# Geometry of relativistic particles with torsion 

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#### Abstract

We consider the motion of relativistic particles described by an action that is linear in the torsion (second curvature) of the particle path. The Euler-Lagrange equations and the dynamical constants of the motion associated with the Poincaré group, the mass and the spin of the particle, are expressed in a simple way in terms of the curvatures of the embedded worldline. The moduli spaces of solutions are completely exhibited in 4-dimensional background spaces and in the 5 -dimensional case we explicitly obtain the curvatures of the worldline.


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## 1 Introduction

There exists a very nice literature concerning geometrical models that describe a relativistic particle. The Poincaré and invariance requirements imply that an admissible Lagrangian density $f$ must depend on the extrinsic curvatures of particle trajectories in the background gravitational field. The general model considers a particle moving on a $d$-dimensional manifold with position coordinates parametrized by the arc-length and the action is given by

$$
\mathcal{L}(\gamma)=\int_{\gamma} f\left(k_{1}, \ldots, k_{n}\right) d s
$$

where $k_{i}$ are the curvature functions of the curve $\gamma$. The simplest model of this kind has a Lagrangian very special: the action is proportional to the proper time along the worldline. This model describes a massive free particle.

The study of these mechanical systems depending on the first curvatures became intensively studied in the late eighties (after the work by Polyakov) as toy models of rigid strings and $(2+1)$-dimensional field theories with the Chern-Simons term. Since then, mainly due to the studies of Plyushchay, those systems were of independent interest and had taken on a life of its own.

[^0]The model of a particle with torsion was investigated in (2+1)-Minkowski space [1]. It was shown that, at classical level, the squared mass of the system is restricted from above and that, besides the massive solutions of the equations of motion, the model must also have massless and tachyonic solutions. A relativistic model of the anyon, describing the states of the particle with torsion with the maximum value of the mass, was constructed in [2]. In [3] the author obtains the classical equations of motion of the model whose Lagrangian is $f=-m+\alpha \tau$. This model of relativistic particle with torsion (whose action appears in the Bose-Fermi transmutation mechanism) is also studied in [4], where it is canonically quantized in the $(2+1)$-Minkowski and 3 -Euclidean spaces. The solutions of the equations of motion in the massive, massless and tachyonic sectors are found by using Hamiltonian formalism. In [5] the author reconsiders the simplest models describing spinning particles with rigidity, both massive and massless, and describes the moduli spaces of solutions in $(2+1)$-backgrounds with constant curvature.

In $d=(3+1)$ there are also some geometrical models of relativistic particles. It seems interesting to investigate these models and establish which of them have a maximal symmetry [6]. For instance, $f=a+b k_{1}, a \neq 0$, describes a massive relativistic boson [7]; $f=c k_{1}$ models a massless particle with an arbitrary (both integer and half-integer) helicity [8]. In [9] the author studied the consequences of coupling of the highest curvature $k_{3}$ to the Lagrangian of a massive spinless particle. More recently, [10], the authors consider a relativistic particle whose dynamics is determined by an action depending on the torsion $k_{2}$. The Euler-Lagrange equations are obtained but unfortunately, as the authors pointed out, these higher order differential equations do not appear to be tractable in general.

This paper is organized as follows. In Section 2 we briefly recall some basic facts about the geometry of curves in pseudo-Euclidean spaces and present the model, whose action is given by

$$
\mathcal{L}(\gamma)=\int_{\gamma}\left(p k_{2}+q\right) d s
$$

where $p$ and $q$ are constants, $s$ denotes the arc-length parameter on the curve and $k_{2}$ stands for its torsion (second curvature). The motion equations for these Lagrangians are obtained and solved in $d$-background gravitational fields, showing that the motion will be restricted to, at most, a 5 -dimensional subspace. In Section 3 we obtain three Killing vector fields along the curve from the boundary conditions in $d=4$. We will use these vector fields to integrate the Frenet equations and two of them can be interpreted as the linear and rotational momenta. In Section 4 we completely integrate the Frenet equations in $d=4$ and obtain parametrizations of the particle paths. Section 5 is devoted to discussion and concluding remarks.

## 2 The model and the equations of motion

Let $\mathbb{R}_{\nu}^{d}$ be the $d$-dimensional pseudo-Euclidean space $\mathbb{R}^{d}$ with background gravitational field $\mathrm{d} s^{2}=\langle$,$\rangle given by$

$$
\mathrm{d} s^{2}=-\sum_{i=1}^{\nu} \mathrm{d} x_{i}^{2}+\sum_{i=\nu+1}^{d} \mathrm{~d} x_{i}^{2},
$$

where $\left(x_{1}, \ldots, x_{d}\right)$ denote the usual rectangular coordinates in $\mathbb{R}^{d}$. As usual, let $\nabla$ denote the pseudo-Riemannian connection on $\mathbb{R}_{\nu}^{d}$. In this section we describe the geometry of non-degenerate curves in $\mathbb{R}_{\nu}^{d}$ in terms of the Frenet apparatus of the curve.

Let $\gamma:[0, L] \rightarrow \mathbb{R}_{\nu}^{d}$ be a non-degenerate curve parametrized by the arc-length. The existence of this natural parameter, called proper time, allows us to introduce the $d$ dimensional Frenet apparatus as a generalization of the classical 3-dimensional Frenet equations. When the index $\nu$ is greater than zero (so our background space is at least of Lorentzian nature) we must take some precautions.

The Frenet frame in the Euclidean case can be obtained as follows. Let $T=\gamma^{\prime}$ be the unit tangent vector to the curve. Clearly $T^{\prime}$ is an orthogonal vector to $T$, so there exist a function $k$ and a unit vector field $N$ along $\gamma$ such that $T^{\prime}=k N$. Now as above $N^{\prime}$ is orthogonal to $N$ so there exist a function $\tau$ and a vector field $B$ such that $N^{\prime}=-k T+\tau B$. And so on. The problems can appear e.g. when the ambient space is Lorentzian and $\gamma$ is a space-like curve. In this case $T^{\prime}$ (or $N^{\prime}$ ) may be a light-like vector, so that the above construction is not permitted. This family of curves are called $s$-degenerate curves and they are studied in [12]. In general, when the index of the metric of the ambient space is greater than the index of the curve, substantial differences with regard to the classical Frenet frame can appear. To avoid these inconveniences we will work out only with Frenet curves.

A non-null curve $\gamma$ is said to be a Frenet curve if there exist functions $\left\{k_{1}, \ldots, k_{d-1}\right\}$ and a system of orthonormal vector fields $\left\{T=\gamma^{\prime}, N_{1}, \ldots, N_{d-1}\right\}$ along $\gamma$ such that they fulfill the following equations:

$$
\begin{aligned}
\nabla_{T} T & =\varepsilon_{1} k_{1} N_{1} \\
\nabla_{T} N_{1} & =-\varepsilon_{0} k_{1} T+\varepsilon_{2} k_{2} N_{2} \\
\nabla_{T} N_{i} & =-\varepsilon_{i-1} k_{i} N_{i-1}+\varepsilon_{i+1} k_{i+1} N_{i+1}, \quad i=2, \ldots, d-2 \\
\nabla_{T} N_{d-1} & =-\varepsilon_{d-2} k_{d-1} N_{d-2}
\end{aligned}
$$

Here $\varepsilon_{0}=\langle T, T\rangle$ and $\varepsilon_{i}=\left\langle N_{i}, N_{i}\right\rangle$ for $i=1, \ldots, d-1$. Observe that curves in Euclidean space $\mathbb{R}^{d}$ and time-like curves in Lorentzian space $\mathbb{L}^{d}$ are always Frenet curves.

The fundamental theorem for Frenet curves tells us that the curvatures $k_{1}, \ldots, k_{d-1}$ completely determine the curve up to pseudo-Euclidean transformations (this can be shown with a similar technique, almost word for word, to that used in the Euclidean case, see e.g. [11]). Even more, given functions $k_{1}, \ldots, k_{d-1}$ we can always construct a Frenet curve, parametrized by the arc length, with these functions as curvature functions. Then any local geometrical scalar defined along Frenet curves can always be expressed as a function of their curvatures and derivatives.

We are going to consider dynamics with Lagrangians which linearly depend on the torsion of the relativistic particle. Our space $\Lambda$ of elementary fields in this theory is that of Frenet curves fulfilling given first order boundary data to drop out the boundary terms which appear when computing the first order variation of the action. We consider the action $\mathcal{L}: \Lambda \rightarrow \mathbb{R}$ given by

$$
\mathcal{L}(\gamma)=\int_{\gamma}\left(p k_{2}+q\right) d s
$$

where $p$ and $q$ are constants. The simplest action describing the motion of a particle is achieved when $p=0$, so that it is proportional to the proper time. The worldlines of the
particles are geodesic curves in the background space. Thus, from now on we suppose $p \neq 0$ and $k_{1} \neq 0$.

To compute the first-order variation of this action along the elementary fields space $\Lambda$, and so the field equations describing the dynamics of the particles, we use a standard argument involving some integrations by parts. Then by using the Frenet equations we have

$$
\begin{equation*}
\mathcal{L}^{\prime}(0)=[\mathcal{B}(\gamma, W)]_{0}^{L}-\int_{0}^{L}\left\langle\nabla_{T} P, W\right\rangle v d t, \tag{1}
\end{equation*}
$$

where the vector field $P$ is given by

$$
P=\varepsilon_{1} \varepsilon_{3} p \nabla_{T}\left(\frac{k_{3}}{k_{1}} N_{3}\right)+\varepsilon_{0} p k_{1} N_{2}+\varepsilon_{0} q T
$$

and the boundary term reads

$$
\mathcal{B}(\gamma, W)=\left\langle\nabla_{T}^{2} W, \varepsilon_{1} \frac{p}{k_{1}} N_{2}\right\rangle+\left\langle\nabla_{T} W,-\varepsilon_{1} \varepsilon_{3} p \frac{k_{3}}{k_{1}} N_{3}\right\rangle+\langle W, P\rangle,
$$

$W$ standing for a generic variational vector field along $\gamma$. We take curves with the same endpoints and having there the same Frenet frame, so that $[\mathcal{B}(\gamma, W)]_{0}^{L}$ vanishes. From here we obtain the following result.

## The trajectory $\gamma \in \Lambda$ is the worldline of a relativistic particle in the $d$ dimensional background $\mathbb{R}_{\nu}^{d}$ if and only if

## (i) The Frenet apparatus is well defined on the whole world trajectory. <br> (ii) The vector field $P$ is constant along $\gamma$.

In some sense, the vector field $P$ can be interpreted as the linear momentum of the particle and then the above is a consequence of the conserved linear momentum law.

A straightforward computation shows that $P$ is constant if and only if the following equations of motion hold:

$$
\begin{align*}
p k_{2}\left(1-\varepsilon \varphi^{2}\right)-q & =0  \tag{2}\\
k_{1}^{\prime}\left(1-\varepsilon \varphi^{2}\right)-3 \varepsilon k_{1} \varphi \varphi^{\prime} & =0  \tag{3}\\
-\varepsilon_{2} \varepsilon_{3} \varphi^{\prime \prime}+\varepsilon_{2} \varepsilon_{4} \varphi k_{4}^{2}-\varepsilon k_{1}^{2} \varphi\left(1-\varepsilon \varphi^{2}\right) & =0  \tag{4}\\
2 k_{4} \varphi^{\prime}+\varphi k_{4}^{\prime} & =0  \tag{5}\\
k_{3} k_{4} k_{5} & =0, \tag{6}
\end{align*}
$$

where $\varepsilon=\varepsilon_{0} \varepsilon_{1} \varepsilon_{2} \varepsilon_{3}$ and $\varphi=k_{3} / k_{1}$. The last equation yields $k_{5}=0$, so that the motion will be restricted to, at most, a 5 -dimensional subspace. On the other hand, from Eq. (5) we easily find that $\varphi^{2} k_{4}$ is a constant $B$, which determines $k_{4}$ in terms of the lower order curvatures. From Eq. (3) we obtain that $k_{1}^{2}\left(1-\varepsilon \varphi^{2}\right)^{3}$ is a constant $A$. Finally, using Eq. (2) we can also get $k_{2}$ in terms of $\varphi$. Then we have shown that all of curvatures depend on $\varphi$.

First, we are going to study what happens when $\varphi^{\prime}=0$. In this case there are only three possibilities for the critical curves:
(i) $\varepsilon=1, q=0, k_{4}=0$ and $k_{1}^{2}=k_{3}^{2}$,
(ii) they are helices with $k_{2}=\frac{q}{p}$ and $k_{3}=k_{4}=0$,
(iii) they are helices with $k_{2}=\frac{q}{p\left(1-\varepsilon \varphi^{2}\right)}$ and $k_{4}=\varepsilon_{2} \varepsilon_{4} \varepsilon k_{1}^{2}\left(1-\varepsilon \varphi^{2}\right)$.

If $\varphi^{\prime} \neq 0$, we can get a nice expression for $\varphi$. Observe that $\langle P, P\rangle=\varepsilon_{P} u^{2}$ is constant, because $\nabla_{T} P=0$. This fact along with the expression of $P$ in the Frenet frame

$$
P=\varepsilon_{0} q T+\varepsilon_{0} p k_{1}\left(1-\varepsilon \varphi^{2}\right) N_{2}+\varepsilon_{1} \varepsilon_{3} p \varphi^{\prime} N_{3}+\varepsilon_{1} \varepsilon_{3} \varepsilon_{4} p \varphi k_{4} N_{4}
$$

yields the following ordinary differential equation

$$
\varepsilon_{3} p^{2}\left(\varphi^{\prime}\right)^{2}=\frac{\left(\varepsilon_{P} u^{2}-\varepsilon_{0} q^{2}\right)\left(1-\varepsilon \varphi^{2}\right) \varphi^{2}-\varepsilon_{2} p^{2} A \varphi^{2}-\varepsilon_{4} B^{2}\left(1-\varepsilon \varphi^{2}\right)}{\left(1-\varepsilon \varphi^{2}\right) \varphi^{2}}
$$

It is easy to see that, whenever $\varphi^{\prime} \neq 0$, this equation, together with the expressions of curvatures in terms of $\varphi$, is equivalent to the Euler-Lagrange equations.

When $B=0$, we have that $k_{4}=0$. Then $\varphi$ can be integrated and its solution reads

$$
a \mathrm{E}\left(\arcsin (b \varphi), \frac{\varepsilon}{b^{2}}\right)=t+C_{1}
$$

where $C_{1}$ is an arbitrary constant, $a=\sqrt{\frac{\varepsilon_{3} p^{2}}{\varepsilon\left(\varepsilon_{P} u^{2}-\varepsilon_{0} q^{2}\right)}}, b=\sqrt{\frac{\varepsilon\left(\varepsilon_{P} u^{2}-\varepsilon_{0} q^{2}\right)}{-\varepsilon_{2} p^{2} A+\varepsilon_{P} u^{2}-\varepsilon_{0} q^{2}}}$ and E stands for the elliptic function of second kind.

## 3 Linear and rotational momenta in $\mathbb{R}_{\nu}^{4}$

In this section we use the boundary conditions to obtain three Killing vector fields along $\gamma$ which allow us to integrate the Frenet equations.

Let $Z_{1}$ and $Z_{2}$ be constant vector fields and consider $W=\gamma \wedge Z_{1} \wedge Z_{2}$. Then the boundary term reads

$$
\mathcal{B}(\gamma, W)=\left\langle\left(p N_{1} \wedge N_{2}-\varepsilon_{1} \varepsilon_{3} p \varphi T \wedge N_{3}+\gamma \wedge P\right) \wedge Z_{1}, Z_{2}\right\rangle
$$

Let $\Phi: \mathfrak{X}(\gamma) \rightarrow \mathfrak{X}(\gamma)$ be the mapping defined by

$$
\Phi(Z)=\left(p N_{1} \wedge N_{2}-\varepsilon_{1} \varepsilon_{3} p \varphi T \wedge N_{3}+\gamma \wedge P\right) \wedge Z
$$

where $\mathfrak{X}(\gamma)$ denotes the algebra of differentiable vector fields along the trajectory of the particle. Observe that $\langle\Phi(Z), Z\rangle=0$, for any vector field $Z$. It is not difficult to see that $\Phi$ is covariantly constant, i.e. $\nabla_{T} \Phi=0$, so that the vector fields $Q=\Phi(P)$ and $V=\Phi(Q)$ are also constant vectors and write down as

$$
\begin{aligned}
Q & =p N_{1} \wedge N_{2} \wedge P-\varepsilon_{1} \varepsilon_{3} p \varphi T \wedge N_{3} \wedge P \\
V & =p N_{1} \wedge N_{2} \wedge Q-\varepsilon_{1} \varepsilon_{3} p \varphi T \wedge N_{3} \wedge Q+\gamma \wedge P \wedge Q
\end{aligned}
$$

Then $J=-\gamma \wedge P \wedge Q+V$ is also a Killing vector field along $\gamma$. The vector fields $P, Q$ and $J$ are rewritten as

$$
\begin{align*}
P & =\varepsilon_{0} q T+\varepsilon_{0} p k_{1}\left(1-\varepsilon \varphi^{2}\right) N_{2}+\varepsilon_{1} \varepsilon_{3} p \varphi^{\prime} N_{3},  \tag{7}\\
Q & =-\varepsilon_{0} \varepsilon_{1} \varepsilon_{3} p^{2} \varphi^{\prime} T+\varepsilon_{0} \varepsilon_{3} p^{2} \varphi k_{1}\left(1-\varepsilon \varphi^{2}\right) N_{1}+\varepsilon_{0} \varepsilon_{3} p q N_{3},  \tag{8}\\
J & =p^{2}\left(-\varepsilon_{3} q T-\varepsilon_{0} \varepsilon_{1} \varepsilon_{2} p \varphi^{2} k_{1}\left(1-\varepsilon \varphi^{2}\right) N_{2}-\varepsilon_{0} \varepsilon_{1} p \varphi^{\prime} N_{3}\right), \tag{9}
\end{align*}
$$

and they can be interpreted as generators of the particle mass $M$ and spin $S$, with the massshell condition and the Majorana-like relation between $M$ and $S$ given by $\langle P, P\rangle=M^{2}$ and $\langle P, J\rangle=M S$. Note that there will be the possibility of tachyonic energy flow, since the mass could be positive, negative or zero, according to the causal character of the vector field $P$. Time-like and light-like trajectories are the natural ones in space-time geometries, but some recent experiments point out the existence of superluminal particles (space-like trajectories) without any breakdown of the principle of relativity; theoretical developments exist suggesting that neutrinos might be instances of "tachyons" as their square mass appears to be negative.

## 4 Integration of the Frenet equations

In this section we are going to integrate the Frenet equations in $d=4$ dimensions. Since $P$ and $Q$ determine privileged directions, it is natural to introduce cylindrical coordinates in $\mathbb{R}_{\nu}^{4}$ with $P$ and $Q$ as axes. It can be shown that $P$ and $Q$ are linearly independent when $q \neq 0$. This is trivial provided $\varphi^{\prime} \neq 0$. Otherwise the linear independence follows from

$$
\left\langle P \wedge Q \wedge N_{1}, N_{2}\right\rangle=\varepsilon_{3} p q^{2} \neq 0
$$

However, when $q=0$, there are the following possibilities
(i) the solutions are circles and $P$ is normal to the plane where the circle is lying;
(ii) $\varepsilon=1$ and $k_{1}^{2}=k_{3}^{2}$. In this case $P$ and $Q$ both are zero and so the integration can not be made.

Therefore, we have to consider three cases according to either $P$ and $Q$ both are non-null or one of them is null. Notice that $P$ and $Q$ can not be simultaneously null, as they are orthogonal and we have excluded the case where they are linearly dependent.

## 4.1 $P$ and $Q$ both are non-null

We can choose an orthonormal coordinate system $\left\{z_{1}, z_{2}, z_{3}, z_{4}\right\}$ in such a way that $P$ and $Q$ are collinear with $\partial_{z_{1}}$ and $\partial_{z_{2}}$, respectively. Let $\varepsilon_{r}=\left\langle\partial_{z_{3}}, \partial_{z_{3}}\right\rangle$ and $\varepsilon_{\theta}=\left\langle\partial_{z_{4}}, \partial_{z_{4}}\right\rangle$, and consider the family of rotations $R_{\theta}=e^{\theta L}, L$ being the matrix

$$
L=\left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\varepsilon_{r} \\
0 & 0 & \varepsilon_{\theta} & 0
\end{array}\right)
$$

Then we can define

$$
\Psi\left(z_{1}, z_{2}, r, \theta\right)=R_{\theta}\left(z_{1} \partial_{z_{1}}+z_{2} \partial_{z_{2}}+r \partial_{z_{3}}\right)
$$

a system of cylindrical coordinates $\left(z_{1}, z_{2}, r, \theta\right)$ in $\mathbb{R}_{\nu}^{4}$ such that $\gamma \wedge \partial_{z_{1}} \wedge \partial_{z_{2}}=\partial_{\theta}$.
By using the invariance under translations, it is not difficult to see that $J$ can be written as

$$
J=-\frac{\varepsilon_{P} \varepsilon_{Q} v^{2}}{u} \partial_{z_{1}}-u v \partial_{\theta},
$$

where $v^{2}=\varepsilon_{Q}\langle Q, Q\rangle$. Here we have used that $\langle P, J\rangle=-\langle Q, Q\rangle$ and $\langle Q, J\rangle=0$. It is also easy to see that $v^{2}=\varepsilon_{0} \varepsilon_{3} \varepsilon_{Q} p^{2}\left(\varepsilon_{P} u^{2}-\varepsilon_{2} p^{2} A\right)$. Finally, writing $T$ in the basis $\left\{\partial_{z_{1}}, \partial_{z_{2}}, \partial_{r}, \partial_{\theta}\right\}$, we obtain $P, Q, J$ and $T$ in both Frenet and cylindrical frames. Then take the products of each two of them and integrate to get the equations

$$
\begin{aligned}
& z_{1}(s)=\varepsilon_{P} \frac{q}{u} s+c_{1}, \\
& z_{2}(s)=-\varepsilon_{1} \varepsilon_{3} \varepsilon_{Q} \frac{p^{2} \varphi(s)}{v}+c_{2}, \\
& r(s)^{2}=\frac{\varepsilon_{\theta}}{u^{2} v^{2}}\left(\varepsilon_{P} \frac{p^{4} u^{4}-v^{4}}{u^{2}}-\varepsilon_{2} p^{6} A\left(1+\varepsilon \varphi(s)^{2}\right)\right), \\
& \theta(s)=\frac{\varepsilon_{1} \varepsilon \varepsilon_{Q} \varepsilon_{P} p^{4} A}{u^{3} v} \int_{0}^{s} \frac{1}{r(\mu)^{2}} d \mu+c_{3},
\end{aligned}
$$

where $c_{1}, c_{2}$ and $c_{3}$ are arbitrary constants. Note that here we have used the relation

$$
\langle J, J\rangle=p^{4}\left(\varepsilon_{P} u^{2}-\varepsilon_{2} p^{2} A\left(1+\varepsilon \varphi^{2}\right)\right)
$$

to get the integration.

## 4.2 $P$ is null and $Q$ is non-null

Take an orthonormal coordinate system $\left(z_{1}, z_{2}, z_{3}, z_{4}\right)$ in such a way that $P=\frac{1}{\sqrt{2}}\left(\partial_{z_{1}}-\partial_{z_{3}}\right)$, with $\left\langle\partial_{z_{1}}, \partial_{z_{1}}\right\rangle=1$ and $\left\langle\partial_{z_{3}}, \partial_{z_{3}}\right\rangle=-1$, and $Q$ being collinear with $\partial_{z_{2}}$. Denote $\varepsilon_{\theta}=$ $\left\langle\partial_{z_{4}}, \partial_{z_{4}}\right\rangle$ and set $R_{\theta}=e^{\theta L}$, where $L$ is

$$
L=\left(\begin{array}{rrrr}
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\varepsilon_{\theta} & 0 & \varepsilon_{\theta} & 0
\end{array}\right)
$$

and the pseudo-orthonormal frame $\left\{v_{1}=P, v_{2}=\partial_{z_{2}}, v_{3}=\frac{1}{\sqrt{2}}\left(\partial_{z_{1}}+\partial_{z_{3}}\right), v_{4}=\partial_{z_{4}}\right\}$. The cylindrical coordinates in $\mathbb{R}_{\nu}^{4}$ defined by

$$
\Psi\left(x_{1}, x_{2}, r, \theta\right)=R_{\theta}\left(x_{1} v_{1}+x_{2} v_{2}+r v_{3}\right)
$$

satisfy $\gamma \wedge v_{1} \wedge v_{2}=\partial_{\theta}$.
By using the invariance under translations, we see that $J$ can be written as

$$
J=-\varepsilon_{Q} v^{2} v_{3}-v \partial_{\theta} .
$$

Furthermore, one has $v_{3}=-\varepsilon_{\theta} \frac{\theta^{2}}{2} P+\partial_{r}-\frac{\theta}{r} \partial_{\theta}$, so that

$$
J=\varepsilon_{Q} \varepsilon_{\theta} v^{2} \frac{\theta^{2}}{2} P-\varepsilon_{Q} v^{2} \partial_{r}+v\left(\frac{\varepsilon_{Q} v \theta}{r}-1\right) \partial_{\theta} .
$$

Proceeding as above we find

$$
\begin{aligned}
x_{1}(s) & =\frac{\varepsilon_{\theta}}{v^{2}}\left(\varepsilon_{0} \varepsilon_{3} q p^{2} s+\varepsilon_{\theta} v \theta(s) r(s)\left(\varepsilon_{Q} \frac{v \theta(s)}{2}-r(s)\right)+2 \varepsilon_{\theta} v q \int_{0}^{s} r(\mu) \theta(\mu) d \mu\right)+c_{1}, \\
x_{2}(s) & =\varepsilon_{P} \frac{q}{u} s+c_{2}, \\
r(s) & =q s+c_{3}, \\
\theta(s) & =\frac{1}{2 v}\left(\varepsilon_{Q} r(s)-\frac{\varepsilon_{0} \varepsilon_{3} \varepsilon_{\theta} p^{2}\left(1+\varepsilon \varphi(s)^{2}\right)}{r(s)}\right),
\end{aligned}
$$

where $c_{1}, c_{2}$ and $c_{3}$ are arbitrary constants.

## 4.3 $P$ is non-null and $Q$ is null

Let $\left(z_{1}, z_{2}, z_{3}, z_{4}\right)$ be an orthonormal coordinate system such that $P$ is collinear with $\partial_{z_{1}}$ and $Q=\frac{1}{\sqrt{2}}\left(\partial_{z_{2}}-\partial_{z_{3}}\right)$, being $\partial_{z_{2}}$ spacelike and $\partial_{z_{3}}$ timelike. Set $\varepsilon_{\theta}=\left\langle\partial_{z_{4}}, \partial_{z_{4}}\right\rangle$ and define $R_{\theta}=e^{\theta L}, L$ being

$$
L=\left(\begin{array}{rrrr}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 \\
0 & \varepsilon_{\theta} & \varepsilon_{\theta} & 0
\end{array}\right)
$$

and the pseudo-orthonormal frame $\left\{v_{1}=\partial_{z_{1}}, v_{2}=Q, v_{3}=\frac{1}{\sqrt{2}}\left(\partial_{z_{2}}+\partial_{z_{3}}\right), v_{4}=\partial_{z_{4}}\right\}$. Then the cylindrical coordinates defined by

$$
\Psi\left(x_{1}, x_{2}, r, \theta\right)=R_{\theta}\left(x_{1} v_{1}+x_{2} v_{2}+r v_{3}\right)
$$

provide a new coordinate system $\left(x_{1}, x_{2}, r, \theta\right)$ in $\mathbb{R}_{\nu}^{4}$ verifying $\gamma \wedge v_{1} \wedge v_{2}=\partial_{\theta}$.
Using again the invariance under translations, we have

$$
J=-u \partial_{\theta},
$$

yielding the equations

$$
\begin{aligned}
x_{1}(s) & =\varepsilon_{P} \frac{q}{u} s+c_{1} \\
x_{2}(s) & =-\frac{\varepsilon_{2} \varepsilon_{3} \varepsilon_{P}\left(1+\varepsilon s^{2}\right)}{2 u^{2} s}+c_{2}, \\
r(s)^{2} & =p^{4} \varphi(s)^{2} \\
\theta(s) & =\frac{\varepsilon_{0} \varepsilon_{3} \varepsilon_{\theta} q}{u p^{2}} \int_{0}^{s} \frac{1}{\varphi(\mu)^{2}} d \mu+c_{3},
\end{aligned}
$$

where $c_{1}, c_{2}$ and $c_{3}$ are arbitrary constants.

## 5 Conclusions and final remarks

We have studied actions in $d$-dimensional background spaces whose Lagrangian depends linearly on the torsion (second curvature) of the particle path, completing previous works. We have shown that the motion will be restricted to, at most, a 5 -dimensional subspace and completely solved the Euler-Lagrange equations in $d=4,5$. Finally we have integrated the Frenet equations in four dimensions.

These techniques can be extended and applied to other models that depend on any other curvature. However the system of Euler-Lagrange equations obtained are quite complicated to study and solve.

We encourage to go into the 5-dimensional case in order to completely integrate the Cartan equations of the worldlines and provide parametrizations of the trajectory paths.

Finally, even though we have got an explicit description of the motion equation in spaces with zero constant curvature, we note that a priori there is no restriction to apply these ideas in background gravitational fields with non-zero constant curvature or even with non-constant curvature. The extension to a curved background may be a stimulating task.

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