# Null generalized helices in Lorentz-Minkowski spaces 

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#### Abstract

We obtain a Lancret-type theorem for null generalized helices in Lorentz-Minkowski spaces. In the 3 -dimensional space we get that the only null generalized helices are the ordinary null helices. In the 5-dimensional space, we distinguish between null generalized helices with non-null or null axis and in both cases we solve the natural equations problem.


## 1 Introduction

The study of generalized helices in 3-dimensional Euclidean space $\mathbb{R}^{3}$ amounts to 1802 when M.A. Lancret stated that (see [13] for details) "a necessary and sufficient condition in order to a curve be a generalized helix is that its torsion is a constant multiple of its curvature". Here a generalized helix is a curve of constant slope, that is, a curve whose tangent indicatrix is a planar curve.

The $n$-dimensional case ( $n$ odd) was considered by Hayden in 1931 (see [8]) where he introduced the "generalized helix" as the curve defined by the properties

$$
\frac{\kappa_{n-1}}{\kappa_{n-2}}=\text { const. }, \quad \frac{\kappa_{n-3}}{\kappa_{n-4}}=\text { const. }, \quad \ldots, \quad \frac{\kappa_{2}}{\kappa_{1}}=\text { const. }
$$

where $\kappa_{1}, \kappa_{2}, \ldots, \kappa_{n-1}$ are the curvatures of the curve.
In $\mathbb{R}^{3}$, a generalized helix satisfies that its tangent makes a constant angle with a fixed direction (called the axis). In the general case, we must replace "fixed" direction by "parallel vector field". Hayden proved in $[8,9]$ that a curve is a generalized helix is there exists a parallel vector field lying in the osculating space of the curve and making constant angles with the tangent and the principal normals.

When the ambient space is a Lorentzian space form, some results have been obtained. For example, in [2], a non-null curve $\gamma$ immersed in $\mathbb{L}^{3}$ is called a generalized helix if its tangent indicatrix is contained in some plane, say $\pi$, of $\mathbb{L}^{3}$. Since $\pi$ can be either degenerate or non-degenerate, then both cases are distinguished by calling degenerate and

[^0]non-degenerate generalized helices, respectively. Then the authors give a sort of Lancret theorem for generalized helices in $\mathbb{L}^{3}$ which formally agrees with the classical one. In fact they proved that "generalized helices in $\mathbb{L}^{3}$ correspond with non-null curves in $\mathbb{L}^{3}$ for which the ratio of torsion to curvature is constant".

In the geometry of null curves difficulties arise because the arc length vanishes, so that it is not possible to normalize the tangent vector in the usual way. A method of proceeding is to introduce a new parameter called the pseudo-arc which normalizes the derivative of the tangent vector (see [4] and [16]). In [6] we generalize the results of Bonnor, since for a null curve in an $n$-dimensional Lorentzian space form we introduce a Frenet frame with the minimun number of curvature functions (which we call the Cartan frame), and then we study the null helices in those spaces, that is, null curves with constant curvatures. In this paper we use the Cartan frame (and the Cartan curvatures) introduced in [6] to define and study null generalized helices in the Lorentz-Minkowski spaces (some results for null curves in 3-dimensional spaces are obtained in [1]).

To point out the interest of generalized helices it should be mentioned that they arise in the context of the interplay between geometry and integrable Hamiltonian systems (see [10] and [11]). In [3] we have found parametrized solutions of the localized induction equation (LIE) in the 3-dimensional anti De Sitter space, so that the soliton solutions are the null geodesics of the Lorentzian Hopf cylinders. Therefore there is a natural geometric evolution on generalized helices inducing a mKdV curvature evolution equation coming from the LIE. The role of generalized helices here is probably similar to that of curves of constant torsion or constant natural curvature (see [11]).

Other applications of generalized helices can be found in [12], where the author proposed a mathematical model of the auditory process in the cochlea that doesn't neglect the effect of cochlea coiling, and [7], where the authors obtained, for inhomogeneous electromagnetic waves in isotropic media, the operator evolution solutions of Maxwell equations; in the case of homogeneous waves an evolution operator is associated with a set of righthanded and left-handed generalized helices.

This paper is organized as follows. First we remember the Frenet references for null curves in an orientable Lorentzian manifold and obtain similar equations for spacelike curves in lightlike totally geodesic submanifolds of a Lorentzian space form. By using those equations, in the next section we define the null generalized helices in odd dimensional spaces and obtain a Lancret-type theorem (see Theorem 5). In sect. 4 we get that the only generalized helices in 3-dimensional Lorentz-Minkowski space are the ordinary helices (Proposition 6); as for null generalized helices in 5 -dimensional Lorentz-Minkowski space we prove the following result (Theorem 8):
Let $\Gamma$ be a null Cartan curve in $\mathbb{R}_{1}^{5}, v \neq 0$ a constant unit vector (spacelike or timelike) and $\Sigma$ the orthogonal hyperplane (resp. timelike or spacelike) to $v$ in $\mathbb{R}_{1}^{5}$. Let $\Upsilon$ denotes the projection of $\Gamma$ onto $\Sigma$. Then $\Gamma$ is a generalized helix with axis $v$ if and only if $\Upsilon$ is a curve in $\Sigma$ with constant curvature and torsion.

From that result we can solve the natural equations problem for generalized helices with non-degenerate axis (Theorem 9):

Let $\Gamma$ be a null Cartan curve in $\mathbb{R}_{1}^{5}$. Then $\Gamma$ is a generalized helix with non-degenerate axis if and only if it is a null geodesic of a Lorentzian cylinder $C_{1}=\Upsilon \times \mathbb{R}_{1}^{1}$ or $C_{2}=\Upsilon \times \mathbb{R}^{1}$, where $\Upsilon$ is a non-degenerate curve (spacelike in $\mathbb{R}^{4}$ or timelike in $\mathbb{R}_{1}^{4}$ ) with constant curvature and torsion.

Finally, for null generalized helices $\Gamma$ with null axis, we prove the following (Theorem 10):

If $\Gamma$ is a generalized helix with null axis $v \neq 0$ and $\Sigma$ denotes its lightlike orthogonal hyperplane, then the curvatures $\rho_{1}, \rho_{2}$ and $\rho_{3}$ of the projected spacelike curve $\Upsilon$ are given by:

$$
\rho_{1}(s)=\frac{\tilde{r}}{\sqrt{s}}, \quad \rho_{3}=\tilde{r}_{1} \rho_{2},
$$

for certain constants $\tilde{r}$ and $\tilde{r}_{1}$. Conversely, if $\Upsilon$ is a spacelike curve in a lightlike hyperplane of $\mathbb{R}_{1}^{5}$ whose curvatures satisfy the above relations, then there exist a null generalized helix $\Gamma$ in $\mathbb{R}_{1}^{5}$ whose projection onto $\Sigma$ is just exactly $\Upsilon$.

The natural equations problem for null generalized helices with null axis can be solved as follows (Theorem 11):
Let $\Gamma$ be a null Cartan curve in $\mathbb{R}_{1}^{5}$. Then $\Gamma$ is a null generalized helix with null axis if and only if it is a geodesic in a Lorentzian surface $\Upsilon \times \mathbb{R}_{0}^{1}$, where $\Upsilon$ is a timelike generalized helix in $\mathbb{R}_{1}^{5}$ with null axis and with constant first curvature, and $\mathbb{R}_{0}^{1}$ stands for the direction of the axis.

## 2 Preliminaries

Let $E$ be a real vector space with a symmetric bilinear mapping $g: E \times E \rightarrow \mathbb{R}$. We say that $g$ is degenerate on $E$ if there exists a vector $\xi \neq 0$ in $E$ such that

$$
g(\xi, v)=0, \quad \text { for all } v \in E
$$

otherwise, $g$ is said to be non-degenerate. The radical (also called the null space) of $E$, with respect to $g$, is the subspace $\operatorname{rad}(E)$ of $E$ defined by

$$
\operatorname{rad}(E)=\{\xi \in E \mid g(\xi, v)=0, v \in E\} .
$$

The dimension of $\operatorname{rad}(E)$ is called the nullity degree of $g$ (or $E$ ) and is denoted by $r_{E}$.
If $F$ is a subspace of $E$, then we can consider $g_{F}$ the symmetric bilinear mapping on $F \times F$ obtained by restricting $g$ and define $r_{F}$ as the nullity degree of $F$ (or $g_{F}$ ). For simplicity, we will use $\langle$,$\rangle instead of g$ or $g_{F}$.

A vector $v$ is said to be timelike, lightlike or spacelike provided that $g(v, v)<0$, $g(v, v)=0$ (and $v \neq 0$ ), or $g(v, v)>0$, respectively. The vector $v=0$ is assumed to be spacelike. A unit vector is a vector $u$ such that $g(u, u)= \pm 1$.

Two vectors $u$ and $v$ are said to be orthogonal, written $u \perp v$, if $g(u, v)=0$. Similarly, two subsets $U$ and $V$ of $E$ are said to be orthogonal if $u \perp v$ for any $u \in U$ and $v \in V$.

Given two orthogonal subspaces $F_{1}$ and $F_{2}$ in $E$ with $F_{1} \cap F_{2}=\{0\}$, the orthogonal direct sum of $F_{1}$ and $F_{2}$ will be denoted by $F_{1} \perp F_{2}$.

Lemma 1 Let $(E,\langle\rangle$,$) be a bilinear space and let F$ be a hyperplane of $E$. Let $r_{F}=$ $\operatorname{dim} \operatorname{rad}(F)$ and $r_{E}=\operatorname{dim} \operatorname{rad}(E)$. Then the following statements hold:
(i) If $r_{F}=0$ and $r_{E}=1$, then there exists a null vector $L$ such that

$$
E=F \perp \operatorname{span}\{L\}
$$

(ii) If $r_{F}=r_{E} \in\{0,1\}$, then there exists a non-null unit vector $V$ such that

$$
E=F \perp \operatorname{span}\{V\}
$$

Moreover, if $\operatorname{rad}(E)=\{0\}$ then $V$ is unique, up to the sign.
(iii) If $r_{F}=1$ and $r_{E}=0$, and $F=F_{1} \perp L$, where $L \in \operatorname{rad}(F)$ and $F_{1}$ is non-degenerate, then there exists a unique null vector $N$ such that $\langle L, N\rangle=\varepsilon, \varepsilon= \pm 1$, and

$$
E=(\operatorname{span}\{L\} \oplus \operatorname{span}\{N\}) \perp F_{1} .
$$

Proof. We only need to make some algebraic computations.
(i) Since $F$ is non-degenerate, then $E=F \perp F^{\perp}$, where $F^{\perp}=\operatorname{span}\{L\}$ for a certain vector $L$. The inclusion $\operatorname{rad}(E) \subset F^{\perp}$ implies $\operatorname{rad}(E)=F^{\perp}$ and so $L$ is a null vector.
(ii) We may assume that $r_{F}=r_{E}=1$. By considering $F=F_{1} \perp \operatorname{span}\{L\}$, where $F_{1}$ is non-degenerate and $L$ is null, then $E=F_{1} \perp F_{1}^{\perp}$. Since $\operatorname{dim} F_{1}^{\perp}=2$, then $F_{1}^{\perp}=\operatorname{span}\{L\} \oplus \operatorname{span}\{V\}$, where $\operatorname{rad}(E)=\operatorname{span}\{L\}$ and $V$ is a non-null vector in $F^{\perp}$, so that the required splitting is fulfilled.
(iii) By a similar reasoning we may assume that $F=F_{1} \perp \operatorname{span}\{L\}$, where $F_{1}^{\perp}=\operatorname{span}\{L\} \oplus$ $\operatorname{span}\{V\}$. Since $\operatorname{rad}(E)=\{0\}$ then $\langle L, V\rangle \neq 0$. Let $N$ be the vector defined by

$$
N=\frac{\varepsilon}{\langle L, V\rangle}\left(V-\frac{\langle V, V\rangle}{2\langle L, V\rangle} L\right)
$$

It is easy to see that $N$ is the only vector satisfying $\langle N, N\rangle=0,\langle L, N\rangle=\varepsilon$ and $N \in F_{1}^{\perp}$, and the splitting follows.

## 3 Frenet references for null curves

Let $M_{1}^{n}$ be an orientable Lorentzian manifold and consider $\Gamma$ a null curve locally parametrized by $\gamma: I \subset \mathbb{R} \longrightarrow M_{1}^{n}$. Assume that $\left\{\gamma^{\prime}, \gamma^{\prime \prime}, \ldots, \gamma^{(n)}\right\}$ is linearly independent and positively oriented. Then following similar computations to that given in [6] we can construct
a (non unique) general Frenet reference satisfying the following equations:

$$
\begin{align*}
\gamma^{\prime} & =\bar{k}_{1} \bar{L}, \\
\bar{L}^{\prime} & =-\bar{k}_{2} \bar{L}+\bar{k}_{3} \bar{W}_{1}, \\
\bar{W}_{1}^{\prime} & =-\bar{k}_{4} \bar{L}+\bar{k}_{3} \bar{N}, \\
\bar{N}^{\prime} & =\bar{k}_{2} \bar{N}-\bar{k}_{4} \bar{W}_{1}+\bar{k}_{5} \bar{W}_{2}, \\
\bar{W}_{2}^{\prime} & =\bar{k}_{5} \bar{L}+\bar{k}_{6} \bar{W}_{3},  \tag{1}\\
\bar{W}_{i}^{\prime} & =-\bar{k}_{i+3} \bar{W}_{i-1}+\bar{k}_{i+4} \bar{W}_{i+1}, \quad 3 \leq i \leq m-2, \\
\bar{W}_{m-1}^{\prime} & =-\bar{k}_{m+2} \bar{W}_{m-2}+\bar{k}_{m+3} \bar{W}_{m}, \\
\bar{W}_{m}^{\prime} & =-\bar{k}_{m+3} \bar{W}_{m-1} .
\end{align*}
$$

where $\langle\bar{L}, \bar{L}\rangle=\langle\bar{N}, \bar{N}\rangle=0$ and $\langle\bar{L}, \bar{N}\rangle=-1$.
The set $F=\left\{\bar{L}, \bar{W}_{1}, \bar{N}, \bar{W}_{2}, \ldots, \bar{W}_{m}\right\}$ satisfying the above equations is called the Frenet reference on $M_{1}^{n}$ along $\Gamma$ with respect to the screen vector bundle span $\left\{\bar{W}_{i}\right\}$. The functions $\left\{\bar{k}_{1}, \ldots, \bar{k}_{m+3}\right\}$ are called the curvature functions of $\Gamma$ with respect to $F$. Those equations are called the Frenet equations of $\Gamma$ with respect to $F$.

The Frenet reference $F$ is said to be distinguished if $\bar{k}_{1}=1$ and $\bar{k}_{2}=0$. There are many distinguished Frenet references, but for each given parameter on the curve, we can uniquelly construct an associated distinguished reference. In particular, if we choose the pseudo-arc parameter, the associated distinguished Frenet reference is called the Cartan reference of the curve. We have the following result.

Theorem 2 ([6]) Let $\gamma: I \longrightarrow M_{1}^{n}, n=m+2$, be a null curve parametrized by the pseudo-arc such that $\left\{\gamma^{\prime}(t), \gamma^{\prime \prime}(t), \ldots, \gamma^{(n)}(t)\right\}$ is a basis of $T_{\gamma(t)} M_{1}^{n}$ for all $t$. Then there exists only one Frenet reference satisfying the equations

$$
\begin{align*}
L^{\prime} & =W_{1} \\
W_{1}^{\prime} & =-k_{1} L+N, \\
N^{\prime} & =-k_{1} W_{1}+k_{2} W_{2}, \\
W_{2}^{\prime} & =k_{2} L+k_{3} W_{3},  \tag{2}\\
W_{i}^{\prime} & =-k_{i} W_{i-1}+k_{i+1} W_{i+1} \quad i \in\{3, \ldots, m-1\}, \\
W_{m}^{\prime} & =-k_{m} W_{m-1},
\end{align*}
$$

and verifying
(i) $\left\{\gamma^{\prime}, \gamma^{\prime \prime}, \ldots, \gamma^{(i)}\right\}$ and $\left\{L, W_{1}, N, W_{2}, \ldots, W_{i-2}\right\}$ have the same orientation for $1 \leqslant$ $i \leqslant m-1$,
(ii) $\left\{L, W_{1}, N, W_{2}, \ldots, W_{m}\right\}$ is positively oriented.

Furthermore, the curvature functions satisfy $k_{i}>0$ for all $i \geq 2$.

## 4 Frenet references for spacelike curves in lightlike totally geodesic submanifolds of a Lorentzian space form

Let $N^{m}$ be a lightlike totally geodesic submanifold in an oriented Lorentzian manifold $M_{1}^{n}$ and consider a spacelike curve $\Upsilon$ locally parametrized by $\beta: J \subset \mathbb{R} \longrightarrow N^{m}$, with $s$ denoting the arc-length parameter. Let us assume that $\left\{\beta^{\prime}(s), \ldots, \beta^{(m)}(s)\right\}$ is a basis of $T_{\beta(s)} N^{m}$ for all $s \in J$. Write $E_{i}(s)=\operatorname{span}\left\{\beta^{\prime}(s), \ldots, \beta^{(i)}(s)\right\}$, for $1 \leq i \leq m$, and assume that $\operatorname{dim} \operatorname{rad}\left(E_{i}(s)\right)$ is constant for all $s \in J$. Since $\Upsilon$ is contained in a lightlike submanifold, there exist an index $1 \leq i_{0} \leq m$ satisfying $\operatorname{dim} \operatorname{rad}\left(E_{i_{0}}\right)=1$. Let us denote $r=\min \left\{i: \operatorname{dim} \operatorname{rad}\left(E_{i}\right)=1\right\}$, then $r>1$ because $\Upsilon$ is spacelike, and $\operatorname{dim} \operatorname{rad}\left(E_{j}\right)=1$ for $j>r$.

Now we are going to construct a Frenet reference for this kind of curves. The first vector of the reference is $T(s)=\beta^{\prime}(s)$. In a similar way as in the non-degenerate case, and using the Gram-Schmidt method applied to $E_{r-1}$, we can construct a set of orthonormal spacelike vectors $\left\{T, V_{1}, \ldots, V_{r-2}\right\}$ satisfying that $E_{j+1}=\operatorname{span}\left\{T, V_{1}, \ldots, V_{j}\right\}$, for $1 \leq j \leq$ $r-2$. As $\operatorname{dim} \operatorname{rad}\left(E_{r}\right)=1$, by using Lemma 1 we can find a vector $\mathcal{L}$ (not unique) such that

$$
E_{r}=E_{r-1} \perp \operatorname{span}\{\mathcal{L}\}=\operatorname{span}\left\{T, V_{1}, \ldots, V_{r-2}\right\} \perp \operatorname{span}\{\mathcal{L}\} .
$$

A straightforward computation shows that the following equations hold:

$$
\begin{align*}
\beta^{\prime}(s) & =T \\
T^{\prime} & =\rho_{1} V_{1} \\
V_{1}^{\prime} & =-\rho_{1} T+\rho_{2} V_{2}  \tag{3}\\
V_{i}^{\prime} & =-\rho_{i} V_{i-1}+\rho_{i+1} V_{i+1} \quad 2 \leq i \leq r-3 \\
V_{r-2}^{\prime} & =-\rho_{r-2} V_{r-3}+\rho_{r-1} \mathcal{L}
\end{align*}
$$

where $\rho_{j}: J \longrightarrow \mathbb{R}$ are differentiable functions and ( $)^{\prime}$ denotes covariant derivative in $N^{m}$, which in this case agrees with the covariant derivative of the ambient space $M_{1}^{n}$.

Our goal is to show that $r=m$. Let us assume that $r=m-1$, then $\operatorname{dim} \operatorname{rad}\left(E_{m-1}\right)=$ $\operatorname{dim} \operatorname{rad}\left(E_{m}\right)=1$. Taking into account Lemma 1 , there exist a unit spacelike vector $V_{r-1}$ such that $E_{m}=E_{m-1} \perp \operatorname{span}\left\{V_{r-1}\right\}$. By derivating we can add to (3) the following equations:

$$
\begin{align*}
\mathcal{L}^{\prime} & =\rho_{r} \mathcal{L}+\rho_{r+1} V_{r-1},  \tag{4}\\
V_{r-1}^{\prime} & =\rho_{r+2} \mathcal{L},
\end{align*}
$$

from which we deduce

$$
\rho_{r+1}=\left\langle\mathcal{L}^{\prime}, V_{r-1}\right\rangle=\frac{d}{d s}\left\langle\mathcal{L}, V_{r-1}\right\rangle-\left\langle\mathcal{L}, V_{r-1}^{\prime}\right\rangle=0,
$$

and then $\mathcal{L}^{\prime}=\rho_{r} \mathcal{L}$. Since $\mathcal{L} \in \operatorname{span}\left\{\beta^{\prime}, \ldots, \beta^{(r)}\right\}$, we can write $\mathcal{L}=\lambda_{1} \beta^{\prime}+\cdots+\lambda_{r} \beta^{(r)}$, with $\lambda_{r} \neq 0$, and then $\mathcal{L}^{\prime} \in \operatorname{span}\left\{\beta^{\prime}, \ldots, \beta^{(r)}\right\}$. We deduce $\beta^{(r+1)} \in \operatorname{span}\left\{\beta^{\prime}, \ldots, \beta^{(r)}\right\}$, which contradicts the hypothesis. Hence $r \neq m-1$.

Now let us assume $r<m-1$. Then we can find two unit spacelike vectors $V_{r-1}$ and $V_{r}$ satisfying

$$
E_{r+2}=E_{r+1} \perp \operatorname{span}\left\{V_{r}\right\}=E_{r} \perp \operatorname{span}\left\{V_{r-1}\right\} \perp \operatorname{span}\left\{V_{r}\right\},
$$

and such that $\mathcal{L}^{\prime}=\rho_{r} \mathcal{L}+\rho_{r+1} V_{r-1}$. Since $\mathcal{L} \in \operatorname{rad}\left(E_{r+2}\right)$, then $\left\langle\mathcal{L}, \beta^{(r+1)}\right\rangle=\left\langle\mathcal{L}, \beta^{(r+2)}\right\rangle=$ 0 , and hence $\left\langle\mathcal{L}^{\prime}, \beta^{(r+1)}\right\rangle=0$. The hypothesis implies that $\rho_{r+1} \neq 0$ and then $\left\langle V_{r-1}, \beta^{(r+1)}\right\rangle=$ 0 , that is equivalent $V_{r-1} \in \operatorname{rad}\left(E_{r+1}\right)$, which is a contradiction. Hence we conclude that $r=m$ and the Frenet reference is $\left\{T, V_{1}, \ldots, V_{m-2}, \mathcal{L}\right\}$, satisfying the following equations

$$
\begin{align*}
\beta^{\prime}(s) & =T \\
T^{\prime} & =\rho_{1} V_{1} \\
V_{1}^{\prime} & =-\rho_{1} T+\rho_{2} V_{2} \\
V_{i}^{\prime} & =-\rho_{i} V_{i-1}+\rho_{i+1} V_{i+1} \quad 2 \leq i \leq m-3  \tag{5}\\
V_{m-2}^{\prime} & =-\rho_{m-2} V_{m-3}+\rho_{m-1} \mathcal{L} \\
\mathcal{L}^{\prime} & =\rho_{m} \mathcal{L}
\end{align*}
$$

That reference, as in the non-degenerate case, can be constructed in a unique way (up to orientation) except $\mathcal{L}$. The vector $\mathcal{L}$ can be arbitrarily chosen depending on each situation; a good choice is $\rho_{m-1}= \pm 1$. In any case we need at least $m-1$ curvature functions in order to determine completely the curve. Moreover, the most natural criterion to choose the orientation is to consider that $\left\{\beta^{\prime}, \ldots, \beta^{(i+1)}\right\}$ and $\left\{T, V_{1}, \ldots, V_{i}\right\}, 1 \leq i \leq m-2$ have the same orientation, and that $\left\{T, V_{1}, \ldots, V_{m-2}, \mathcal{L}\right\}$ is positively oriented.

The following theorems of existence, uniqueness and congruence can be proved in a similar way as in [6]. Now $M_{1}^{n}(c)$ denotes a Lorentzian space form of constant curvature c.

Theorem 3 Let $\rho_{1}, \rho_{2}, \ldots, \rho_{m}:[-\varepsilon, \varepsilon] \longrightarrow \mathbb{R}$ be differentiable functions. Let $N^{m}$ be a lightlike totally geodesic submanifold of $M_{1}^{n}(c), p$ a point in $N^{m}$ and consider a positively oriented pseudo-orthonormal basis $\left\{T^{0}, V_{1}^{0}, \ldots, V_{m-2}^{0}, \mathcal{L}^{0}\right\}$ of $T_{p} N^{m}$. Then there exists a unique spacelike Cartan curve $\alpha$ in $M_{1}^{n}(c)$, contained in $N^{m}$ with $\alpha(0)=p$, whose Cartan reference $\left\{T, V_{1}, \ldots, V_{m-2}, \mathcal{L}\right\}$ satisfies

$$
T(0)=T^{0}, V_{1}(0)=V_{1}^{0}, \ldots, V_{m-2}(0)=V_{m-2}^{0}, \mathcal{L}(0)=\mathcal{L}^{0}
$$

Theorem 4 If two spacelike Cartan curves $C$ and $\bar{C}$ in $M_{1}^{n}(c)$, contained in a lightlike totally geodesic submanifold $N^{m}$, have Cartan curvatures in $N^{m}\left\{\rho_{1}, \ldots, \rho_{m}\right\}$, where $\rho_{i}:[-\varepsilon, \varepsilon] \longrightarrow \mathbb{R}$ are differentiable functions, then there exists a Lorentzian transformation of $M_{1}^{n}$ (or of $N^{m}$ ) which maps $C$ into $\bar{C}$.

## 5 Null generalized helices in the Lorentz-Minkowski space

Let $\Gamma \subset \mathbb{R}_{1}^{n}, n=2 q+3$, be a null Cartan curve locally parametrized by $\gamma: I \longrightarrow \mathbb{R}_{1}^{n}$, and consider $m=n-2=2 q+1$. In a similar way to the non-degenerate case (see [8], [9], [14], [15], [17], [18]), we present the following definition.

Definition 1 A null Cartan curve $\gamma: I \longrightarrow \mathbb{R}_{1}^{n}, n=2 q+3$, is said to be a generalized helix if there exist a non-zero constant vector $v$ such that the products $\left\langle\gamma^{\prime}(t), v\right\rangle \neq 0$, $\langle N(t), v\rangle$ and $\left\langle W_{2 i+1}(t), v\right\rangle \neq 0,1 \leq i \leq q-1$, are constant.

In the non-degenerate case, the above vectors appearing in the definition are unitary and the constancy of those products imply that the curve $\gamma$ makes a constant angle with some of the vectors of the Cartan reference. The straight line generated by $v$ is uniquely determined and will be called the axis of $\gamma$. That line can be spacelike, timelike or lightlike. When $v$ is a non-null vector (i.e. spacelike or timelike), we can assume without loss of generality that $v$ is unitary.

A classical result stated by M.A. Lancret in 1802 and first proved by B. de Saint Venant in 1845 (see [13] for details) says that "a curve in $\mathbb{R}^{3}$ is a generalized helix if and only if the ratio of curvature to torsion is constant". The following proposition is a generalization of this result to null curves in Lorentz-Minkowski spaces.

Theorem 5 (The Lancret theorem for null curves) Let $\gamma: I \longrightarrow \mathbb{R}_{1}^{n}$ be a null Cartan curve. Then the following statements are equivalent:
(i) There exist constants $\left\{r, r_{1}, \ldots, r_{q}\right\}\left(r_{i} \neq 0\right)$ such that

$$
k_{1}(t)=r \quad \text { and } \quad k_{2 i+1}(t)=r_{i} k_{2 i}(t) \quad \text { for } 1 \leq i \leq q, \quad n=2 q+3
$$

(ii) $\gamma$ is a generalized helix.

Proof. "(i) $\Rightarrow$ (ii)" Bearing in mind (2) we can write

$$
\begin{align*}
L^{\prime} & =W_{1} \\
N^{\prime} & =-k_{1} W_{1}+k_{2} W_{2} \\
W_{2 i+1}^{\prime} & =-k_{2 i+1} W_{2 i}+k_{2 i+2} W_{2 i+2}, \quad 1 \leq i \leq q-1  \tag{6}\\
W_{m}^{\prime} & =-k_{m} W_{m-1}
\end{align*}
$$

Let us denote $R_{i}=\prod_{j \geq i} r_{j}$, and consider the vector field along the curve $\gamma$ given by

$$
v(t)=R_{1}(r L+N)+\sum_{i=1}^{q-1} R_{i+1} W_{2 i+1}+W_{m}
$$

Taking into account (6), statement (i) and that $r_{i} R_{i+1}=R_{i}$, we have the following:

$$
\begin{aligned}
\frac{d v}{d t}(t)= & R_{1}\left(r L^{\prime}+N^{\prime}\right)+\sum_{i=1}^{q-1} R_{i+1} W_{2 i+1}^{\prime}+W_{m}^{\prime} \\
= & R_{1}\left(r W_{1}-k_{1} W_{1}+k_{2} W_{2}\right)+\sum_{i=1}^{q-1}\left(-R_{i} k_{2 i} W_{2 i}+R_{i+1} k_{2 i+2} W_{2 i+2}\right)-r_{q} k_{m-1} W_{m-1} \\
= & R_{1} k_{2} W_{2}+\left(-R_{1} k_{2} W_{2}+R_{2} k_{4} W_{4}\right)+\left(-R_{2} k_{4} W_{4}+R_{3} k_{6} W_{6}\right) \\
& +\cdots+\left(-R_{\frac{m-3}{2}} k_{m-3} W_{m-3}+R_{\frac{m-1}{2}} k_{m-1} W_{m-1}\right)-R_{\frac{m-1}{2}} k_{m-1} W_{m-1} \\
= & 0
\end{aligned}
$$

showing that $v(t)$ is a constant vector. Moreover, it is easy to see that $\langle v, L\rangle,\langle v, N\rangle$ and $\left\langle v, W_{2 i+1}\right\rangle$ are constant, and that concludes the first part of the proof.
"(ii) $\Rightarrow$ (i)" Now there exist a non-zero constant vector $v$ satisfying the conditions of the definition. By derivating we have

$$
0=\frac{d}{d t}\langle L, v\rangle=\left\langle L^{\prime}, v\right\rangle=\left\langle W_{1}, v\right\rangle
$$

which implies that $v \in \operatorname{span}\left\{W_{1}(t)\right\}^{\perp}$ for all $t \in I$. On the other and,

$$
0=\frac{d}{d t}\langle N, v\rangle=\left\langle-k_{1} W_{1}+k_{2} W_{2}, v\right\rangle=k_{2}\left\langle W_{2}, v\right\rangle .
$$

Since $k_{2} \neq 0$, that equation yields $v \in \operatorname{span}\left\{W_{2}(t)\right\}^{\perp}$ for all $t \in I$. Finally, applying the same ideas we have

$$
0=\frac{d}{d t}\left\langle v, W_{2 i+1}\right\rangle=k_{2 i+2}\left\langle v, W_{2 i+2}\right\rangle=0, \quad 1 \leq i \leq q-1,
$$

and then $v \in \operatorname{span}\left\{W_{2 i+2}\right\}^{\perp}, 1 \leq i \leq q-1$. Then we obtain

$$
v \in(\operatorname{span}\{L\} \oplus \operatorname{span}\{N\}) \perp\left(\bigoplus_{1 \leq i \leq q} \operatorname{span}\left\{W_{2 i+1}\right\}\right)
$$

so that we can write

$$
v=\varrho L+\varrho_{1} N+\sum_{i=1}^{q} \varrho_{2 i+1} W_{2 i+1},
$$

for certain functions $\varrho$ and $\varrho_{j}$. From the hypothesis we deduce that $\varrho$ and $\varrho_{j}$ are constant, for $j<m$. But for $j=m$ we have

$$
\frac{d \varrho_{m}}{d t}=\frac{d}{d t}\left\langle v, W_{m}\right\rangle=-k_{m}\left\langle v, W_{m-1}\right\rangle=0,
$$

showing that $\varrho_{m}$ is also constant. All these computations lead to the following

$$
\begin{aligned}
0=\frac{d v}{d t}= & \varrho L^{\prime}+\varrho_{1} N^{\prime}+\sum_{i=1}^{q} \varrho_{2 i+1} W_{2 i+1}^{\prime} \\
= & \varrho W_{1}+\varrho_{1}\left(-k_{1} W_{1}+k_{2} W_{2}\right)+\sum_{i=1}^{q-1} \varrho_{2 i+1}\left(-k_{2 i+1} W_{2 i}+k_{2 i+2} W_{2 i+2}\right)-\varrho_{m} k_{m} W_{m-1} \\
= & \left(\varrho-\varrho_{1} k_{1}\right) W_{1}+\left(\varrho_{1} k_{2}-\varrho_{3} k_{3}\right) W_{2}+\left(\varrho_{3} k_{4}-\varrho_{5} k_{5}\right) W_{4} \\
& +\cdots+\left(\varrho_{m-2} k_{m-1}-\varrho_{m} k_{m}\right) W_{m_{1}} .
\end{aligned}
$$

But that equation holds if and only if

$$
k_{1}=\frac{\varrho}{\varrho_{1}}=r, \quad k_{2 i+1}=\frac{\varrho_{2 i-1}}{\varrho_{2 i+1}} k_{2 i}=r_{i} k_{2 i} .
$$

Observe that, since $\gamma$ is fully immersed, the constants $\varrho_{i} \neq 0$ for all $i$.

## 6 Null generalized helices in low dimensions

## The 3-dimensional case

Let $\gamma: I \longrightarrow \mathbb{R}_{1}^{3}$ be a null Cartan curve with reference $\{L, W, N\}$. The Cartan equations write down as follows:

$$
\begin{align*}
L^{\prime} & =W, \\
W^{\prime} & =-k L+N  \tag{7}\\
N^{\prime} & =-k W .
\end{align*}
$$

Then $\gamma: I \longrightarrow \mathbb{R}_{1}^{3}$ is a generalized helix if there exist a constant vector $v \neq 0$ such that $\left\langle\gamma^{\prime}, v\right\rangle$ is constant. That means that the tangent indicatrix lies in a plane, or equivalently, there exists a vector $v \neq 0$ in $\mathbb{R}_{1}^{3}$ which is orthogonal to the acceleration vector field of $\gamma$. The following result is an easy consequence of Theorem 5 .

Proposition 6 Let $\gamma: I \longrightarrow \mathbb{R}_{1}^{3}$ be a null Cartan curve. Then $\gamma$ is a generalized helix if and only if $\gamma$ is a Cartan helix.

It is well-known that, up to congruences, there are exactly three types of helices, according to its curvature function (or its axis):

| Curve | Curvature | Axis |
| :--- | :--- | :--- |
| $\gamma(t)=\left(-\frac{t}{\sigma}, \frac{1}{\sigma^{2}} \sin \sigma t, \frac{1}{\sigma^{2}} \cos \sigma t\right)$ | $k=\frac{1}{2} \sigma^{2}>0$ | $v=(1,0,0)$ timelike |
| $\gamma(t)=\left(\frac{1}{\omega^{2}} \sinh \omega t, \frac{1}{\omega^{2}} \cosh \omega t,-\frac{t}{\omega}\right)$ | $k=-\frac{1}{2} \omega^{2}<0$ | $v=(0,0,1)$ spacelike |
| $\gamma(t)=\left(\frac{t^{3}}{4}+\frac{t}{3}, \frac{t^{2}}{2}, \frac{t^{3}}{4}-\frac{t}{3}\right)$ | $k=0$ | $v=(1,0,1)$ null. |

## The 5-dimensional case with non-null axis

Let $\Gamma$ be a null Cartan curve in $\mathbb{R}_{1}^{5}$ with Cartan reference $\left\{L, W_{1}, N, W_{2}, W_{3}\right\}$. In this case, $\Gamma$ is a generalized helix if there exist a constant vector $v \neq 0$ satisfying that $\langle v, L\rangle=\lambda$ and $\langle v, N\rangle=\lambda_{1}$ are constant. The Cartan equations in this situation write down as follows:

$$
\begin{align*}
L^{\prime} & =W_{1}, \\
W_{1}^{\prime} & =-k_{1} L+N, \\
N^{\prime} & =-k_{1} W_{1}+k_{2} W_{2},  \tag{8}\\
W_{2}^{\prime} & =k_{2} L+k_{3} W_{3}, \\
W_{3}^{\prime} & =-k_{3} W_{2} .
\end{align*}
$$

Consider $v \neq 0$ a non-zero unit vector (spacelike or timelike), and put $\langle v, v\rangle=\varepsilon$. Let $\Sigma$ denotes the orthogonal hyperplane (resp. timelike or spacelike) to $v$. Then we have the
following decomposition:

$$
\mathbb{R}_{1}^{5}=\operatorname{span}\{v\} \perp \operatorname{span}\{v\}^{\perp}=\operatorname{span}\{v\} \perp \Sigma
$$

Let $P$ denotes the projection map onto the hyperplane $\Sigma$ and let $\Upsilon$ be the curve in $\Sigma$ obtained by projecting $\Gamma$, i.e. $\Upsilon=P(\Gamma)$. Let $\gamma: I \longrightarrow \mathbb{R}_{1}^{5}$ be the pseudo-arc parametrization of $\gamma$ and denote $\bar{\beta}: I \longrightarrow \Sigma$ the parametrization of $\Upsilon$ with respect to the same parameter $t$. Then we can write

$$
\begin{equation*}
\gamma(t)=\bar{\beta}(t)+\bar{\mu}(t) v \tag{9}
\end{equation*}
$$

where $\bar{\mu}: I \longrightarrow \mathbb{R}$ is a non constant differentiable function. Taking derivatives in the last equation we have

$$
L(t)=\bar{\beta}^{\prime}(t)+\bar{\mu}^{\prime}(t) v
$$

and by multiplication we obtain

$$
\left\langle\bar{\beta}^{\prime}(t), \bar{\beta}^{\prime}(t)\right\rangle=-\bar{\mu}^{\prime}(t)^{2} \varepsilon
$$

Then the projected curve $\Upsilon$ is spacelike (resp. timelike) according to $v$ is timelike (resp. spacelike). Let $\beta: J \longrightarrow \Sigma$ be the arc-length parametrization of $\Upsilon$ and let $s$ denotes the arc-length parameter. Let us consider its Frenet reference $\left\{\ell(s), n_{1}(s), n_{2}(s), n_{3}(s)\right\}$ with $\ell(s)=\beta^{\prime}(s)$ and $\langle\ell(s), \ell(s)\rangle=-\varepsilon$, satisfying the following Frenet equations:

$$
\begin{align*}
\ell^{\prime} & =\tilde{k}_{1} n_{1} \\
n_{1}^{\prime} & =\varepsilon \tilde{k}_{1} \ell+\tilde{k}_{2} n_{2} \\
n_{2}^{\prime} & =-\tilde{k}_{2} n_{1}+\tilde{k}_{3} n_{3}  \tag{10}\\
n_{3}^{\prime} & =-\tilde{k}_{3} n_{2}
\end{align*}
$$

The following result relates the curvatures of the null curve $\Gamma$ in $\mathbb{R}_{1}^{5}$ with the curvatures of its projection $\Upsilon$ in a hyperplane $\Sigma$.

Lemma 7 Let $\Gamma$ be a null curve in the 5 -dimensional Lorentz-Minkowski $\mathbb{R}_{1}^{5}$ and consider $\Upsilon$ the orthogonal projection of $\Gamma$ onto a non-degenerate hyperplane $\Sigma$. Let us denote by $t$ and $s$ the pseudo-arc and arc parameters of $\Gamma$ and $\Upsilon$, respectively. Then $s$ and $t$ are linearly related if and only if the first curvature $\tilde{k}_{1}$ of $\Upsilon$ is constant. Moreover, in this case, $k_{1}$ is constant if and only if $\tilde{k}_{2}$ is constant.

Proof. With respect to $s$, equation (9) writes down as follows

$$
\gamma(t(s))=\beta(s)+\mu(s) v
$$

where $\beta(s)=\bar{\beta}(t(s))$ and $\mu(s)=\bar{\mu}(t(s))$. By taking derivatives with respect to $s$ we have:

$$
\begin{equation*}
L(t(s)) t^{\prime}(s)=\ell(s)+\mu^{\prime}(s) v \tag{11}
\end{equation*}
$$

which implies

$$
0=-\varepsilon+\mu^{\prime}(s)^{2} \varepsilon=\varepsilon\left(\mu^{\prime}(s)^{2}-1\right)
$$

and so $\mu(s)$ is a linear function. By derivating again we deduce

$$
W_{1}(t(s)) t^{\prime}(s)^{2}+L(t(s)) t^{\prime \prime}(s)=\tilde{k}_{1} n_{1}
$$

that yields

$$
t^{\prime}(s)^{4}=\tilde{k}_{1}(s)^{2}
$$

This shows the first part of Lemma. The last claim of Lemma follows from the fact that $t^{\prime \prime}(s)=0$ and $\tilde{k}_{1}$ is constant; an easy and similar reasoning yields the conclusion.

We are in position to state a result that relates null generalized helices in $\mathbb{R}_{1}^{5}$ with non-degenerate axis and non-degenerate curves in a hyperplane of $\mathbb{R}_{1}^{5}$.

Theorem 8 Let $\Gamma$ be a null Cartan curve in $\mathbb{R}_{1}^{5}, v \neq 0$ a constant unit vector (spacelike or timelike) and $\Sigma$ the orthogonal hyperplane (resp. timelike or spacelike) to $v$ in $\mathbb{R}_{1}^{5}$. Let $\Upsilon$ denotes the projection of $\Gamma$ onto $\Sigma$. Then $\Gamma$ is a generalized helix with axis $v$ if and only if $\Upsilon$ is a curve in $\Sigma$ with constant curvature and torsion.

Proof. From Lemma 7, to prove the first implication we only need to show that $s$ and $t$ are linearly related. From equation (11) we obtain $\lambda t^{\prime}(s)=\varepsilon \mu^{\prime}(s)=\varepsilon$ and since $\langle L, v\rangle=\lambda$ is constant we deduce that $t(s)$ is a linear function.

Conversely, let us assume that $\tilde{k}_{1}$ and $\tilde{k}_{2}$ are constant. From Lemma 7 we have that $t$ and $s$ are linearly related, and $k_{1}$ is constant. Now we are going to find a constant $r$ such that $k_{3}=r k_{2}$. A straightforward computation leads to the following expressions for the curvature functions:

$$
\begin{aligned}
k_{1}(t(s)) & =-\frac{\tilde{k}_{2}^{2}-\varepsilon \tilde{k}_{1}^{2}}{2 \tilde{k}_{1}} \\
k_{2}(t(s))^{2} & =\frac{\tilde{k}_{2}^{2}}{\tilde{k}_{1}^{2}} \tilde{k}_{3}(s)^{2} \\
k_{3}(t(s))^{2} & =\frac{1}{\tilde{k}_{1}} \tilde{k}_{3}(s)^{2}
\end{aligned}
$$

Hence $k_{3}=r k_{2}$ with $r^{2}=\frac{\tilde{k}_{1}}{\widehat{k}_{2}^{2}}$.
The following result can be deduced from the last theorem.
Theorem 9 (Solving natural equation for generalized helices with non-degenerate axis.) Let $\Gamma$ be a null Cartan curve in $\mathbb{R}_{1}^{5}$. Then $\Gamma$ is a generalized helix with non-degenerate axis if and only if it is a null geodesic of a Lorentzian cylinder $C_{1}=\Upsilon \times \mathbb{R}_{1}^{1}$ or $C_{2}=\Upsilon \times \mathbb{R}^{1}$, where $\Upsilon$ is a non-degenerate curve (spacelike in $\mathbb{R}^{4}$ or timelike in $\mathbb{R}_{1}^{4}$ ) with constant curvature and torsion.

## The 5-dimensional case with null axis

In the following we are going to study null generalized helices with null axis. The main difficulty now is that the orthogonal hyperplane to the axis is also lightlike and then we can project in different ways.

Let $v$ be the axis of the helix and $\Sigma$ the orthogonal hyperplane, so that $v \in \Sigma$. By the general theory of lightlike hypersurfaces (see [5]) we have the decomposition

$$
T_{p} \mathbb{R}_{1}^{5}=T_{p} \Sigma \oplus \operatorname{tr}\left(T_{p} \Sigma\right)=\left(\operatorname{span}\{v\} \perp S\left(T_{p} \Sigma\right)\right) \oplus \operatorname{tr}\left(T_{p} \Sigma\right), \quad \forall p \in \Sigma
$$

where $\operatorname{tr}(T \Sigma)=\bigcup_{p \in \Sigma} \operatorname{tr}\left(T_{p} \Sigma\right)$ is called a screen transversal vector bundle and $S(T \Sigma)=$ $\bigcup_{p \in \Sigma} S\left(T_{p} \Sigma\right)$ is called a screen distribution. Then each choice of a screen distribution provides a projection map on $\Sigma$, so that the problem is finding a canonical screen distribution (or the most canonical screen distribution in some sense).

Let $\Gamma$ be a null generalized helix with null axis $v$ locally parametrized by $\gamma: I \longrightarrow \mathbb{R}_{1}^{5}$. Since $\langle L, v\rangle=\lambda$ is constant, where $L$ is the tangent vector, then $\tilde{L}=\frac{1}{\lambda} L$ is a transversal section along $\Gamma$ satisfying $\langle\tilde{L}, v\rangle=1$. Let $\Upsilon$ denotes the projection of $\Gamma$ with respect to $\tilde{L}$, which is locally parametrized by

$$
\bar{\beta}(t)=\gamma(t)-\langle\gamma(t), v\rangle \tilde{L}(t)
$$

Since $\left\langle\gamma^{\prime}(t), v\right\rangle=\lambda$, then $\langle\gamma(t), v\rangle=\lambda(t+\sigma)$ where $\sigma$ is a constant, from which we have

$$
\begin{equation*}
\bar{\beta}(t)=\gamma(t)-(t+\sigma) L(t) \tag{12}
\end{equation*}
$$

The Frenet equations (5) write down as follows

$$
\begin{align*}
T^{\prime} & =\rho_{1} V_{1} \\
V_{1}^{\prime} & =-\rho_{1} T+\rho_{2} V_{2}  \tag{13}\\
V_{2}^{\prime} & =-\rho_{2} V_{1}+\rho_{3} v \\
v^{\prime} & =0
\end{align*}
$$

where we have choosen $\mathcal{L}=v$.

Theorem 10 Let $\Gamma$ be a null Cartan curve in $\mathbb{R}_{1}^{5}$. If $\Gamma$ is a generalized helix with null axis $v \neq 0$ and $\Sigma$ denotes its lightlike orthogonal hyperplane, then the curvatures $\rho_{1}, \rho_{2}$ and $\rho_{3}$ of the projected spacelike curve $\Upsilon$, obtained as in (12), are given by:

$$
\begin{equation*}
\rho_{1}(s)=\frac{\tilde{r}}{\sqrt{s}}, \quad \rho_{3}=\tilde{r}_{1} \rho_{2} \tag{14}
\end{equation*}
$$

for certain constants $\tilde{r}$ and $\tilde{r}_{1}$. Conversely, if $\Upsilon$ is a spacelike curve in a lightlike hyperplane of $\mathbb{R}_{1}^{5}$ whose curvatures satisfy (14), then there exist a null generalized helix $\Gamma$ in $\mathbb{R}_{1}^{5}$ whose projection onto $\Sigma$ is just exactly $\Upsilon$.

Proof. Let $s$ denotes the arc-length parameter of $\Upsilon$, then (12) can be rewritten as

$$
\beta(s)=\gamma(t(s))-(t(s)+\sigma) L(t(s))
$$

where $t$ stands for the pseudo-arc parameter of $\Gamma$. Taking derivatives and using the Frenet equations we get

$$
T(s)=-(t(s)+\sigma) t^{\prime}(s) W_{1}(t(s))
$$

from which we deduce that

$$
t(s)=-\sigma+\sqrt{\sigma^{2}+2(s+\omega)}
$$

for a constant $\omega$, and so $t^{\prime}(s)=1 / \sqrt{\sigma^{2}+2(s+\omega)}$. Without loss of generality, let us assume that $\sigma=\omega=0$. By derivating, taking into account the Frenet equations and using that $k_{1}$ is constant and $k_{3}=r_{1} k_{2}$, with $r_{1}$ constant, we deduce the following formulae:

$$
\rho_{1}(s)=\frac{\sqrt{k_{1}}}{\sqrt{s}}, \quad \rho_{2}(s)=\frac{k_{2}(s)}{2 \sqrt{k_{1}} \sqrt{s}}, \quad \rho_{3}(s)=-\frac{r_{1} k_{2}(s)}{\sqrt{2 s}} .
$$

Then we can take $\tilde{r}=\sqrt{k_{1}}$ and $\tilde{r}_{1}=-\sqrt{2 k_{1}} r_{1}$, and this concludes the proof.
Conversely, let $\Upsilon$ be a spacelike curve in a lightlike hyperplane $\Sigma$ locally parametrized by $\beta: J \longrightarrow \Sigma$. Put $\Sigma=\operatorname{span}\{v\}^{\perp}$, where $v$ is a null vector. Let $\left\{T, V_{1}, V_{2}, \mathcal{L}=v\right\}$ be the Frenet reference satisfying (13), where the curvatures verify (14) ( $s$ denoting the arc-length parameter). The Frenet reference $\left\{T, V_{1}, V_{2}, \mathcal{L}=v\right\}$ can be completed (in a unique way) to a basis of $T_{\beta(s)} \mathbb{R}_{1}^{5}$, for all $s \in J$, by adding a vector field $\mathcal{N}(s)$ along $\beta(s)$ satisfying

$$
\langle\mathcal{L}, \mathcal{N}\rangle=-1, \quad\langle T, \mathcal{N}\rangle=\left\langle V_{1}, \mathcal{N}\right\rangle=\left\langle V_{2}, \mathcal{N}\right\rangle=\langle\mathcal{N}, \mathcal{N}\rangle=0 .
$$

An easy computation shows that $\mathcal{N}^{\prime}(s)=\rho_{3}(s) V_{2}(s)=\tilde{r}_{1} \rho_{2}(s) V_{2}(s)$. Then it is a straightforward computation to see that the curve

$$
\bar{\gamma}(s)=\beta(s)+\frac{\sqrt{s}}{\tilde{r}}\left(V_{1}(s)-\frac{\tilde{r}_{1}}{2} v-\frac{1}{\tilde{r}_{1}} \mathcal{N}(s)\right)
$$

is a parametrization of a null generalized helix $\Gamma$ with axis $v$. First, $\bar{\gamma}$ is a null curve since

$$
\bar{\gamma}^{\prime}(s)=\frac{1}{2 \tilde{r} \sqrt{s}}\left(V_{1}(s)-\frac{\tilde{r}_{1}}{s} v-\frac{1}{\tilde{r}_{1}} \mathcal{N}(s)\right) .
$$

Now let $t$ be the pseudo-arc parameter of $\Gamma$ and put $\bar{\gamma}(s)=\gamma(t(s))$. By taking derivatives we get

$$
\begin{aligned}
\bar{\gamma}^{\prime \prime}(s) & =\gamma^{\prime \prime}(t(s)) t^{\prime}(s)^{2}+\gamma^{\prime}(t(s)) t^{\prime \prime}(s) \\
& =-\frac{1}{2 s} T(s)+\frac{1}{4 \tilde{r} s \sqrt{s}}\left(-V_{1}(s)+\frac{\tilde{r}_{1}}{2} v+\frac{1}{\tilde{r}_{1}} \mathcal{N}(s)\right),
\end{aligned}
$$

from which we deduce

$$
\left\langle\bar{\gamma}^{\prime \prime}(s), \bar{\gamma}^{\prime \prime}(s)\right\rangle=t^{\prime}(s)^{4}=\frac{1}{4 s^{2}},
$$

and so $t^{\prime}(s)=1 / \sqrt{2 s}$. From here, a long and messy computation yields the Frenet reference of $\Gamma$ :

$$
\begin{aligned}
L & =\frac{1}{\sqrt{2} \tilde{r}}\left(V_{1}-\frac{\tilde{r}_{1}}{2} v-\frac{1}{\tilde{r}_{1}} \mathcal{N}\right), \\
W_{1} & =-T, \\
N & =\frac{\tilde{r}}{\sqrt{2}}\left(V_{1}-\frac{\tilde{r}_{1}}{2} v-\frac{1}{\tilde{r}_{1}} \mathcal{N}\right), \\
W_{2} & =-V_{2}, \\
W_{3} & =-\frac{\tilde{r}_{1}}{2}+\frac{1}{\tilde{r}_{1}} \mathcal{N} .
\end{aligned}
$$

These equations imply that $\Gamma$ is a null generalized helix with null axis $v$, since $\langle L, v\rangle=$ $\frac{1}{\sqrt{2} \tilde{r} \tilde{r}_{1}}$ and $\langle N, v\rangle=\frac{\tilde{r}}{\sqrt{2} \tilde{r}_{1}}$ are constant. Moreover, the curvatures are given by

$$
k_{1}=\tilde{r}^{2}, \quad k_{2}=2 \tilde{r} \sqrt{s} \rho_{2}, \quad k_{3}=\sqrt{2} \sqrt{s} \rho_{2}
$$

which concludes the proof.
Now let $\Gamma$ be a null generalized helix with null axis, whose curvatures satisfy $k_{1}(t)=r$ and $k_{3}(t)=r_{1} k_{2}(t)$, then the axis is given by

$$
v=-\frac{1}{2}\left(r L+N+\frac{1}{r_{1}} W_{3}\right)
$$

where $r=\frac{1}{2 r_{1}^{2}}$.
Let us consider a timelike curve $\Upsilon$ in $\mathbb{R}_{1}^{5}$ parametrized by $\beta: J \longrightarrow \mathbb{R}_{1}^{5}$, with Frenet reference $\left\{\ell, n_{1}, n_{2}, n_{3}, n_{4}\right\}$ satisfying the following equations:

$$
\begin{align*}
\ell^{\prime} & =\tilde{k}_{1} n_{1} \\
n_{1}^{\prime} & =\tilde{k}_{1} \ell+\tilde{k}_{2} n_{2} \\
n_{2}^{\prime} & =-\tilde{k}_{2} n_{1}+\tilde{k}_{3} n_{3}  \tag{15}\\
n_{3}^{\prime} & =-\tilde{k}_{3} n_{2}+\tilde{k}_{4} n_{4} \\
n_{4}^{\prime} & =-\tilde{k}_{4} n_{3}
\end{align*}
$$

with $\ell(s)=\beta^{\prime}(s), s$ standing for the arc parameter. Following [9], the curve $\Upsilon$ is said to be a generalized helix if there exist a constant vector $v \neq 0$ such that the products $\left\langle\beta^{\prime}, v\right\rangle$ and $\left\langle n_{2}, v\right\rangle$ are constant. In this case, the Lancret theorem assures us that $\tilde{k}_{2}=\tilde{r}_{1} \tilde{k}_{1}$ and $\tilde{k}_{4}=\tilde{r}_{3} \tilde{k}_{3}$. Moreover, the axis is given by

$$
v=\frac{1}{2}\left(\ell+\frac{1}{\tilde{r}_{1}} n_{2}+\frac{1}{\tilde{r}_{1} \tilde{r}_{3}} n_{4}\right) .
$$

If the axis $v$ is null $(\langle v, v\rangle=0)$, then we have $\tilde{r}_{1}=\sqrt{1+1 / \tilde{r}_{3}^{2}}$.
Let us consider the surface $S$ locally parametrized by

$$
\begin{equation*}
X(s, \omega)=\beta(t)+\omega v \tag{16}
\end{equation*}
$$

then

$$
\frac{\partial X}{\partial s}(s, \omega)=\ell(s), \quad \frac{\partial X}{\partial \omega}=v
$$

showing that $S$ is a Lorentzian surface of $\mathbb{R}_{1}^{5}$. The null geodesics $\Gamma$ in $S$ can be parametrized by

$$
\begin{equation*}
\bar{\gamma}(s)=\beta(s)-(s+\sigma) v \tag{17}
\end{equation*}
$$

where $\sigma$ is constant. Let $t$ be the pseudo-arc parameter of $\Gamma$ (as a curve of $\mathbb{R}_{1}^{5}$ ), then $\bar{\gamma}(s)=\gamma(t(s))$. A long and messy computation, from equation (17), yields the following
relations among the Cartan curvatures of $\Gamma$ and the generalized helix $\Upsilon$ :

$$
\begin{aligned}
& k_{1}(t(s))=\frac{1}{2} \frac{\tilde{k}_{1}(s)}{\tilde{r}_{3}^{2}}-\frac{7}{8} \tilde{k}_{1}^{\prime}(s)^{2} \\
& \tilde{k}_{1}(s)^{3}
\end{aligned} \frac{1}{2} \frac{\tilde{k}_{1}^{\prime \prime}(s)}{\tilde{k}_{1}(s)^{2}}, ~ \begin{aligned}
& k_{2}(t(s))=\frac{\sqrt{1+\tilde{r}_{3}^{2}}}{\tilde{r}_{3}} \tilde{k}_{3}^{2}(s), \\
& k_{3}(t(s))=\sqrt{\frac{1+\tilde{r}_{3}^{2}}{\tilde{k}_{1}(s)}} \tilde{k}_{3}(s) .
\end{aligned}
$$

As a consequence, we obtain that $\Gamma$ is a null generalized helix with null axis if and only if $\tilde{k}_{1}(s)$ is constant. In this case,

$$
k_{1}=r=\frac{1}{2} \frac{\tilde{k}_{1}}{\tilde{r}_{3}^{2}}, \quad r_{1}=\frac{\tilde{r}_{3}}{\sqrt{\tilde{k}_{1}}} .
$$

From here and by using the theorem for existence and uniqueness of timelike curves $\mathbb{R}_{1}^{5}$, we can prove the following theorem.

Theorem 11 (Solving natural equation for generalized helices with degenerate axis.) Let $\Gamma$ be a null Cartan curve in $\mathbb{R}_{1}^{5}$. Then $\Gamma$ is a generalized helix with null axis if and only if it is a geodesic in a Lorentzian surface $\Upsilon \times \mathbb{R}_{0}^{1}$, where $\Upsilon$ is a timelike generalized helix in $\mathbb{R}_{1}^{5}$ with null axis and with constant first curvature, and $\mathbb{R}_{0}^{1}$ stands for the direction of the axis.

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