ have Cartan curvatures $\{k_1, \ldots, k_m\}$, where $k_i : [-\varepsilon, \varepsilon] \longrightarrow \mathbb{R}$ are differentiable functions, then there exists a Lorentzian transformation of $\mathbb{H}_1^n$ which maps C into $\bar{C}$.*

6.1. Null helices in $\mathbb{H}_1^4$

Let γ be a null curve in $\mathbb{H}_1^4$, then its Cartan frame satsifies the following equations

$$\begin{aligned} \gamma' &= L, \\ L' &= W_1, \\ N' &= \gamma + k_1 W_1 + k_2 W_2, \\ W_1' &= -k_1 L - N, \\ W_2' &= -k_2 L. \end{aligned} \tag{4}$$

If γ is a helix, then it is easy to see that it verifies the following ordinary differential equation

$$\gamma^{(5)} + 2k_1\gamma^{(3)} + (1 - k_2^2)\gamma' = 0.$$

Before we state the main result of this section we present some examples of helices in the 4-dimensional anti De Sitter space.

Example 6.3 [Helices of Type A1] Let $0 < \omega^2 < 1$ and let γ_ω be the curve in $\mathbb{H}_1^4$ defined by

$$\begin{aligned} \gamma_\omega(t) = \Bigg(&\frac{t}{2\omega}\cosh\omega t, \frac{1}{\omega^2}\left(\cosh\omega t - \frac{1}{2}\omega t\sinh\omega t\right), \frac{1}{\omega^2}\left(\sinh\omega t - \frac{1}{2}\omega t\cosh\omega t\right), \\ &\frac{t}{2\omega}\sinh\omega t, \frac{\sqrt{1-\omega^4}}{\omega^2}\Bigg). \end{aligned}$$

Then γ_ω is a helix with curvatures

$$k_1 = -\omega^2 \quad \text{and} \quad k_2^2 = 1 - \omega^4.$$

Example 6.4 [Helices of Type A2] Let $0 < \sigma^2 < 1$ and let γ_σ be the curve in $\mathbb{H}_1^4$ defined by

$$\begin{aligned} \gamma_\sigma(t) = \Bigg(&\frac{1}{\sigma^2}\left(\sin\sigma t - \frac{1}{2}\sigma t\cos\sigma t\right), \frac{1}{\sigma^2}\left(\cos\sigma t + \frac{1}{2}\sigma t\sin\sigma t\right), -\frac{t}{2\sigma}\cos\sigma t, \\ &\frac{t}{2\sigma}\sin\sigma t, \frac{\sqrt{1-\sigma^4}}{\sigma^2}\Bigg). \end{aligned}$$

Then γ_σ is a helix with curvatures

$$k_1 = \sigma^2 \quad \text{and} \quad k_2^2 = 1 - \sigma^4.$$

Example 6.5 [Helices of Type A3] Let $\omega^2 = 1$ and let γ be the null curve in $\mathbb{H}_1^4$ defined by

$$\gamma(t) = \left(1 - \frac{t^4}{24}, \frac{\omega(t^3+t)}{2\sqrt{3}}, \frac{t^4}{24}, \frac{\omega(t^3-t)}{2\sqrt{3}}, \frac{t^2}{2}\right).$$

Then γ is a helix with curvatures

$$k_1 = 0 \quad \text{and} \quad k_2^2 = 1.$$

γ will be called the *null quartic* in $\mathbb{H}_1^4$.

Example 6.6 [Helices of Type B1] Let $0 < \omega^2 < \sigma^2$ and $\omega^2\sigma^2 < 1$, and let $\gamma_{\omega,\sigma}$ be the curve in $\mathbb{H}_1^4$ defined by

$$\gamma_{\omega,\sigma}(t) = \sqrt{\frac{1}{\sigma^2-\omega^2}}\left(\frac{1}{\omega}\sin\omega t, \frac{1}{\omega}\cos\omega t, \frac{1}{\sigma}\sin\sigma t, \frac{1}{\sigma}\cos\sigma t, \sqrt{\frac{1+\omega^4}{\omega^2} - \frac{1+\sigma^4}{\sigma^2}}\right).$$

Then $\gamma_{\omega,\sigma}$ is a helix with curvatures

$$k_1 = \frac{1}{2}\left(\omega^2+\sigma^2\right) \quad \text{and} \quad k_2^2 = 1 - \omega^2\sigma^2.$$

Example 6.7 [Helices of Type B2] Let $0 < \sigma^2 < \omega^2$ and $\omega^2\sigma^2 < 1$, and let$\gamma_{\omega,\sigma}$ be the curve in $\mathbb{H}_1^4$ defined by

$$\gamma_{\omega,\sigma}(t) = \sqrt{\frac{1}{\omega^2-\sigma^2}}\left(\frac{1}{\omega}\sinh\omega t, \frac{1}{\sigma}\cosh\sigma t, \frac{1}{\sigma}\sinh\sigma t, \frac{1}{\omega}\cosh\omega t, \sqrt{\frac{1+\sigma^4}{\sigma^2} - \frac{1+\omega^4}{\omega^2}}\right).$$

Then $\gamma_{\omega,\sigma}$ is a helix with curvatures

$$k_1 = -\frac{1}{2}\left(\omega^2+\sigma^2\right) \quad \text{and} \quad k_2^2 = 1 - \omega^2\sigma^2.$$

Example 6.8 [Helices of Type B3] Let $\sigma \neq 0$ and let γ_σ be the curve in $\mathbb{H}_1^4$ defined by

$$\gamma_\sigma(t) = \left(\frac{2+2\sigma^4-\sigma^2t^2}{2\sigma^2\sqrt{1+\sigma^4}}, \frac{t}{\sigma}, \frac{t^2}{2\sqrt{1+\sigma^4}}, \frac{1}{\sigma^2}\sin\sigma t, \frac{1}{\sigma^2}\cos\sigma t\right).$$

Then γ_σ is a helix with curvatures

$$k_1 = \frac{\sigma^2}{2} \quad \text{and} \quad k_2^2 = 1.$$

Example 6.9 [Helices of Type B4] Let $\omega \neq 0$ and let γ_ω be the curve in $\mathbb{H}_1^4$ defined by

$$\gamma_\omega(t) = \left(\frac{2+2\omega^4+\omega^2t^2}{2\omega^2\sqrt{1+\omega^4}}, \frac{1}{\omega^2}\sinh\omega t, \frac{1}{\omega^2}\cosh\omega t, \frac{t^2}{2\sqrt{1+\omega^4}}, \frac{t}{\omega}\right).$$

Then γ_ω is a helix with curvatures

$$k_1 = -\frac{\omega^2}{2} \quad \text{and} \quad k_2^2 = 1.$$

Example 6.10 [Helices of Type B5] Let $\omega\sigma \neq 0$ and let $\gamma_{\omega,\sigma}$ be the curve in $\mathbb{H}_1^4$ defined by

$$\gamma_{\omega,\sigma}(t) = \sqrt{\frac{1}{\omega^2+\sigma^2}}\left(\sqrt{\frac{1+\omega^4}{\omega^2}+\frac{1+\sigma^4}{\sigma^2}}, \frac{1}{\omega}\sinh\omega t, \frac{1}{\omega}\cosh\omega t, \frac{1}{\sigma}\sin\sigma t, \frac{1}{\sigma}\cos\sigma t\right).$$

Then $\gamma_{\omega,\sigma}$ is a helix with curvatures

$$k_1 = \frac{1}{2}\left(\sigma^2-\omega^2\right) \quad \text{and} \quad k_2^2 = 1+\omega^2\sigma^2.$$

Example 6.11 [Helices of Type C] Let $\omega^2+\sigma^2 < 1$ and let $\gamma_{\omega,\sigma}$ be the curve in $\mathbb{H}_1^4$ defined by

$$\begin{aligned}\gamma_{\omega,\sigma}(t) = \frac{1}{2\omega\sigma\left(\omega^2+\sigma^2\right)} & \left(2\omega\sigma\cosh\omega t\sin\sigma t + \left(\omega^2-\sigma^2\right)\sinh\omega t\cos\sigma t,\right. \\ & -2\omega\sigma\cosh\omega t\cos\sigma t + \left(\omega^2-\sigma^2\right)\sinh\omega t\sin\sigma t, \left(\omega^2+\sigma^2\right)\sinh\omega t\cos\sigma t, \\ & \left.\left(\omega^2+\sigma^2\right)\sinh\omega t\sin\sigma t, 2\omega\sigma\sqrt{1-\left(\omega^2+\sigma^2\right)^2}\right).\end{aligned}$$

Then $\gamma_{\omega,\sigma}$ is a helix with curvatures

$$k_1 = -\omega^2+\sigma^2 \quad \text{and} \quad k_2^2 = 1-\left(\omega^2+\sigma^2\right)^2.$$

Theorem 6.12 *Let γ be a null curve fully immersed in $\mathbb{H}_1^4$. Then γ is a helix if and only if it is congruent to one helix of the families described in Examples 6.3–6.11.*

Proof. That result can be obtained following a similar reasoning as in the precedent cases. The idea of the proof consists of finding, for any constants k_1 and $k_2 \neq 0$, a helix in one of the above nine examples with curvatures k_1 and k_2. The following table collects all the possibilities and figure 8 represents a diagram of them. □

Case	Type	Parameters
$k_1^2 + k_2^2 = 1$, $k_1 < 0$, $0 < k_2^2 < 1$	A1	$\omega^2 = -k_1$
$k_1^2 + k_2^2 = 1$, $k_1 > 0$, $0 < k_2^2 < 1$	A2	$\sigma^2 = k_1$
$k_1 = 0$, $k_2^2 = 1$	A3	
$k_1^2 + k_2^2 > 1$, $k_1 > 0$, $0 < k_2^2 < 1$	B1	$\omega^2 = k_1 - \sqrt{k_1^2 + k_2^2 - 1}$, $\sigma^2 = k_1 + \sqrt{k_1^2 + k_2^2 - 1}$
$k_1^2 + k_2^2 > 1$, $k_1 < 0$, $0 < k_2^2 < 1$	B2	$\omega^2 = -k_1 + \sqrt{k_1^2 + k_2^2 - 1}$, $\sigma^2 = -k_1 - \sqrt{k_1^2 + k_2^2 - 1}$
$k_1^2 + k_2^2 > 1$, $k_1 > 0$, $k_2^2 = 1$	B3	$\sigma^2 = 2k_1$
$k_1^2 + k_2^2 > 1$, $k_1 < 0$, $k_2^2 = 1$	B4	$\omega^2 = -2k_1$
$k_2^2 > 1$	B5	$\omega^2 = -k_1 + \sqrt{k_1^2 + k_2^2 - 1}$, $\sigma^2 = k_1 + \sqrt{k_1^2 + k_2^2 - 1}$
$k_1^2 + k_2^2 < 1$	C	$\omega^2 = \frac{1}{2}\left(-k_1 + \sqrt{1 - k_2^2}\right)$, $\sigma^2 = \frac{1}{2}\left(k_1 + \sqrt{1 - k_2^2}\right)$

Bibliography

[1] B. O'Neill. *Semi-Riemannian Geometry*. Academic Press, New York - London, 1983.

[2] Aurel Bejancu, Angel Ferrández, and Pascual Lucas. A new viewpoint on geometry of a lightlike hypersurface in a semi-Euclidean space. *Saitama Math. J.*, 16:31–38 (1999), 1998.

[3] E. Vessiot. Sur les curbes minima. *Comptes Rendus*, 140:1381–1384, 1905.

[4] W.B. Bonnor. Null curves in a Minkowski space-time. *Tensor, N.S.*, 20:229–242, 1969.

[5] M. Castagnino. Sulle formule di Frenet-Serre per le curve nulle di una V_4 riemanniana a metrica iperbolica normale. *Rendiconti di matematica, Roma, Ser. 5*, 23:438–461, 1964.

[6] J.L. Synge. Geometry of dinamical null lines. *Tensor, N.S.*, 24:69–74, 1972.

[7] L.P. Hughston and W.T. Shaw. Real classical strings. *Proc. Roy. Soc. London Ser. A*, 414:415–422, 1987.

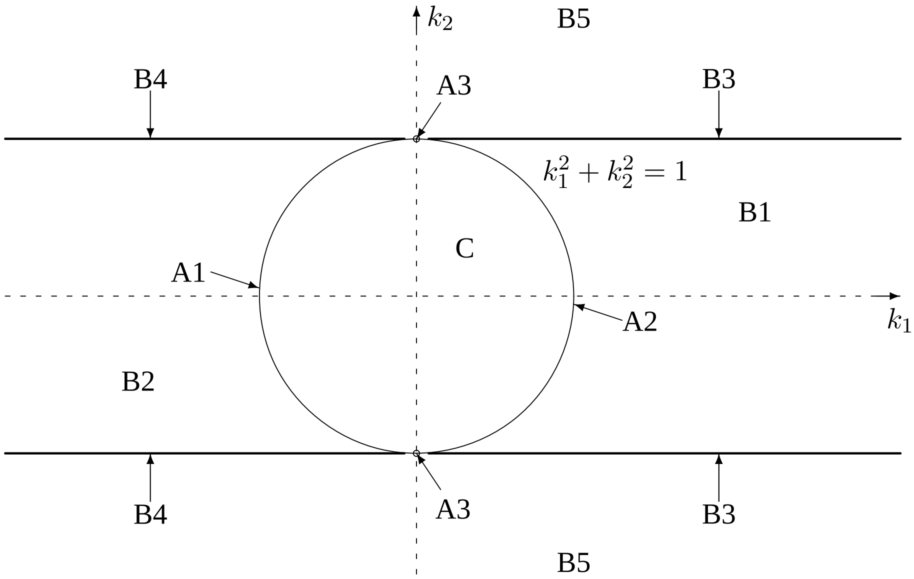

Figure 8: Regions in $\mathbb{R}^2$ with its corresponding types of helices.

[8] L.P. Hughston and W.T. Shaw. Classical strings in ten dimensions. *Proc. Roy. Soc. London Ser. A*, 414:423–431, 1987.

[9] L.P. Hughston and W.T. Shaw. Constraint-free analysis of relativistic strings. *Classical Quatum Gravity*, 5:69–72, 1988.

[10] L.P. Hughston and W.T. Shaw. Spinor parametrizations of minimal surfaces. In *The Mathematics of Surfaces, III (Oxford, 1989)*, pages 359–372. Oxford Univ. Press, New York, 1989.

[11] William T. Shaw. Twistors and strings. In *Mathematics and General Relativity (Santa Cruz, CA, 1986)*, pages 337–363. Amer. Math. Soc., RI, 1988.

[12] H. Urbantke. On Pinl's representation of null curves in n dimensions. In *Relativity Today (Budapest, 1987)*, pages 34–36. World Sci. Publ., Teaneck, New York, 1988.

[13] Armen Nersessian and Eduardo Ramos. Massive spinning particles and the geometry of null curves. *Phys. Lett. B*, 445:123–128, 1998.

[14] Armen Nersessian and Eduardo Ramos. A geometrical particle model for anyons. *Modern Phys. Lett. A*, 14(29):2033–2037, 1999.

[15] Aurel Bejancu. Lightlike curves in Lorentz manifolds. *Publ. Math. Debrecen*, 44:145–155, 1994.

[16] Krishan L. Duggal and Aurel Bejancu. *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, volume 364 of *Mathematics and its Aplications*. Kluwer Academic Publishers Group, Dordrecht, The Netherlands, 1996.

[17] Manuel Barros, Angel Ferrández, Pascual Lucas, and Miguel A. Meroño. General helices in the 3-dimensional Lorentzian space forms. To appear in Rocky Mountain J. Math., 2001.