# $s$-degenerate curves in Lorentzian space forms 

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#### Abstract

In this paper we introduce $s$-degenerate curves in Lorentzian space forms as those ones whose derivative of order $s$ is a null vector provided that $s>1$ and all derivatives of order less than $s$ are spacelike (see the exact definition in section 2). In this sense classical null curves are 1-degenerate curves. We obtain a reference along an $s$-degenerate curve in an $n$-dimensional Lorentzian space with the minimun number of curvatures. That reference generalizes the reference of Bonnor for null curves in Minkowski spacetime and it will be called the Cartan frame of the curve. The associated curvature functions are called the Cartan curvatures of the curve. We characterize the $s$-degenerate helices (that is, $s$-degenerate curves with constant Cartan curvatures) in $n$-dimensional Lorentzian space forms and we obtain a complete classification of them in dimension four.


## 1 Introduction

The geometry of null hypersurfaces in spacetimes has played an important role in the development of general relativity, as well as in mathematics and physics of gravitation. It is necessary, for example, to understand the causal structure of spacetimes, black holes, assymptotically flat systems and gravitational waves.

A starting point to study null surfaces, or in general null hypersurfaces, consists of investigating the curves that live in those hypersurfaces. In this sense, the null curves in Lorentzian space forms has been studied by several authors (see, for example, [2], [3], [9], and references therein).

However, in a null hypersurface there are many other curves distinct from the null ones. They are spacelike curves with a null higher derivative, that is, $s$-degenerate curves (see section 2 for details). In this paper we study $s$-degenerate curves in Lorentzian space forms $\mathbb{M}_{1}^{n}$ and obtain existence, uniqueness and congruence theorems for that kind of curves. Notice that they must be spacelike curves.

Timelike and lightlike trajectories are the natural ones in spacetime geometries, but some recent experiments point out the existence of superluminal particles (spacelike trayectories) without any breakdown of the principle of relativity; theoretical developments exist suggesting that neutrinos might be instances of "tachyons" as their square mass appears to be negative. A model has recently been presented to fit the cosmic ray spectrum at

[^0]$E \approx 1-4 \mathrm{PeV},[\mathbf{6}, \mathbf{7}, \mathbf{8}]$, using the hypothesis that the electron neutrino is a tachyon. This model yields a value for $m^{2}\left(\nu_{e}\right) \approx-3 e V^{2}$, which is consistent with the results from recent mesurements in tritium beta decay experiments, $[\mathbf{4}],[\mathbf{1 1}],[\mathbf{1 4}]$. Moreover, the muon neutrino also exhibits a negative mass-squared, $[\mathbf{1}]$. However, as it is pointed out in [5], at present time we have not a satisfactory quantum theory for tachyonic fermions, so more theoretical work would be needed to determine a physically acceptable theory.

In [12] the author considers a model of a $D$-dimensional massless particle described by a Lagrangian proportional to the $N$ th extrinsic curvature of the world-line. He presents the Hamiltonian formulation of the system and shows that its trajectories are spacelike curves.

Therefore, it is required to construct a complete (at least local) theory of spacelike trajectories for neutrinos. Here, we are intended to provide a suitable mathematical macchinery to support the recent advances in theoretical physics.

In this paper we prove the following theorems.
Theorem 4. Let $k_{1}, \ldots, k_{m}:[-\delta, \delta] \longrightarrow \mathbb{R}$ be differentiable functions with $k_{i}>0$ for $i \neq$ $s, m$. Let $p$ be a point in $\mathbb{M}_{1}^{n}, n=m+2$, and let $\left\{W_{1}^{0}, \ldots, W_{s-1}^{0}, L^{0}, W_{s}^{0}, N^{0}, W_{s+1}^{0}, \ldots, W_{m}^{0}\right\}$ be a positively oriented pseudo-orthonormal basis of $T_{p} \mathbb{M}_{1}^{n}(c)$. Then there exists a unique $s$-degenerate Cartan curve $\gamma$ in $\mathbb{M}_{1}^{n}(c)$, with $\gamma(0)=p$, whose Cartan reference satisfies:

$$
L(0)=L^{0}, N(0)=N^{0}, W_{i}(0)=W_{i}^{0}, \quad i \in\{1, \ldots, m\}
$$

Theorem 5. If two s-degenerate Cartan curves $C$ and $\bar{C}$ in $\mathbb{M}_{1}^{n}(c)$ have Cartan curvatures $\left\{k_{1}, \ldots, k_{m}\right\}$, where $k_{i}:[-\delta, \delta] \longrightarrow \mathbb{R}$ are differentiable functions, then there exists a Lorentzian transformation of $\mathbb{M}_{1}^{n}(c)$ which maps bijectively $C$ into $\bar{C}$.

In section 5 we characterize the 2-degenerate helices (that is, 2-degenerate curves with constant Cartan curvatures) in 4-dimensional Lorentzian space forms and we obtain a complete classification of them.

## 2 Frenet frames for $s$-degenerate curves

The goal of this section is to find Frenet frames for $s$-degenerate curves in Lorentzian space forms. Before to do that, we need a technical result.

Let $E$ be a real vector space with a symmetric bilinear mapping $g: E \times E \rightarrow \mathbb{R}$. We say that $g$ is degenerate on $E$ if there exists a vector $\xi \neq 0$ in $E$ such that

$$
g(\xi, v)=0, \quad \text { for all } v \in E
$$

otherwise, $g$ is said to be non-degenerate. The radical (also called the null space) of $E$, with respect to $g$, is the subspace $\operatorname{Rad}(E)$ of $E$ defined by

$$
\operatorname{Rad}(E)=\{\xi \in E \mid g(\xi, v)=0, v \in E\}
$$

The dimension of $\operatorname{Rad}(E)$ is called the nullity degree of $g$ (or $E$ ) and is denoted by $r_{E}$.

If $F$ is a subspace of $E$, then we can consider $g_{F}$ the symmetric bilinear mapping on $F \times F$ obtained by restricting $g$ and define $r_{F}$ as the nullity degree of $F$ (or $g_{F}$ ). For simplicity, we will use $\langle$,$\rangle instead of g$ or $g_{F}$.

A vector $v$ is said to be timelike, lightlike or spacelike provided that $g(v, v)<0$, $g(v, v)=0$ (and $v \neq 0$ ), or $g(v, v)>0$, respectively. The vector $v=0$ is assumed to be spacelike. A unit vector is a vector $u$ such that $g(u, u)= \pm 1$.

Two vectors $u$ and $v$ are said to be orthogonal, written $u \perp v$, if $g(u, v)=0$. Similarly, two subsets $U$ and $V$ of $E$ are said to be orthogonal if $u \perp v$ for any $u \in U$ and $v \in V$. Given two orthogonal subspaces $F_{1}$ and $F_{2}$ in $E$ with $F_{1} \cap F_{2}=\{0\}$, the orthogonal direct sum of $F_{1}$ and $F_{2}$ will be denoted by $F_{1} \perp F_{2}$.

Lemma 1 Let $(E,\langle\rangle$,$) be a bilinear space and let F$ be a hyperplane of $E$. Let $r_{F}=$ $\operatorname{dim} \operatorname{Rad}(F)$ and $r_{E}=\operatorname{dim} \operatorname{Rad}(E)$. Then the following statements hold:
(i) If $r_{F}=0$ and $r_{E}=1$, then there exists a null vector $L$ such that

$$
E=F \perp \operatorname{span}\{L\}
$$

(ii) If $r_{F}=r_{E} \in\{0,1\}$, then there exists a non-null unit vector $V$ such that

$$
E=F \perp \operatorname{span}\{V\}
$$

Moreover, if $\operatorname{Rad}(E)=\{0\}$ then $V$ is unique, up to the sign.
(iii) If $r_{F}=1$ and $r_{E}=0$, and $F=F_{1} \perp L$, where $L \in \operatorname{Rad}(F)$ and $F_{1}$ is non-degenerate, then there exists a unique null vector $N$ such that $\langle L, N\rangle=\varepsilon, \varepsilon= \pm 1$, and

$$
E=(\operatorname{span}\{L\} \oplus \operatorname{span}\{N\}) \perp F_{1}
$$

Proof. We only need to make some algebraic computations.
(i) Since $F$ is non-degenerate, then $E=F \perp F^{\perp}$, where $F^{\perp}=\operatorname{span}\{L\}$ for a certain vector $L$. The inclusion $\operatorname{Rad}(E) \subset F^{\perp}$ implies $\operatorname{Rad}(E)=F^{\perp}$ and so $L$ is a null vector.
(ii) We may assume that $r_{F}=r_{E}=1$. By considering $F=F_{1} \perp \operatorname{span}\{L\}$, where $F_{1}$ is non-degenerate and $L$ is null, then $E=F_{1} \perp F_{1}^{\perp}$. Since $\operatorname{dim} F_{1}^{\perp}=2$, then $F_{1}^{\perp}=\operatorname{span}\{L\} \oplus \operatorname{span}\{V\}$, where $\operatorname{Rad}(E)=\operatorname{span}\{L\}$ and $V$ is a non-null vector in $F^{\perp}$, so that the required splitting is fulfilled.
(iii) By a similar reasoning we may assume that $F=F_{1} \perp \operatorname{span}\{L\}$, where $F_{1}^{\perp}=\operatorname{span}\{L\} \oplus$ span $\{V\}$. Since $\operatorname{Rad}(E)=\{0\}$ then $\langle L, V\rangle \neq 0$. Let $N$ be the vector defined by

$$
N=\frac{\varepsilon}{\langle L, V\rangle}\left(V-\frac{\langle V, V\rangle}{2\langle L, V\rangle} L\right)
$$

It is easy to see that $N$ is the only vector satisfying $\langle N, N\rangle=0,\langle L, N\rangle=\varepsilon$ and $N \in F_{1}^{\perp}$, and the splitting follows.

Let $\left(M_{1}^{n}, \nabla\right)$ be an oriented Lorentzian manifold and let $\gamma: I \longrightarrow M_{1}^{n}$ be a differentiable curve in $M_{1}^{n}$. For any vector field $V$ along $\gamma$, let $V^{\prime}$ be the covariant derivative of $V$ along $\gamma$. Write $E_{i}(t)=\operatorname{span}\left\{\gamma^{\prime}(t), \gamma^{\prime \prime}(t), \ldots, \gamma^{(i)}(t)\right\}$, where $t \in I$ and $i=1,2, \ldots, n$. Let $d$ be the number defined by $d=\max \left\{i: \operatorname{dim} E_{i}(t)=i\right.$ for all $\left.t\right\}$.

Definition 1 With the above notations, the curve $\gamma: I \longrightarrow M_{1}^{n}$ is said to be an $s$ degenerate (or s-lightlike) curve if for all $1 \leq i \leq d$, $\operatorname{dim} \operatorname{Rad}\left(E_{i}(t)\right)$ is constant for all $t$, and there exists $s, 0<s \leq d$, such that $\operatorname{Rad}\left(E_{s}\right) \neq\{0\}$ and $\operatorname{Rad}\left(E_{j}\right)=\{0\}$ for all $j<s$.

Remark 1 Note that 1-degenerate curves are precisely the null (or lightlike) curves (see, for instance, $[\mathbf{2}],[\mathbf{3}],[\mathbf{9}]$, and references therein). In this paper we will focus on $s$-degenerate curves, $s>1$ ), in Lorentzian spaces. Notice that they must be spacelike curves.

To find the Frenet frames, we will distinguish four cases separately:

1) $d=n$ and $s \leq d$.
2) $d<n$ and $s=d$.
3) $d<n$ and $s=d-1$.
4) $d<n$ and $s<d-1$.

Case 1: $d=n$ and $s \leq d$
First of all, write $\gamma^{\prime}=\bar{k}_{1} W_{1}$, where $W_{1}$ is a unit spacelike vector such that $\bar{k}_{1}>0$. Then $E_{2}=\operatorname{span}\left\{W_{1}\right\} \oplus \operatorname{span}\left\{\gamma^{\prime \prime}\right\}$, so that from Lemma 1 there exists a unit spacelike vector $W_{2}$ such that $E_{2}=\operatorname{span}\left\{W_{1}\right\} \perp \operatorname{span}\left\{W_{2}\right\}$. Furthermore, $W_{2}$ is unique by choosing it in such a way that $\left\{\gamma^{\prime}, \gamma^{\prime \prime}\right\}$ and $\left\{W_{1}, W_{2}\right\}$ have the same orientation. By proceeding iteratively, using Lemma 1, we obtain a set $\left\{W_{1}, \ldots, W_{s-1}\right\}$ of orthonormal spacelike sections along $\gamma$ such that $\left\{\gamma^{\prime}, \ldots, \gamma^{(i)}\right\}$ and $\left\{W_{1}, \ldots, W_{i}\right\}$ have the same orientation for all $i, 1 \leq i \leq s-1$. Now we have $E_{s}=E_{s-1} \oplus \operatorname{span}\left\{\gamma^{(s)}\right\}$ and $\operatorname{dim} \operatorname{Rad}\left(E_{s}\right)=1$. By using again Lemma 1 we can find a (not unique) null vector field $L$ such that $E_{s}=E_{s-1} \oplus \operatorname{span}\{L\}$. As $s \neq n$, because $E_{n}$ is non-degenerate, then $E_{s+1}=E_{s} \oplus \operatorname{span}\left\{\gamma^{(s+1)}\right\}$. Now we will prove that $\operatorname{dim} \operatorname{Rad}\left(E_{s+1}\right)=1$. By assuming that $\operatorname{dim} \operatorname{Rad}\left(E_{s+1}\right)=0$, then there exists a unique null vector field $N$ satisfying $\left\langle W_{i}, N\right\rangle=\langle N, N\rangle=0,\langle L, N\rangle=\varepsilon, \varepsilon= \pm 1$, and $E_{s+1}=\operatorname{span}\left\{W_{1}, \ldots, W_{s-1}, L, N\right\}$. By taking derivatives we obtain the following equations:

$$
\begin{aligned}
\gamma^{\prime} & =\bar{k}_{1} W_{1} \\
W_{1}^{\prime} & =\bar{k}_{2} W_{2} \\
W_{i}^{\prime} & =-\bar{k}_{i} W_{i-1}+\bar{k}_{i+1} W_{i+1}, \quad 2 \leq i \leq s-2 \\
W_{s-1}^{\prime} & =-\bar{k}_{s-1} W_{s-2}+\varepsilon \bar{k}_{s} L \\
L^{\prime} & =\varepsilon \bar{k}_{s+1} L
\end{aligned}
$$

for certain functions $\bar{k}_{j}, j=1, \ldots, s+1$. As $L \in \operatorname{span}\left\{\gamma^{\prime}, \ldots, \gamma^{(s)}\right\}$, we can write $L=\lambda_{1} \gamma^{\prime}+\ldots+\lambda_{s} \gamma^{(s)}$, with $\lambda_{s} \neq 0$, and therefore $L^{\prime}=(*)+\lambda_{s} \gamma^{(s+1)}=\varepsilon \bar{k}_{s+1} L \in$ $\operatorname{span}\left\{\gamma^{\prime}, \ldots, \gamma^{(s)}\right\}$. We conclude that $\gamma^{(s+1)} \in \operatorname{span}\left\{\gamma^{\prime}, \ldots, \gamma^{(s)}\right\}$, which can not hold.

Then $\operatorname{dim} \operatorname{Rad}\left(E_{s+1}\right)=1$, and using Lemma 1 once more there exists a (not unique) vector field $W_{s}$ such that $\left\{\gamma^{\prime}, \ldots, \gamma^{(s+1)}\right\}$ and $\left\{W_{1}, \ldots, W_{s-1}, L, W_{s}\right\}$ have the same orientation. Since $n>s+1$, we claim that $\operatorname{dim} \operatorname{Rad}\left(E_{s+2}\right)=0$. Otherwise, there exists a unit spacelike vector field $W_{s+1}$ orthogonal to $E_{s+1}$. By differentiating we obtain

$$
\begin{align*}
W_{s-1}^{\prime} & =-\bar{k}_{s-1} W_{s-2}+\varepsilon \bar{k}_{s} L  \tag{1}\\
L^{\prime} & =\varepsilon \bar{k}_{s+1} L+\bar{k}_{s+2} W_{s}
\end{align*}
$$

Since $\operatorname{Rad}\left(E_{s+2}\right)=\operatorname{span}\{L\}$ we get $\left\langle L, \gamma^{(s+1)}\right\rangle=\left\langle L, \gamma^{(s+2)}\right\rangle=0$, so that $\left\langle L^{\prime}, \gamma^{(s+1)}\right\rangle=0$. From here and (1) we find that $\left\langle W_{s}, \gamma^{(s+1)}\right\rangle=0$ (i.e., $W_{s}$ lies in $\operatorname{Rad}\left(E_{s+1}\right)$ ), which is a contradiction. Hence $\operatorname{dim} \operatorname{Rad}\left(E_{s+2}\right)=0$ and there exists a unique $N$ satisfying $\langle N, L\rangle=\varepsilon$ and $\left\langle N, W_{i}\right\rangle=0$. We choose $\varepsilon$ in such a way that $\left\{\gamma^{\prime}, \ldots, \gamma^{(s+2)}\right\}$ and $\left\{W_{1}, \ldots, W_{s-1}, L, W_{s}, N\right\}$ have the same orientation. If $s+2=n$, the process concludes; otherwise, $\operatorname{dim} \operatorname{Rad}\left(E_{i}\right)=0$ for $i>s+2$ and we can obtain orthonormal spacelike sections $\left\{W_{s+1}, \ldots, W_{m}\right\}, m=n-2$, with the same orientation rule. The vector field $W_{m}$ is chosen in order that $\left\{W_{1}, \ldots, W_{s-1}, L, W_{s}, N, W_{s+1}, \ldots, W_{m}\right\}$ is positively oriented. Regarding this reference, we have the following equations

$$
\begin{aligned}
\gamma^{\prime} & =\bar{k}_{1} W_{1} \\
W_{1}^{\prime} & =\bar{k}_{2} W_{2} \\
W_{i}^{\prime} & =-\bar{k}_{i} W_{i-1}+\bar{k}_{i+1} W_{i+1}, \quad 2 \leq i \leq s-2 \\
W_{s-1}^{\prime} & =-\bar{k}_{s-1} W_{s-2}+\varepsilon \bar{k}_{s} L \\
L^{\prime} & =\varepsilon \bar{k}_{s+1} L+\bar{k}_{s+2} W_{s} \\
W_{s}^{\prime} & =\varepsilon \bar{k}_{s+3} L-\varepsilon \bar{k}_{s+2} N \\
N^{\prime} & =-\bar{k}_{s} W_{s-1}-\varepsilon \bar{k}_{s+1} N-\bar{k}_{s+3} W_{s}+\bar{k}_{s+4} W_{s+1} \\
W_{s+1}^{\prime} & =-\varepsilon \bar{k}_{s+4} L+\bar{k}_{s+5} W_{s+2} \\
W_{j}^{\prime} & =-\bar{k}_{j+3} W_{j-1}+\bar{k}_{j+4} W_{j+1}, \quad s+2 \leq j \leq m-1 \\
W_{m}^{\prime} & =-\bar{k}_{m+3} W_{m-1}
\end{aligned}
$$

for certain functions $\left\{\bar{k}_{1}, \ldots, \bar{k}_{m+3}\right\}$. The set $\mathcal{F}=\left\{W_{1}, \ldots, W_{s-1}, L, W_{s}, N, W_{s+1}, \ldots, W_{m}\right\}$ is said to be a Frenet reference along $\gamma$. The functions $\left\{\bar{k}_{1}, \ldots, \bar{k}_{m+3}\right\}$ are called the curvature functions of $\gamma$ with respect to $\mathcal{F}$. The above equations are called the Frenet equations of $\gamma$ with respect to $\mathcal{F}$.

Case 2: $d<n$ and $s=d$
A similar reasoning as in Case 1 shows that there exists a set $\left\{W_{1}, \ldots, W_{s-1}, L\right\}$ such that $L$ is a null vector, $\left\{W_{1}, \ldots, W_{s-1}\right\}$ is an orthonormal set of spacelike vectors and $E_{d}=\operatorname{span}\left\{W_{1}, \ldots, W_{s-1}, L\right\}$. Then we can obtain the following equations:

$$
\begin{aligned}
\gamma^{\prime} & =\bar{k}_{1} W_{1} \\
W_{1}^{\prime} & =\bar{k}_{2} W_{2} \\
W_{i}^{\prime} & =-\bar{k}_{i} W_{i-1}+\bar{k}_{i+1} W_{i+1}, \quad 2 \leq i \leq s-2 \\
W_{s-1}^{\prime} & =-\bar{k}_{s-1} W_{s-2}+\varepsilon \bar{k}_{s} L \\
L^{\prime} & =\varepsilon \bar{k}_{s+1} L
\end{aligned}
$$

for certain functions $\left\{\bar{k}_{1}, \ldots, \bar{k}_{s+1}\right\}$. If $M_{1}^{n}$ is a Lorentzian space form, then $\gamma$ lies in a $d$-dimensional totally geodesic lightlike submanifold. This can be proved by adapting the proofs of Theorems 5 and 9 of Chapter 7 in [13]. This case has been treated in Sect. 4 of [10].

Case 3. $d<n$ and $s=d-1$

As above again, we obtain $E_{d}=\operatorname{span}\left\{W_{1}, \ldots, W_{s-1}, L, W_{s}\right\}$ and equations

$$
\begin{aligned}
W_{s-1}^{\prime} & =-\bar{k}_{s-1} W_{s-2}+\varepsilon \bar{k}_{s} L \\
L^{\prime} & =\varepsilon \bar{k}_{s+1} L+\bar{k}_{s+2} W_{s} \\
W_{s}^{\prime} & =\varepsilon \bar{k}_{s+3} L
\end{aligned}
$$

Since $W_{s}$ lies in $E_{s}^{\perp}$, we have $\left\langle W_{s}, \gamma^{(s)}\right\rangle=0$. By differentiating here we deduce that $\left\langle W_{s}, \gamma^{(s+1)}\right\rangle=0$, which is a contradiction.

Case 4: $d<n$ and $s<d-1$

Now we have $E_{d}=\left\{W_{1}, \ldots, W_{s-1}, L, W_{s}, N, W_{s+1}, \ldots, W_{d-2}\right\}$. Working as in case of nondegenerate curves (see, for example, [13, Vol. IV]), if $M_{1}^{n}$ is a Lorentzian space form we deduce that $\gamma$ lies in a $d$-dimensional non-degenerate totally geodesic submanifold of $M_{1}^{n}$. So this case reduces to Case 1.

Remark 2 Before going any further, we note that the type $s$ does not depend on the parameter of the curve. To see that let $\bar{t}$ be another parameter and write $\gamma(t)=\beta(\bar{t}(t))$. By differentiating with respect to $t$ we get $\gamma^{(i)}(t)=\sum_{j=1}^{i} x_{i j}(t) \beta^{j}(\bar{t})$, that is, $E_{i}=$ $\operatorname{span}\left\{\gamma^{\prime}(t), \ldots, \gamma^{(i)}(t)\right\}=\operatorname{span}\left\{\beta^{\prime}(\bar{t}), \ldots, \beta^{(i)}(\bar{t})\right\}$, which shows the claim.

On the other hand, let $\Phi: M_{1}^{n} \longrightarrow M_{1}^{n}$ be an isometry and $\bar{\gamma}(t)=(\Phi \circ \gamma)(t)$. Then for all vector field $V$ along $\gamma$ we have

$$
\frac{\bar{D}}{d t}\left(d \Phi_{\gamma(t)}(V(t))\right)=d \Phi_{\gamma(t)}\left(\frac{D}{d t} V(t)\right)
$$

where $D_{t}$ and $\bar{D}_{t}$ stand for the covariant derivatives along $\gamma$ and $\bar{\gamma}$, respectively.
Hence $\left\langle\gamma^{(i)}(t), \gamma^{(j)}(t)\right\rangle=\left\langle\bar{\gamma}^{(i)}(t), \bar{\gamma}^{(j)}(t)\right\rangle$, showing that this kind of curves are invariant under Lorentzian transformations, in the sense that the type $s$ does not change under a Lorentzian transformation.

## 3 The Cartan reference of an $s$-degenerate curve

The goal of this section is to find a Frenet frame with the minimal number of curvatures and such that they are invariant under Lorentzian transformations. We will restrict ourselves to Case 1. Without loss of generality, let us assume that $\gamma$ is arc-length parametrized, so that $W_{1}=\gamma^{\prime}$ and $\bar{k}_{1}=1$. By taking $\bar{k}_{s}=\varepsilon$, Lemma 1 leads to a uniquely determined set $\left\{W_{1}, \ldots, W_{s-1}, L\right\}$. Therefore we only need to find $W_{s}$.

Suppose that $W_{s}$ and $W_{s}^{*}$ are two distinct vector fields generating two distinct Frenet frames, that is,

$$
\begin{aligned}
\left\{W_{1}, \ldots, W_{s-1}, L, W_{s}, N, W_{s+1}, \ldots, W_{m}\right\} & \longrightarrow\left\{\bar{k}_{1}=1, \bar{k}_{2}, \ldots, \bar{k}_{s}=1, \bar{k}_{s+1}, \ldots, \bar{k}_{m+3}\right\} \\
\left\{W_{1}, \ldots, W_{s-1}, L, W_{s}^{*}, N^{*}, W_{s+1}^{*}, \ldots, W_{m}^{*}\right\} & \longrightarrow\left\{\bar{k}_{1}=1, \bar{k}_{2}, \ldots, \bar{k}_{s}=1, \bar{k}_{s+1}^{*}, \ldots, \bar{k}_{m+3}^{*}\right\}
\end{aligned}
$$

A straightforward computation shows that

$$
\begin{equation*}
W_{s}^{*}=f L+W_{s}, \quad N^{*}=-\frac{1}{2} f^{2} L+N-f W_{s}, \quad \bar{k}_{s+1}^{*}=\bar{k}_{s+1}-f \bar{k}_{s+2} \tag{2}
\end{equation*}
$$

where $f: I \longrightarrow R$ is a differentiable function. We can choose $f$ in such a way that $\bar{k}_{s+1}^{*}=0$. Then by reordering the curvature functions we have the following equations:

$$
\begin{align*}
\gamma^{\prime} & =W_{1} \\
W_{1}^{\prime} & =k_{1} W_{2} \\
W_{i}^{\prime} & =-k_{i-1} W_{i-1}+k_{i} W_{i+1}, \quad 2 \leq i \leq s-2 \\
W_{s-1}^{\prime} & =-k_{s-2} W_{s-2}+L \\
L^{\prime} & =k_{s-1} W_{s} \\
W_{s}^{\prime} & =\varepsilon k_{s} L-\varepsilon k_{s-1} N  \tag{3}\\
N^{\prime} & =-\varepsilon W_{s-1}-k_{s} W_{s}+k_{s+1} W_{s+1}, \\
W_{s+1}^{\prime} & =-\varepsilon k_{s+1} L+k_{s+2} W_{s+2}, \\
W_{j}^{\prime} & =-k_{j} W_{j-1}+k_{j+1} W_{j+1}, \quad s+2 \leq j \leq m-1, \\
W_{m}^{\prime} & =-k_{m} W_{m-1}
\end{align*}
$$

for certain functions $\left\{k_{1}, \ldots, k_{m}\right\}$. Bearing in mind (2) we can easily deduce the following result.

Theorem 2 Let $\gamma: I \longrightarrow M_{1}^{n}, n=m+2$, be an $s$-degenerate unit curve, $s>1$, and suppose that $\left\{\gamma^{\prime}(t), \gamma^{\prime \prime}(t), \ldots, \gamma^{(n)}(t)\right\}$ spans $T_{\gamma(t)} M_{1}^{n}$, for all $t$. Then there exists a unique Frenet frame satisfying the equations (3).

Definition 2 An $s$-degenerate curve, $s>1$, satisfying the above conditions is said to be an $s$-degenerate Cartan curve. The reference and curvature functions given by (3) will be called the Cartan reference and Cartan curvatures of $\gamma$, respectively.

Observe that when $m>s$ then $\varepsilon=-1$ and $k_{i}>0$ for $i \neq s$, and $k_{m}>0\left(k_{m}<0\right.$, resp. $)$ according to $\left\{\gamma^{\prime}, \gamma^{\prime \prime}, \ldots, \gamma^{(n)}\right\}$ is positively or negatively oriented, respectively. However, when $m=s$ then $\varepsilon=-1$ or $\varepsilon=1$ according to $\left\{\gamma^{\prime}, \gamma^{\prime \prime}, \ldots, \gamma^{(n)}\right\}$ is positively or negatively oriented, respectively, and $k_{i}>0$ for $i \neq s$.

Definition 3 An s-degenerate helix in $M_{1}^{n}$ is an $s$-degenerate Cartan curve having constant Cartan curvatures.

## $4 s$-degenerate curves in Lorentzian space forms

Let $\gamma: I \longrightarrow \mathbb{M}_{1}^{n}(c)$ be an $s$-degenerate Cartan curve, $\mathbb{M}_{1}^{n}(c)$ standing for $\mathbb{R}_{1}^{n}$, $\mathbb{S}_{1}^{n}$ o $\mathbb{H}_{1}^{n}$, according to $c=0, c=1$ or $c=-1$, respectively. Let $D_{t}$ denote the covariant derivative in $\mathbb{M}_{1}^{n}(c)$ along $\gamma$. Then for any vector field $V$ along $\gamma$ we have $D_{t} V=V^{\prime}+c\left\langle V, \gamma^{\prime}\right\rangle \gamma$, where $\langle$,$\rangle denotes the standard metric in \mathbb{R}_{1}^{n}, \mathbb{R}_{1}^{n+1}$ or $\mathbb{R}_{2}^{n+1}$. If $\left\{W_{1}, \ldots, W_{s-1}, L, W_{s}, N, W_{s+1}, \ldots, W_{m}\right\}$ is the Cartan reference, then equations (3) write down as follows:

$$
\begin{align*}
\gamma^{\prime} & =W_{1} \\
W_{1}^{\prime} & =k_{1} W_{2}-c \gamma \\
W_{i}^{\prime} & =-k_{i-1} W_{i-1}+k_{i} W_{i+1}, \quad 2 \leq i \leq s-2 \\
W_{s-1}^{\prime} & =-k_{s-2} W_{s-2}+L \\
L^{\prime} & =k_{s-1} W_{s} \\
W_{s}^{\prime} & =\varepsilon k_{s} L-\varepsilon k_{s-1} N  \tag{4}\\
N^{\prime} & =-\varepsilon W_{s-1}-k_{s} W_{s}+k_{s+1} W_{s+1}, \\
W_{s+1}^{\prime} & =-\varepsilon k_{s+1} L+k_{s+2} W_{s+2}, \\
W_{j}^{\prime} & =-k_{j} W_{j-1}+k_{j+1} W_{j+1}, \quad s+2 \leq j \leq m-1, \\
W_{m}^{\prime} & =-k_{m} W_{m-1}
\end{align*}
$$

Now we state the following question:
Let $\left\{W_{1}, \ldots, W_{s-1}, L, W_{s}, N, W_{s+1}, \ldots, W_{m}\right\}$ be a reference satisfying (3) for certain functions $k_{j}$. Is there an s-degenerate Cartan curve $\gamma$ having $\left\{W_{1}, \ldots\right.$, $\left.W_{s-1}, L, W_{s}, N, W_{s+1}, \ldots, W_{m}\right\}$ as Cartan reference and $k_{j}$ as Cartan curvatures?

The answer is affirmative, as we will show in this section. But before to do that, we are going to state and prove an algebraic result.

Definition 4 A basis $\mathcal{B}=\left\{L_{1}, N_{1}, \ldots, L_{r}, N_{r}, W_{1}, \ldots, W_{m}\right\}$ of $\mathbb{R}_{q}^{n}$, with $2 r \leq 2 q \leq n$ and $m=n-2 r$, is said to be pseudo-orthonormal if it satisfies the following equations:

$$
\begin{aligned}
\left\langle L_{i}, L_{j}\right\rangle & =\left\langle N_{i}, N_{j}\right\rangle=0, \quad\left\langle L_{i}, N_{i}\right\rangle=\varepsilon_{i}, \quad\left\langle L_{i}, N_{j}\right\rangle=0 \quad i \neq j \\
\left\langle L_{i}, W_{\alpha}\right\rangle & =\left\langle N_{i}, W_{\alpha}\right\rangle=0, \quad\left\langle W_{\alpha}, W_{\beta}\right\rangle=\varepsilon_{\alpha} \delta_{\alpha \beta}
\end{aligned}
$$

where $i, j \in\{1, \ldots, r\}, \alpha, \beta \in\{1, \ldots, m\}, \varepsilon_{\alpha}=-1$ if $1 \leq \alpha \leq q-r$ and $\varepsilon_{\alpha}=1$ if $q-r+1 \leq \alpha \leq m$.

Lemma 3 Let $\mathcal{B}=\left\{L_{1}, N_{1}, \ldots, L_{r}, N_{r}, W_{1}, \ldots, W_{m}\right\}$ be a basis of $\mathbb{R}_{q}^{n}$, with $2 r \leq 2 q \leq n$ and $m=n-2 r$. Consider $\mathcal{B}^{\prime}=\left\{V_{1}, \ldots, V_{q}, V_{q+1}, \ldots, V_{n}\right\}$ where

$$
V_{i}= \begin{cases}\frac{1}{\sqrt{2}}\left(L_{i}-\varepsilon_{i} N_{i}\right) & i=1, \ldots, r  \tag{5}\\ W_{i-r} & i=r+1, \ldots, q \\ \frac{1}{\sqrt{2}}\left(L_{i-q}+\varepsilon_{i-q} N_{i-q}\right) & i=q+1, \ldots, q+r \\ W_{i-2 r} & i=q+r+1, \ldots, n\end{cases}
$$

The following conditions are equivalent:
(i) $\mathcal{B}$ is a pseudo-orthonormal basis.
(ii) $\mathcal{B}^{\prime}$ is an orthonormal basis.
(iii) $\mathcal{B}^{\prime}$ satisfies

$$
-\sum_{\alpha=1}^{q} V_{\alpha i} V_{\alpha j}+\sum_{\beta=q+1}^{n} V_{\beta i} V_{\beta j}=\eta_{i j}
$$

(iv) $\mathcal{B}$ satisfies

$$
\sum_{\alpha=1}^{r} \varepsilon_{\alpha}\left(L_{\alpha i} N_{\alpha j}+L_{\alpha j} N_{\alpha i}\right)-\sum_{\beta=1}^{q-r} W_{\beta i} W_{\beta j}+\sum_{\theta=q-r+1}^{m} W_{\theta i} W_{\theta j}=\eta_{i j}
$$

Here $V_{\rho k}, L_{\rho k}, N_{\rho k}$ and $W_{\rho k}$ stand for the components of vectors $V_{\rho}, L_{\rho}, N_{\rho}$ and $W_{\rho}$, respectively, and $\left(\eta_{i j}\right)$ denotes the matrix of the canonical metric in the standard coordinates.

Proof. (i) $\Leftrightarrow$ (ii) It is obvious.
(ii) $\Leftrightarrow$ (iii) Consider the matrices $V=\left(V_{i j}\right), B=\left(b_{i j}\right)$ and $C=\left(c_{i j}\right)$ in $\mathcal{M}_{n \times n}(\mathbb{R})$ given by

$$
\begin{aligned}
b_{i j} & =\left\langle V_{i}, V_{j}\right\rangle \\
c_{i j} & =-\sum_{\alpha=1}^{q} V_{\alpha i} V_{\alpha j}+\sum_{\beta=q+1}^{n} V_{\beta i} V_{\beta j} .
\end{aligned}
$$

Put

$$
V=\left(\begin{array}{ll}
A_{1} & A_{2} \\
A_{3} & A_{4}
\end{array}\right), \quad B=\left(\begin{array}{cc}
B_{1} & B_{2} \\
B_{3} & B_{4}
\end{array}\right) \quad \text { and } \quad C=\left(\begin{array}{cc}
C_{1} & C_{2} \\
C_{3} & C_{4}
\end{array}\right)
$$

where $A_{1}, B_{1}$ and $C_{1}$ are matrices in $\mathcal{M}_{q \times q}(\mathbb{R})$. Consider the complex matrix

$$
A=\left(\begin{array}{cc}
A_{1} & i A_{2} \\
-i A_{3} & A_{4}
\end{array}\right) \in \mathcal{M}_{n \times n}(\mathbb{C})
$$

Then a straightforward computation shows that

$$
A A^{T}=\left(\begin{array}{cc}
-B_{1} & i B_{2} \\
i B_{3} & B_{4}
\end{array}\right) \quad \text { and } \quad A^{T} A=\left(\begin{array}{cc}
-C_{1} & -i C_{2} \\
-i C_{3} & C_{4}
\end{array}\right)
$$

Then $\mathcal{B}^{\prime}$ is orthonormal if and only if $C_{1}=-I, C_{4}=I$ and $C_{2}=C_{3}=0$.
(iii) $\Leftrightarrow$ (iv) From (5) we have

$$
L_{\alpha}=\frac{1}{\sqrt{2}}\left(V_{\alpha+q}+V_{\alpha}\right) \quad \text { and } \quad N_{\alpha}=\frac{\varepsilon_{\alpha}}{\sqrt{2}}\left(V_{\alpha+q}-V_{\alpha}\right), \quad \alpha \in\{1, \ldots, r\}
$$

and therefore

$$
\varepsilon_{\alpha}\left(L_{\alpha i} N_{\alpha j}+N_{\alpha i} L_{\alpha j}\right)=-V_{\alpha i} V_{\alpha j}+V_{(\alpha+q) i} V_{(\alpha+q) j}, \quad \alpha \in\{1, \ldots, r\}, \quad i, j \in\{1, \ldots, n\}
$$

which finishes the proof.

Theorem 4 Let $k_{1}, \ldots, k_{m}:[-\delta, \delta] \longrightarrow \mathbb{R}$ be differentiable functions with $k_{i}>0$ for $i \neq$ $s, m$. Let $p$ be a point in $\mathbb{M}_{1}^{n}, n=m+2$, and let $\left\{W_{1}^{0}, \ldots, W_{s-1}^{0}, L^{0}, W_{s}^{0}, N^{0}, W_{s+1}^{0}, \ldots, W_{m}^{0}\right\}$ be a positively oriented pseudo-orthonormal basis of $T_{p} \mathbb{M}_{1}^{n}(c)$. Then there exists a unique $s$-degenerate Cartan curve $\gamma$ in $\mathbb{M}_{1}^{n}(c)$, with $\gamma(0)=p$, whose Cartan reference satisfies:

$$
L(0)=L^{0}, N(0)=N^{0}, W_{i}(0)=W_{i}^{0}, \quad i \in\{1, \ldots, m\} .
$$

Proof. By the general theory of differential equations we know that there exists a unique solution $\left\{W_{1}, \ldots, W_{s-1}, L, W_{s}, N, W_{s+1}, \ldots, W_{m}\right\}$ of (4), defined on the interval $[-\delta, \delta]$, and satisfying the initial conditions of the theorem. Taking into account (4), a straightforward computation leads to

$$
\frac{d}{d t}\left(\varepsilon\left(L_{i}(t) N_{j}(t)+L_{j}(t) N_{i}(t)\right)+c \gamma_{i}(t) \gamma_{j}(t)+\sum_{\beta=1}^{m} W_{\beta i}(t) W_{\beta j}(t)\right)=0
$$

Now, since $\left\{W_{1}, \ldots, W_{s-1}, L, W_{s}, N, W_{s+1}, \ldots, W_{m}\right\}$ is pseudo-orthonormal at $t=0$, Lemma 3 (with $r=1$ ) yields

$$
\varepsilon\left(L_{i}(t) N_{j}(t)+L_{j}(t) N_{i}(t)\right)+c \gamma_{i}(t) \gamma_{j}(t)+\sum_{\beta=1}^{m} W_{\beta i}(t) W_{\beta j}(t)=\nu_{i j}, \quad \forall t \in[\delta, \delta] .
$$

By using again Lemma 3, we deduce that, for all $t,\left\{L, N, W_{1}, \ldots, W_{m}, \gamma\right\}$ is pseudoorthonormal if $c= \pm 1$, and $\left\{L, N, W_{1}, \ldots, W_{m}\right\}$ is pseudo-orthonormal if $c=0$. This concludes the proof.

Theorem 5 (Congruence Theorem) If two $s$-degenerate Cartan curves $C$ and $\bar{C}$ in $\mathbb{M}_{1}^{n}(c)$ have Cartan curvatures $\left\{k_{1}, \ldots, k_{m}\right\}$, where $k_{i}:[-\delta, \delta] \longrightarrow \mathbb{R}$ are differentiable functions, then there exists a Lorentzian transformation of $\mathbb{M}_{1}^{n}(c)$ which maps bijectively $C$ into $\bar{C}$.

## 5 s-degenerate helices in $\mathbb{M}_{1}^{4}(c)$

This section is devoted to the classification of 2-degenerate Cartan helices in Lorentzian space forms $\mathbb{M}_{1}^{4}(c)$. Now, the Cartan equations write down as follows:

$$
\begin{align*}
\gamma^{\prime} & =W_{1}, \\
W_{1}^{\prime} & =L-c \gamma, \\
L^{\prime} & =k_{1} W_{2},  \tag{6}\\
W_{2}^{\prime} & =\varepsilon k_{2} L-\varepsilon k_{1} N, \\
N^{\prime} & =-\varepsilon W_{1}-k_{2} W_{2} .
\end{align*}
$$

If we assume that $k_{1}$ and $k_{2}$ are constant, then $\gamma$ satisfies the following differential equation:

$$
\gamma^{(5)}-\left(2 \varepsilon k_{1} k_{2}-c\right) \gamma^{(3)}-\left(k_{1}^{2}+2 \varepsilon c k_{1} k_{2}\right) \gamma^{\prime}=0 .
$$

Without loss of generality, we can assume that $\gamma$ is positively oriented, that is, $\varepsilon=-1$.
In what follows, we will present examples of 2-degenerate Cartan helices in $\mathbb{M}_{1}^{4}(c)$ and show the corresponding characterization theorems.

### 5.1 Helices in $\mathbb{R}_{1}^{4}$

Example 1 Let $\gamma_{\omega, \sigma}$ be the curve in $\mathbb{R}_{1}^{4}$ defined by

$$
\gamma_{\omega, \sigma}(t)=\frac{1}{\sqrt{\omega^{2}+\sigma^{2}}}\left(\frac{\sigma}{\omega} \cosh \omega t, \frac{\sigma}{\omega} \sinh \omega t, \frac{\omega}{\sigma} \sin \sigma t, \frac{\omega}{\sigma} \cos \sigma t\right),
$$

with $\omega \sigma>0$. Then $\gamma_{\omega, \sigma}$ is a helix with curvatures

$$
k_{1}=\omega \sigma \quad \text { and } \quad k_{2}=\frac{\sigma^{2}-\omega^{2}}{2 \omega \sigma} .
$$

Theorem 6 (Clasification theorem of 2-degenerate helices in $\mathbb{R}_{1}^{4}$ ) Let $\gamma$ be an $s$ degenerate Cartan curve fully immersed in $\mathbb{R}_{1}^{4}$. Then $\gamma$ is a helix if and only if it is congruent to a helix of Example 1.

Proof. Let $k_{1}>0$ and $k_{2}$ be the constant curvatures of $\gamma$. By the congruence theorem 5 it suffices to find a helix of the family given in Example 1 with these curvatures. Take constants $\omega$ and $\sigma$ such that

$$
\omega^{2}=k_{1}\left(-k_{2}+\sqrt{1+k_{2}^{2}}\right) \quad \text { and } \quad \sigma^{2}=k_{1}\left(k_{2}+\sqrt{1+k_{2}^{2}}\right)
$$

with $\omega \sigma>0$. The proof concludes since the curvatures of $\gamma_{\omega, \sigma}$ are $k_{1}$ and $k_{2}$.

### 5.2 Helices in $\mathbb{S}_{1}^{4}$

Example 2 (Helices of Type 1) Let $0<\sigma^{2}<1<\omega^{2}$ and let $\gamma_{\omega, \sigma}$ be the curve in $\mathbb{S}_{1}^{4}$ defined by

$$
\begin{aligned}
\gamma_{\omega, \sigma}(t)= & \left(\sqrt{\frac{\left(\omega^{2}-1\right)\left(1-\sigma^{2}\right)}{\omega^{2} \sigma^{2}}}, \frac{1}{\omega} \sqrt{\frac{1-\sigma^{2}}{\omega^{2}-\sigma^{2}}} \sin \omega t, \frac{1}{\omega} \sqrt{\frac{1-\sigma^{2}}{\omega^{2}-\sigma^{2}}} \cos \omega t\right. \\
& \left.\frac{1}{\sigma} \sqrt{\frac{\omega^{2}-1}{\omega^{2}-\sigma^{2}}} \sin \sigma t, \frac{1}{\sigma} \sqrt{\frac{\omega^{2}-1}{\omega^{2}-\sigma^{2}}} \cos \sigma t\right)
\end{aligned}
$$

Then $\gamma_{\omega, \sigma}$ is a helix with curvatures

$$
k_{1}=\sqrt{\left(\omega^{2}-1\right)\left(1-\sigma^{2}\right)} \quad \text { and } \quad k_{2}=\frac{\omega^{2}+\sigma^{2}-1}{2 \sqrt{\left(\omega^{2}-1\right)\left(1-\sigma^{2}\right)}} .
$$

Example 3 (Helices of Type 2) Let $\sigma^{2}>1$ and let $\gamma_{\omega, \sigma}$ be the curve in $\mathbb{S}_{1}^{4}$ defined by

$$
\begin{aligned}
\gamma_{\omega, \sigma}(t)= & \left(\frac{1}{\omega} \sqrt{\frac{\sigma^{2}-1}{\omega^{2}+\sigma^{2}}} \cosh \omega t, \frac{1}{\omega} \sqrt{\frac{\sigma^{2}-1}{\omega^{2}+\sigma^{2}}} \sinh \omega t, \frac{1}{\sigma} \sqrt{\frac{\omega^{2}+1}{\omega^{2}+\sigma^{2}}} \sin \sigma t,\right. \\
& \left.\frac{1}{\sigma} \sqrt{\frac{\omega^{2}+1}{\omega^{2}+\sigma^{2}}} \cos \sigma t, \frac{1}{\omega \sigma} \sqrt{\left(\omega^{2}+1\right)\left(\sigma^{2}-1\right)}\right), \omega \neq 0
\end{aligned}
$$

Then $\gamma_{\omega, \sigma}$ is a helix with curvatures

$$
k_{1}=\sqrt{\left(\sigma^{2}-1\right)\left(\omega^{2}+1\right)} \quad \text { and } \quad k_{2}=\frac{\sigma^{2}-\omega^{2}-1}{2 \sqrt{\left(\sigma^{2}-1\right)\left(\omega^{2}+1\right)}}
$$

Example 4 (Helices of Type 3) Let $\sigma^{2}>1$ and let $\gamma_{\sigma}$ be the curve in $\mathbb{S}_{1}^{4}$ defined by

$$
\begin{aligned}
\gamma_{\sigma}(t)= & \left(\frac{1}{2} \frac{\sqrt{\sigma^{4}-1}}{\sigma^{2}-1} t^{2}, \sqrt{\frac{\sigma^{2}-1}{\sigma^{2}}} t, \frac{\sqrt{\sigma^{4}-1}}{\sigma^{2}}-\frac{\sqrt{\sigma^{4}-1}}{2\left(\sigma^{2}+1\right)} t^{2},\right. \\
& \left.\frac{1}{\sigma^{2}} \sin \sigma t, \frac{1}{\sigma^{2}} \cos \sigma t\right) .
\end{aligned}
$$

Then $\gamma_{\sigma}$ is a helix with curvatures

$$
k_{1}=\sqrt{\sigma^{2}-1} \quad \text { and } \quad k_{2}=\frac{1}{2} \sqrt{\sigma^{2}-1}
$$

Theorem 7 (Clasification theorem of 2-degenarate helices in $\mathbb{S}_{1}^{4}$ ) Let $\gamma$ be an $s$ degenerate Cartan curve fully immersed in $\mathbb{S}_{1}^{4}$. Then $\gamma$ is a helix if and only if it is congruent to one in the families described in Examples 2-4.

Proof. Let $k_{1}>0$ and $k_{2}$ be the constant curvatures of $\gamma$. We have to find a helix of one of the above types with these curvatures.
Case 1: Assume that $k_{2}>k_{1} / 2$. Take the helix $\gamma_{\omega, \sigma}$ of type 1 determined by
$\omega^{2}=\frac{\left(2 k_{1} k_{2}+1\right)+\sqrt{\left(1-2 k_{1} k_{2}\right)^{2}+4 k_{1}^{2}}}{2}$ and $\quad \sigma^{2}=\frac{\left(2 k_{1} k_{2}+1\right)-\sqrt{\left(1-2 k_{1} k_{2}\right)^{2}+4 k_{1}^{2}}}{2}$.
A straightforward computation shows that $0<\sigma^{2}<1<\omega^{2}$ and the curvatures of $\gamma_{\omega, \sigma}$ are $k_{1}$ and $k_{2}$.
Case 2: Assume that $k_{2}<k_{1} / 2$. Take the helix $\gamma_{\omega, \sigma}$ of type 2 determined by
$\omega^{2}=\frac{-\left(2 k_{1} k_{2}+1\right)+\sqrt{\left(1-2 k_{1} k_{2}\right)^{2}+4 k_{1}^{2}}}{2} \quad$ and $\quad \sigma^{2}=\frac{\left(2 k_{1} k_{2}+1\right)+\sqrt{\left(1-2 k_{1} k_{2}\right)^{2}+4 k_{1}^{2}}}{2}$.
It is easy to show that $\sigma^{2}>1$ and the curvatures of $\gamma_{\omega, \sigma}$ are $k_{1}$ and $k_{2}$.
Case 3: Assume that $k_{2}=k_{1} / 2$. Take the helix $\gamma_{\sigma}$ of type 3 determined by $\sigma^{2}=1+k_{1}^{2}$. It is easy to see that $\sigma^{2}>1$ and the curvatures of $\gamma_{\sigma}$ are $k_{1}$ and $k_{2}$.

The result follows from the congruence theorem 5 .

### 5.3 Helices en $\mathbb{H}_{1}^{4}$

Example 5 (Helices of Type 1) Let $0<\sigma^{2}<1<\omega^{2}$ and let $\gamma_{\omega, \sigma}$ be the curve in $\mathbb{H}_{1}^{4}$ defined by

$$
\begin{aligned}
\gamma_{\omega, \sigma}(t)= & \left(\frac{1}{\omega} \sqrt{\frac{1-\sigma^{2}}{\omega^{2}-\sigma^{2}}} \cosh \omega t, \frac{1}{\sigma} \sqrt{\frac{\omega^{2}-1}{\omega^{2}-\sigma^{2}}} \cosh \sigma t, \frac{1}{\omega} \sqrt{\frac{1-\sigma^{2}}{\omega^{2}-\sigma^{2}}} \sinh \omega t,\right. \\
& \left.\frac{1}{\sigma} \sqrt{\frac{\omega^{2}-1}{\omega^{2}-\sigma^{2}}} \sinh \sigma t,-\frac{1}{\omega \sigma} \sqrt{\left(\omega^{2}-1\right)\left(1-\sigma^{2}\right)}\right)
\end{aligned}
$$

Then $\gamma_{\omega, \sigma}$ is a helix with curvatures

$$
k_{1}=\sqrt{\left(\omega^{2}-1\right)\left(1-\sigma^{2}\right)} \quad \text { and } \quad k_{2}=-\frac{1}{2} \frac{\omega^{2}+\sigma^{2}-1}{\sqrt{\left(\omega^{2}-1\right)\left(1-\sigma^{2}\right)}} .
$$

Example 6 (Helices of Type 2) Let $\omega^{2}>1$ and let $\gamma_{\omega, \sigma}$ be the curve in $\mathbb{H}_{1}^{4}$ defined by

$$
\begin{aligned}
\gamma_{\omega, \sigma}(t)= & \left(\sqrt{\frac{\left(\omega^{2}-1\right)\left(\sigma^{2}+1\right)}{\omega^{2} \sigma^{2}}}, \frac{1}{\omega} \sqrt{\frac{\sigma^{2}+1}{\omega^{2}+\sigma^{2}}} \cosh \omega t, \frac{1}{\omega} \sqrt{\frac{\sigma^{2}+1}{\omega^{2}+\sigma^{2}}} \sinh \omega t\right. \\
& \left.\frac{1}{\sigma} \sqrt{\frac{\omega^{2}-1}{\omega^{2}+\sigma^{2}}} \sin \sigma t, \frac{1}{\sigma} \sqrt{\frac{\omega^{2}-1}{\omega^{2}+\sigma^{2}}} \cos \sigma t\right), \sigma \neq 0
\end{aligned}
$$

Then $\gamma_{\omega, \sigma}$ is a helix with curvatures

$$
k_{1}=\sqrt{\left(\omega^{2}-1\right)\left(\sigma^{2}+1\right)} \quad \text { and } \quad k_{2}=\frac{1}{2} \frac{\sigma^{2}-\omega^{2}+1}{\sqrt{\left(\omega^{2}-1\right)\left(\sigma^{2}+1\right)}}
$$

Example 7 (Helices of Type 3) Let $\omega^{2}>1$ and let $\gamma_{\omega}$ be the curve in $\mathbb{H}_{1}^{4}$ defined by

$$
\begin{aligned}
\gamma_{\omega, \sigma}(t)= & \left(\frac{\sqrt{\omega^{4}-1}}{\omega^{2}}+\frac{\sqrt{\omega^{4}-1}}{2\left(\omega^{2}+1\right)} t^{2}, \frac{1}{\omega^{2}} \cosh \omega t, \frac{1}{\omega^{2}} \sinh \omega t\right. \\
& \left.\sqrt{\frac{\omega^{2}-1}{\omega^{2}}} t, \frac{1-\omega^{4}}{2\left(\omega^{2}+1\right)} t^{2}\right)
\end{aligned}
$$

Then $\gamma_{\omega}$ is a helix with curvatures

$$
k_{1}=\sqrt{\omega^{2}-1} \quad \text { and } \quad k_{2}=-\frac{1}{2} \sqrt{\omega^{2}-1}
$$

Theorem 8 (Clasification theorem of 2-degenerate helices in $\mathbb{H}_{1}^{4}$ ) Let $\gamma$ be an $s$ degenerate Cartan curve fully immersed in $\mathbb{H}_{1}^{4}$. Then $\gamma$ is a helix if and only if it is congruent to one in the families described in Examples 5-7.

Proof. The idea of the proof is exactly alike as that in the precedent cases. Let $k_{1}>0$ and $k_{2}$ be the constant curvatures of $\gamma$. By the congruence theorem we only have to find a helix of one of the above types with these curvatures.
Case 1: Assume that $k_{2}<-k_{1} / 2$. Take the helix $\gamma_{\omega, \sigma}$ of type 1 determined by
$\omega^{2}=\frac{\left(1-2 k_{1} k_{2}\right)+\sqrt{\left(2 k_{1} k_{2}+1\right)^{2}+4 k_{1}^{2}}}{2} \quad$ and $\quad \sigma^{2}=\frac{\left(1-2 k_{1} k_{2}\right)-\sqrt{\left(2 k_{1} k_{2}+1\right)^{2}+4 k_{1}^{2}}}{2}$.
A straightforward computation shows that $0<\sigma^{2}<1<\omega^{2}$ and the curvatures of $\gamma_{\omega, \sigma}$ are $k_{1}$ and $k_{2}$.

Case 2: Assume that $k_{2}>-k_{1} / 2$. Take the helix $\gamma_{\omega, \sigma}$ of type 2 determined by
$\omega^{2}=\frac{\left(1-2 k_{1} k_{2}\right)+\sqrt{\left(2 k_{1} k_{2}+1\right)^{2}+4 k_{1}^{2}}}{2} \quad$ and $\quad \sigma^{2}=\frac{-\left(1-2 k_{1} k_{2}\right)+\sqrt{\left(2 k_{1} k_{2}+1\right)^{2}+4 k_{1}^{2}}}{2}$.
As before we have that $\omega^{2}>1$ and the curvatures of $\gamma_{\omega, \sigma}$ are $k_{1}$ and $k_{2}$.
Case 3: Finally, assume that $k_{2}=-k_{1} / 2$. Take the helix $\gamma_{\omega}$ of type 3 determined by $\omega^{2}=1+k_{1}^{2}$. It is easy to see that $\omega^{2}>1$ and the curvatures of $\gamma_{\omega}$ are $k_{1}$ and $k_{2}$.

## References

[1] K. Assamagan, C. Brönnimann, M. Daum, H. Forrer, R. Frosch, P. Gheno, R. Horisberger, M. Janousch, P. Kettle, T. Spirig and C. Wigger. Upper limit of the muon-neutrino mass and charged-pion mass from momentum analysis of a surface muon beam. Phys. Rev. D, 53 (1996), 6065-6077.
[2] A. Bejancu. Lightlike curves in Lorentz manifolds. Publ. Math. Debrecen, 44 (1994), 145-155.
[3] W. B. Bonnor. Null curves in a Minkowski spacetime. Tensor, N. S., 20 (1969), 229-242.
[4] J. Bonn and C. Weinheimer. Neutrino mass from tritium $\beta$ decay - present limits and perspectives. Acta Physica Polonica B, 31 (2000), 1209-1220.
[5] A. Chodos, A. Hauser and V. Kostelecky, 1985. Phys. Lett. B 150, 296.
[6] R. Ehrlich. Implications for the cosmic ray spectrum of a negative electron neutrino (mass) ${ }^{2}$. Phys. Rev. D, 60 (1999), 017302, 4 pp.
[7] R. Ehrlich. Neutrino mass ${ }^{2}$ inferred from the cosmic ray spectrum and tritium beta decay. Phys. Lett. B, 493 (2000), 229-232.
[8] R. Erhlich. Is there a 4.5 PeV neutron line in the cosmic ray spectrum? Phys. Rev. $D, 60$ (1999), 73005, 5 pp.
[9] A. Ferrández, A. Giménez and P. Lucas. Null helices in Lorentzian space forms. International Journal of Modern Physics A, 16 (2001), 4845-4863.
[10] A. Ferrández, A. Giménez and P. Lucas. Null generalized helices in LorentzMinkowski spaces. Preprint, 2002.
[11] V. Lobashev, V. Aseev, A. Belesev, A. Berlev, E. Geraskin, A. Golubev, O. Kazachenko, Y. Kuznetsov, R. Ostroumov, L. Rivkis, B. Stern, N. Titov, S. Zadorozhny and Y. Zakharov. Direct search for mass of neutrino and anomaly in the tritium beta-spectrum. Physics Letters B, 460 (1999), 227-235.
[12] A. Nersesyan. A Lagrangian model of a massless particle on spacelike curves. Theoret. and Math. Phys., 126 (2001), 147-160.
[13] M. Spivak. A Comprehensive Introduction to Differential Geometry. Publish or Perish, Boston, 1975.
[14] C. Weinheimer, B. Degen, A. Bleile, J. Bonn, L. Bornschein, O. Kazachenko, A. Kovalik and E. Otten. High precision measurement of the tritium $\beta$ spectrum near its endpoint and upper limit on the neutrino mass. Physics Letters B, 460 (1999), 219-226.

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