Relativistic particles with rigidity along light-like curves

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Abstract

We study actions in $(2+1)$-dimensions associated with null curves whose Lagrangians are arbitrary functions f on the curvature of the particle path, showing that null helices are always possible trajectories of the particles for every function f. The vector field P, obtained from the Euler-Lagrange equation, can be interpreted as the linear momentum of the particle since it is constant along the curve, which agrees with the conserved linear momentum law. The cases when f is constant or linear are completely solved and, by using Killing vector fields, we are able to integrate the Cartan equations in cylindrical coordinates around the linear momentum P.

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1 Introduction

For the past fifteen years, many interesting papers concerning Lagrangians describing spinning particles have been published (see e.g. [1]–[18] and references therein). In the general situation, as it is well known, one has to provide the classical model with the extra bosonic variables. To this end, an interesting hypothesis deals with Lagrangians on higher geometrical invariants to supply those extra degrees of freedom. This approach has the interesting point of view that the spinning degrees of freedom are encoded in the geometry of its world trajectories. The PoincarÚ and invariance requirements imply that an admissible Lagrangian density F must depends on the extrinsic curvatures of curves in the background gravitational field. In particular, the Lagrangians depending on the first and second curvatures have been intensively studied in the late eighties and in the nineties. At the beginning those systems were studied as toy models of rigid strings and (2+1)-dimensional field theories with the Chern-Simons term but shortly after, mainly due to the papers by Plyushchay, those systems are of independent interest.

The actions considered before are defined on non-isotropic curves (spacelike or timelike), but on $(d+1)$-spacetimes one can also consider actions defined on null (lightlike)

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curves. The studies of Lagrangians on these curves begin in the late nineties by considering the simplest geometrical particle model associated with null paths in four-dimensional Minkowski spacetime, [19], where the action is proportional to the pseudo-arc of the particle. The authors obtain the equations of motion and show that they are particular examples of null helices. The same authors consider in [20] this geometrical particle model associated with null curves in (2+1)-dimensions.

The next step deals with a more complicated three-dimensional system where the action is a linear function on the curvature of the curve. In [21] the authors show that its mass and spin spectra are defined by one-dimensional nonrelativistic mechanics with a cubic potential. Recently, in [22] we obtain, using geometrical methods, a complete description of the relativistic particle paths.

This paper concerns with actions in (2+1)-dimensions whose Lagrangians are arbitrary functions f on the curvature of the particle path. The paper is organized as follows. In Section 2 we present the model, whose action $\mathfrak{L} : \Lambda \to \mathbb{R}$ is given by $\mathfrak{L}(\gamma) = \int_\gamma f(k)dt$, where f is a differentiable function and k stands for the curvature of the null curve. The equations of motion for these Lagrangians are completely given in (2+1)-background gravitational fields. In Section 3 we solve the motion equations and get the null worldlines of the relativistic particles in cylindrical coordinates. To this end, we distinguish two cases: the linear momentum P is non-null (space-like or time-like) or null. In Section 4 we make a deeper study when f is a quadratic function, including the constant and linear cases. Finally, Section 5 is devoted to discussion and concluding remarks.

2 The model and the equations of motion

Let $\mathbb{L}_1^3$ denote a 3-dimensional Lorent-Minkowski space with background gravitational field $\langle , \rangle$ and Levi-Civita connection ∇. First of all, we describe the geometry of light-like (or null) curves in $\mathbb{L}_1^3$ in terms of the Cartan frame of the curve (see [23] for details).

Let $\gamma : [a, b] \to \mathbb{L}_1^3$ be a null Cartan curve such that $\{\gamma'(t), \gamma''(t), \gamma'''(t)\}$ is positively oriented for all $t \in [a, b]$. Let us consider its Cartan frame $\{L = \gamma', W, N\}$, where

$$\begin{aligned} \langle L, L\rangle = \langle N, N\rangle = 0, \qquad \langle L, N\rangle = -1, \\ \langle W, L\rangle = \langle W, N\rangle = 0, \qquad \langle W, W\rangle = 1, \end{aligned}$$

with the vector product $\times$ given by $L \times W = -L$, $L \times N = -W$ and $W \times N = -N$. The Cartan equations read

$$\begin{aligned} \nabla_L L &= W, \\ \nabla_L W &= -kL + N, \\ \nabla_L N &= -kW, \end{aligned} \tag{1}$$

where ∇ denotes covariant derivative and k is the *curvature* (sometimes called *torsion* since it is obtained from the third derivative of the relativistic null path) of the curve. The fundamental theorem for null curves tells us that k determines completely the null curve up to Lorentzian transformations (see [23, Theorem 4]). Even more, given a function k we can always construct a null curve, parametrized by the pseudo-arc length parameter, whose curvature function is precisely k (see [23, Theorem 3]). Then any local geometrical scalar defined along null curves can always be expressed as a function of its curvature and derivatives.

In this section we analyze mechanical systems with Lagrangians depending arbitrarily on the curvature of the light-like curve. The space of elementary fields in this model is the set Λ of all null Cartan curves fulfilling given first order boundary data to drop out the boundary terms which appear when computing the first order variation of the action.

The letter γ will also denote a variation of null curves $\gamma = \gamma(s, \omega) : [0, 1] \times (-\varepsilon, \varepsilon) \to \mathbb{L}^3$ with $\gamma(s, 0)$ the reparametrization of $\gamma(t)$. Associated with such a variation is the variational vector field $V(s) = V(s, 0)$, where $V = V(s, \omega) = \frac{\partial \gamma}{\partial \omega}(s, \omega)$. We denote by δ the differentiable function verifying $\frac{\partial \gamma}{\partial s}(s, \omega) = \delta(s, \omega) L(s, \omega)$. We write $\gamma(t), k(t, \omega), V(t)$, etc., for the corresponding pseudo-arc length parameter.

The actions $\mathcal{L}$ for the curve depend locally on its geometry and they possess various symmetries, both local and global. The local symmetry is reparametrization invariance and it restricts severely the form of $\mathcal{L}$. We consider the action $\mathcal{L} : \Lambda \to \mathbb{R}$ given by

$$\mathcal{L}(\gamma) = \int_\gamma f(k) dt,$$

where f is a differentiable function. The simplest action describing the motion of a particle is achieved when $f(k)$ is proportional to the pseudo-arc length parameter, and it is studied by Nersessian and Ramos in [19, 20]. When the action is linear on the curvature of the particle path, some advances have been produced in [21, 22]. No other cases appear to have been considered.

A null curve γ will be a critic point of the action $\mathcal{L}$ if

$$\left.\frac{d}{d\omega}\right|_{\omega=0} \mathcal{L}(\gamma_\omega) = \left.\frac{d}{d\omega}\right|_{\omega=0} \int_{\gamma_\omega} f(k_\omega) dt = 0,$$

for all variation of null curves γ_ω of γ. Next we present a necessary Lemma for our computations.

Using the above notation, the following assertions hold:

(a) $\quad \boldsymbol{0 = \langle \nabla_L V, L \rangle}$;

(b) $\quad \boldsymbol{\frac{\partial \delta}{\partial \omega} = V(\delta) = -\frac{1}{2} h \delta, \quad h = -\left\langle \nabla_L^2 V, W \right\rangle}$;

(c) $\quad \boldsymbol{\frac{\partial k}{\partial \omega} = \left\langle \nabla_L^3 V, N \right\rangle + k \langle \nabla_L V, N \rangle + kh - \frac{1}{2} L(L(h))}$.

A vector field V along γ which infinitesimally preserves the causal character, the pseudo-arc length parameter and the curvature of γ is said to be a Killing vector field along γ. Hence Killing vector fields along γ are characterized by the equations

$$\langle \nabla_L V, L \rangle = V(\delta) = V(k) = 0.$$

As we will see, the Killing vector fields plays an important role to integrate the Euler-Lagrange and Cartan equations.

Let γ a immersed null curve in $\mathbb{L}^3$. A vector field V on γ is a Killing vector field along γ if and only if it extends to a Killing field $\tilde{V}$ on $\mathbb{L}^3$.

The same conclusion is true if we consider a complete, simply connnected, Lorentzian space form, but it is not needed in this paper.

To compute the first-order variation of this action along the elementary fields space Λ, and so the field equations describing the dynamics of the particle, we use a standard argument involving some integrations by parts. Then the Cartan equations yield

$$\mathcal{L}'(0) = \frac{1}{2}\left[\Omega(\gamma, V)\right]_a^b - \frac{1}{2}\int_a^b \langle V, \mathcal{E}(\gamma)L\rangle \, dt, \tag{2}$$

where

$$\mathcal{E}(\gamma) = \varphi''' + (k\varphi)' + k\varphi', \qquad \varphi = -f(k) + 2kf'(k) + k''f''(k) + (k')^2 f^{(3)}(k) \tag{3}$$

and the frontier term read

$$\begin{aligned}\Omega(\gamma, V) \;\; = \;\; & \left\langle \nabla_L^3 V, f'(k)W\right\rangle + \left\langle \nabla_L^2 V, f'(k)(3N - kL) - f''(k)k'W\right\rangle \\ & + \left\langle \nabla_L V, (f(k) + f''(k)k'' + f^{(3)}(k)(k')^2)W - 2f''(k)k'N\right\rangle + \langle V, P\rangle \, .\end{aligned}$$

Here the vector field P is given by

$$P = (\varphi'' + k\varphi)L - \varphi' W + \varphi N, \tag{4}$$

V standing for a generic variational vector field along γ.

To drop out $[\Omega(\gamma, V)]_a^b$ we have to consider curves with the same endpoints and having the same Cartan frame in these points. Under these conditions, the first-order variation reads

$$\mathcal{L}'(0) = -\frac{1}{2}\int_a^b \langle V, \mathcal{E}(\gamma)L\rangle \, dt.$$

We obtain the following result.

The trajectory $\gamma \in \Lambda$ is the null worldline of a relativistic particle in the (2+1)-dimensional spacetime if and only if

- **(i) W, N and k are well defined in the whole world trajectory.**
- **(ii) The following differential equation is fulfilled: $\mathcal{E}(\gamma) = 0$.**

A straightforward computation shows that $\mathcal{E}(\gamma)L = \nabla_L P$, so that P is a constant vector field along γ if and only if γ is a critical point of $\mathcal{L}$. In this case, $\langle P, P\rangle = \varepsilon p^2$ is constant, where $p = ||P||$ and $\varepsilon = 1, -1$ according to P is space-like or time-like, respectively. In some sense, the vector field P can be interpreted as the linear momentum of the particle and then the above is a consequence of the conserved linear momentum law.

3 The solutions of the equations of motion

The main goal of this section is to integrate the motion equations of Lagrangians giving models for relativistic particles that involve an arbitrary function on the curvature of the null path.

First of all, we easily see that curves with constant curvature (i.e. null helices, [23, 24]) are always possible trajectories of the particles for every Lagrangian function f, constructed out of the geometrical quantities that characterize the curve (the curvature and its derivatives). When the trajectory is not a helix, note that non-zero vector field P possesses a non-vanishing space-like component orthogonal to the light-like particle trajectory, which seems to be a manifestation of a generic feature of higher-derivative theories.

Secondly, if the action is proportional to the pseudo-arc length of the particle path (i.e. f is a constant function) then we have that its solutions are also null helices, [20]. The classical phase space of this system agrees with that of a massive spinning particle of spin $s = c^2/m$, where m is the particle mass and c is the coupling constant in front of the action.

By using that our local action is invariant under rotations we deduce that the vector field X given by

$$X = \Big((k')^2 f^{(3)}(k) + k'' f''(k) + f(k)\Big) L - 2k' f''(k) W + 2f'(k) N + P \times \gamma$$

is constant along γ. Then

$$J = -P \times \gamma + X = \Big((k')^2 f^{(3)}(k) + k'' f''(k) + f(k)\Big) L - 2k' f''(k) W + 2f'(k) N \qquad (5)$$

is a Killing vector field along γ that jointly with the constant vector field P allow us to find non-trivial first integrals of the Euler-Lagrange equations. Furthermore, it is follows easily that

$$\nabla_L J = -\varphi' L + \varphi W. \qquad (6)$$

We can observe that if $\varphi = 0$, is satisfied the Euler-Lagrange equation $\mathcal{E}(\gamma) = 0$, so we can distinguish two types of solutions of the equation $\mathcal{E}(\gamma) = 0$ depending on $\varphi = 0$ or $\varphi \neq 0$. In the first case and using (4) and (6) we have that $P = 0$ and

$$J = (2f(k) - 2kf'(k))L - 2k' f''(k) W + 2f'(k) N$$

is constant. Therefore $\langle J, J\rangle = \varepsilon j^2$ is a constant of the motion and a first integral of the equation $\varphi = 0$.

If $\varphi \neq 0$, it can be shown that

$$J = -P \times \gamma + \varepsilon\omega P^*, \qquad \varepsilon = \pm 1, \qquad (7)$$

where ω is constant and P^* is a vector field with the same causal character as P and satisfying $\langle P, P^*\rangle = \varepsilon$. Then $\langle P, J\rangle = \omega$. Bearing in mind Eqs. (3) and (5) we obtain that f and k have to fulfill the following ordinary differential equations

$$(\varphi')^2 - 2\varphi(\varphi'' + k\varphi) = \varepsilon p^2,$$
$$-2f'(k)\varphi'' + 2k' f''(k)\varphi' - 2f(k)\varphi - \varphi^2 = \omega.$$

The Killing vector fields P and J can be interpreted as generators of the particle mass m and spin s, with the mass-shell condition and the Majorana-like relation between m and s given by

$$\langle P, P\rangle = \varepsilon p^2 = m^2,$$
$$\langle P, J\rangle = \omega = ms.$$

Note that there will be the possibility of tachyonic energy flow, since the mass could be positive, negative or zero, according to the causal character of the vector field P. Time-like and light-like trajectories are the natural ones in space-time geometries, but some recent experiments point out the existence of superluminal particles (space-like trayectories) without any breakdown of the principle of relativity; theoretical developments exist suggesting that neutrinos might be instances of "tachyons" as their square mass appears to be negative. Then in order to integrate the equations of Cartan when $\varphi \neq 0$ we must consider all possible cases: P is non-null (space-like or time-like) or P is null.

3.1 P is non-null

As P determines a privileged direction, it is natural to introduce cylindrical coordinates in $\mathbb{L}^3$ with P as the z-axis. Then $P = p\partial_z$ and from Eq. (7) we find $J = \varepsilon(\omega/p)\partial_z - p\partial_\theta$. It is easy to see that the only non-zero products between coordinate vector fields are $\langle \partial_z, \partial_z\rangle = \varepsilon$, $\langle \partial_\theta, \partial_\theta\rangle = -\varepsilon r^2$ and $\langle \partial_r, \partial_r\rangle = 1$. From here we get $\langle J, J\rangle = \varepsilon(\omega^2/p^2 - p^2 r^2)$, $\langle L, P\rangle = \varepsilon p z'$ and $\langle L, J\rangle = (\omega/p)z' + \varepsilon p r^2\theta'$. All these equations yield the following result.

Let $\gamma \subset \mathbb{L}^3$ be the null worldline of a relativistic particle in the (2+1)-dimensional space-time, P being a non-null vector field. Then γ can be described in cylindrical coordinates around P as follows

$$r^2 = \frac{\omega^2}{p^4} - \frac{\varepsilon}{p^2}\langle J, J\rangle, \quad z' = \frac{\varepsilon}{p}\langle L, P\rangle, \quad \theta' = \frac{p(p^2\langle L, J\rangle - \varepsilon\omega\langle L, P\rangle)}{\varepsilon\omega^2 - p^2\langle J, J\rangle}, \tag{8}$$

where

$$\begin{aligned}
\langle J, J\rangle &= -4\left(f(k)f'(k) + k''f'(k)f''(k) + (k')^2\left(-f''(k)^2 + f'(k)f^{(3)}(k)\right)\right), \\
\langle L, P\rangle &= f(k) - 2kf'(k) - k''f''(k) - (k')^2 f^{(3)}(k), \\
\langle L, J\rangle &= -2f'(k).
\end{aligned} \tag{9}$$

3.2 P is null

In this case we are going to introduce a coordinate system similar to cylindrical coordinates. Without loss of generality, we may assume that P is collinear with $(1,1,0)$ (in the usual rectangular coordinates of $\mathbb{L}^3$). Then we consider the coordinates (r,θ,z) given by the following parametrization

$$X(r,\theta,z) = \left(z - \frac{\varepsilon r}{2}(\theta^2+1), z - \frac{\varepsilon r}{2}(\theta^2-1), -\varepsilon r\theta\right),$$

where $\theta, z \in \mathbb{R}$ and $r \in \mathbb{R}\backslash\{0\}$. These coordinates are called the *null cylindrical coordinates around* P (or with axis P). It is not difficult to check that $\langle \partial_\theta, \partial_\theta \rangle = r^2$ and $\langle \partial_r, \partial_z \rangle = \varepsilon$, being zero all the remaining metric products. Then we have $P = a\partial_z$, a being a nonzero constant, and we can choose $P^* = (-\varepsilon/a)(1, 0, 1)$. Now it is easy to see that

$$\begin{aligned}
P^* &= -\frac{\varepsilon}{2a}(\theta-1)^2\partial_z - \frac{\theta-1}{ar}\partial_\theta + \frac{1}{a}\partial_r,\\
J &= -\frac{\omega}{2a}(\theta-1)^2\partial_z + \Big(a - \frac{\varepsilon\omega}{ar}(\theta-1)\Big)\,\partial_\theta + \frac{\varepsilon\omega}{a}\partial_r.
\end{aligned}$$

These equations give us the description of the curve γ. More precisely,

Let $\gamma \subset \mathbb{L}^3$ be the null worldline of a relativistic particle in the (2+1)-dimensional space-time, P being a null vector field. Then γ can be described in cylindrical coordinates around P as follows

$$\begin{array}{l|l}
\boldsymbol{\omega = 0} & \boldsymbol{\omega \neq 0} \\
\hline
\boldsymbol{r^2 = \dfrac{1}{a^2}\langle J, J\rangle} & \boldsymbol{r' = \dfrac{\varepsilon}{a}\langle L, P\rangle} \\
\boldsymbol{\theta' = a\dfrac{\langle L, J\rangle}{\langle J, J\rangle}} & \boldsymbol{\theta = \dfrac{a^2r^2 - \langle J, J\rangle}{2\varepsilon\omega r} + 1} \\
\boldsymbol{z' = -\dfrac{a}{2}\dfrac{\langle L, J\rangle^2}{\langle J, J\rangle\langle L, P\rangle}} & \boldsymbol{z' = -\dfrac{a}{2}\dfrac{r^2\theta'^2}{\langle L, P\rangle}}
\end{array} \tag{10}$$

where $\langle J, J\rangle$, $\langle L, P\rangle$ and $\langle L, J\rangle$ are given in Eq. (9)**.**

4 When f is a quadratic function

This section deals with a relativistic particle whose dynamics is described by a local action with a quadratic Lagrangian function $f(k) = \rho k^2 + \mu k + \lambda$, for certain constants ρ, μ and λ. Observe that we are considering a higher-derivative model, with $f = f(\gamma', \gamma'', \gamma''')$. However this dependence of f on the embedding functions γ and their derivatives does not break PoincarÚ invariance.

Now we are going to study the corresponding ordinary differential equations for different values of the constants ρ, μ and λ.

Case 1: $\rho = \mu = 0$, $\lambda \neq 0$ (the constant case)

This case represents the simplest action describing the motion of a particle, since it is proportional to the proper time along the light-like trajectory of the particle in space-time. We have $P = -\lambda(kL + N)$ and $J = \lambda L$, so that $\langle P, P\rangle = -2\lambda^2 k = \varepsilon p^2$ and $\langle P, J\rangle = \lambda^2 = \omega$, and therefore

$$k = -\frac{\varepsilon p^2}{2\omega}.$$

This shows that γ is a Cartan helix, [23, 24], with axis given by the vector P. Note that $\omega \neq 0$, otherwise $\lambda = 0$ which can not hold. Moreover, massive (tachyonic) solutions correspond to the null helices with negative (positive) curvature. This was shown by Nersessian and Ramos using a Hamiltonian formulation for this geometrical model, [20]. Here we offer an alternative proof which exploits the geometry of the particle trajectories.

Case 2: $\rho = 0$, $\mu \neq 0$ (the linear case)

Without loss of generality we normalize the constant μ to be one, then we find $P = (k'' + k^2 - \lambda k)L - k'W + (k - \lambda)N$ and $J = (k + \lambda)L + 2N$. In this case $\varphi = k - \lambda$ and we obtain a first solution when $\varphi = 0$, or equivalent $k = \lambda$, that is, γ is a Cartan helix. So, the constant vector field $J = 2\lambda L + 2N$ provided us a constant of the motion given by $\langle J, J\rangle = -8\lambda$.

If $\varphi \neq 0$, the first integrals provided by the vector fields P and J read

$$\begin{aligned} (k')^2 - 2(k - \lambda)(k'' + k^2 - \lambda k) - \varepsilon p^2 &= 0, \\ -2k'' - 3k^2 + 2\lambda k + \lambda^2 - \omega &= 0. \end{aligned} \tag{11}$$

From that we obtain

$$(k')^2 + k^3 - \lambda k^2 + (\omega - \lambda^2)k + \lambda^3 - \omega\lambda - \varepsilon p^2 = 0, \tag{12}$$

which can be written as $(k')^2 + Q(k) = 0$, Q being the polynomial $Q(X) = X^3 - \lambda X^2 + (\omega - \lambda^2)X + \lambda^3 - \omega\lambda - \varepsilon p^2$. Putting $q = k + \lambda$ we recover Eq. (39) in [21], showing that the system under consideration contains massive and tachyonic branches. Later we will come back to this, when we determine the curvature functions of the particle trajectories in both sectors.

By using standard techniques involving the elliptic Jacobi functions, the solution can be found in terms of the roots α_1, α_2 and α_3 of the equation $Q(X) = 0$. First, assume that all roots of Q are real, $\alpha_1 \leq \alpha_2 \leq \alpha_3$. Then it is well-known that

$$\begin{aligned} \lambda &= \alpha_1 + \alpha_2 + \alpha_3, \\ \omega - \lambda^2 &= \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3, \\ \varepsilon p^2 + \omega\lambda - \lambda^3 &= \alpha_1\alpha_2\alpha_3, \end{aligned} \tag{13}$$

from which we easily deduce

$$\alpha_1 \leq \frac{\lambda}{3}, \quad \alpha_2 \leq \frac{\lambda - \alpha_1}{2}. \tag{14}$$

Before obtaining all solutions, note that since $Q(k) = -(k')^2$ then k takes values only where Q is negative. Trivial solutions are $k(s) = \alpha_i$, where α_i is a real root of Q, so that we find again the null Cartan helices. In this case $\langle P, P\rangle = -2k(k - \lambda)^2$ and $\langle P, J\rangle = -3k^2 + 2\lambda k + \lambda^2$. As before, the massive and tachyonic sectors correspond with negative or positive curvature, respectively. Now we are going to analyze all possible cases.

I. Q has a real root of multiplicity 3: $\alpha = \alpha_1 = \alpha_2 = \alpha_3$

We have that $\alpha = \lambda/3$ and the curvature function is given by

$$k(t) = \frac{\lambda}{3} - \frac{4}{(t+E)^2}, \quad s \in (-\infty, \lambda/3)$$

where E is a constant of integration, depending on the initial conditions, satisfying that $t+E$ is always different from zero. From Eq. (11) or (13) we find the relations $8\lambda^3 + 27\varepsilon p^2 = 0$ and $4\lambda^2 - 3\omega = 0$. Note that the constant of the motion εp^2 and ω are completely determined by the constant λ.

II. Q has two real roots, the lowest with multiplicity 2: $\alpha = \alpha_1 = \alpha_2 < \alpha_3$

The root α_3 is given by $\lambda - 2\alpha$. There are two possibilities:

$$k(t) = \lambda - 2\alpha + (3\alpha - \lambda)\coth^2\left(\frac{1}{2}\sqrt{\lambda - 3\alpha}(t+E)\right), \quad s \in (-\infty, \alpha)$$
$$k(t) = \lambda - 2\alpha + (3\alpha - \lambda)\tanh^2\left(\frac{1}{2}\sqrt{\lambda - 3\alpha}(t+E)\right), \quad s \in (\alpha, \lambda - 2\alpha]$$

where E is a constant. In this case the relations among α, λ, ω and p are $-2\alpha(\alpha-\lambda)^2 = \varepsilon p^2$ and $-(\alpha - \lambda)(3\alpha + \lambda) = \omega$.

III. Q has two real roots, the greatest with multiplicity 2: $\alpha = \alpha_1 < \alpha_2 = \alpha_3$

We obtain that $\alpha_2 = \alpha_3 = (\lambda - \alpha)/2$, and the solution is given by

$$k(t) = \alpha + \frac{3\alpha - \lambda}{2}\tan^2\left(\frac{1}{2}\sqrt{\frac{\lambda - 3\alpha}{2}}(t+E)\right), \quad s \in (-\infty, \alpha]$$

where E is a constant. Now the mass-shell condition and the Majorana-type relation read $(1/4)(\alpha - \lambda)(\alpha + \lambda)^2 = \varepsilon p^2$ and $-(1/4)(\alpha + \lambda)(3\alpha - 5\lambda) = \omega$.

IV. Q has three distinct real roots: $\alpha_1 < \alpha_2 < \alpha_3$

Let us denote $\alpha = \alpha_1$ and $\beta = \alpha_2$, then $\alpha_3 = \lambda - \alpha - \beta$. There are two possibilities for the curvature:

$$k(t) = \alpha - (\beta - \alpha)\,\mathrm{tn}^2\left(\frac{1}{2}\sqrt{\lambda - 2\alpha - \beta}(t+E), \sqrt{\frac{\lambda - \alpha - 2\beta}{\lambda - 2\alpha - \beta}}\right),$$
$$k(t) = \lambda - \alpha - \beta + (\alpha + 2\beta - \lambda)\,\mathrm{sn}^2\left(\frac{1}{2}\sqrt{\lambda - 2\alpha - \beta}(t+E), \sqrt{\frac{\lambda - \alpha - 2\beta}{\lambda - 2\alpha - \beta}}\right),$$

defined in the intervals $(-\infty, \alpha]$ or $[\beta, \lambda - \alpha - \beta]$, respectively. In this case we have the following relations among constants: $-(\alpha + \beta)(\alpha - \lambda)(\beta - \lambda) = \varepsilon p^2$ and $(\alpha + \lambda)(\beta + \lambda) - (\alpha + \beta)^2 = \omega$.

V. Q has complex roots

Let us suppose that α_1 and α_2 are complex (so α_3 is real). Then the curvature is given by

$$k(t) = \alpha_3 - (\alpha_3 - \alpha_2)\,\mathrm{sn}^2\left(\frac{1}{2}\sqrt{\alpha_3 - \alpha_1}(t+E), \sqrt{\frac{\alpha_2 - \alpha_3}{\alpha_1 - \alpha_3}}\right), \quad s \in (-\infty, \alpha_3].$$

Write $\alpha_1 = \alpha + \beta i$ and $\alpha_2 = \alpha - \beta i$, then the mass-shell condition and the Majorana-type relation read $-2\alpha\left((\alpha-\lambda)^2 + \beta^2)\right) = \varepsilon p^2$ and $\lambda^2 + 2\alpha\lambda - 3\alpha^2 + \beta^2 = \omega$.

To integrate the Cartan equations of the curves obtained before we can use the cylindrical coordinates described in Section 3.1 when the axis P is non-null or in Section 3.2 otherwise.

Case 3: $\rho \neq 0$ (the quadratic case)

As before, without loss of generality we can assume that $\rho = 1$. The Euler-Lagrange equation is given by

$$2k^{(5)} + (10k + \mu)\,k^{(3)} + 20k''k' + k'\left(15k^2 + 3\mu k - \lambda\right) = 0, \tag{15}$$

In this case $\varphi = 2k'' + 3k^2 + \mu k - \lambda$ and we have two families of solutions. If $\varphi = 0$, then $P = 0$ and $J = -2(k^2 - \lambda)L - 4k'W + 2(2k+\mu)N$ is a constant vector field verifying $\langle J, J\rangle = \varepsilon j^2$. Then, the first family of solutions satisfies the equation

$$(k')^2 + k^3 + \frac{\mu}{2}k^2 - \lambda k - \left(\frac{\mu}{2} + \frac{\varepsilon}{16}j^2\right) = 0.$$

This equation has the same nature that the equation (12) and the solutions are seemed.

We now suppose that $\varphi \neq 0$, then the vector fields P and J read

$$\begin{aligned} P =& \left(2k^{(4)} + k''(8k+\mu) + 6(k')^2 + 3k^3 + \mu k^2 - \lambda k\right) L \\ &- \left(2k^{(3)} + k'(6k+\mu)\right) W + \left(2k'' + 3k^2 + \mu k - \lambda\right) N, \end{aligned}$$

$$J = \left(2k'' + k^2 + \mu k + \lambda\right) L - 4k'W + 2(2k+\mu)N.$$

If $\varphi \neq 0$, using the above equations we obtain the following first integrals:

$$\begin{aligned} &-2\left(2k^{(4)} + k''(8k+\mu) + 6(k')^2 + 3k^3 + \mu k^2 - \lambda k\right)\left(2k'' + 3k^2 + \mu k - \lambda\right) \\ &+ \left(2k^{(3)} + k'(6k+\mu)\right)^2 - \varepsilon p^2 = 0, \end{aligned}$$

$$\begin{aligned} &-(8k+4\mu)k^{(4)} + 8k'k^{(3)} - 4(k'')^2 - (40k^2 + 24\mu k + 2\mu^2)k'' \\ &- 8\mu(k')^2 - 15k^4 - 14\mu k^3 + (2\lambda - 3\mu^2)k^2 + 2\lambda\mu k + \lambda^2 - \omega = 0. \end{aligned}$$

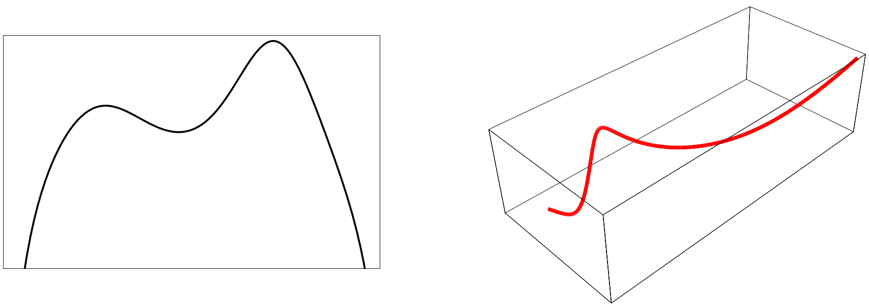
On the other hand, it is easy to see that another first integral is given by

$$2k^{(4)} + 10kk'' + \mu k'' + 5(k')^2 + 5k^3 + \frac{3}{2}\mu k^2 - \lambda k + c = 0,$$

c being a constant. These three first integrals can be combined to obtain the following ordinary differential equation of degree two:

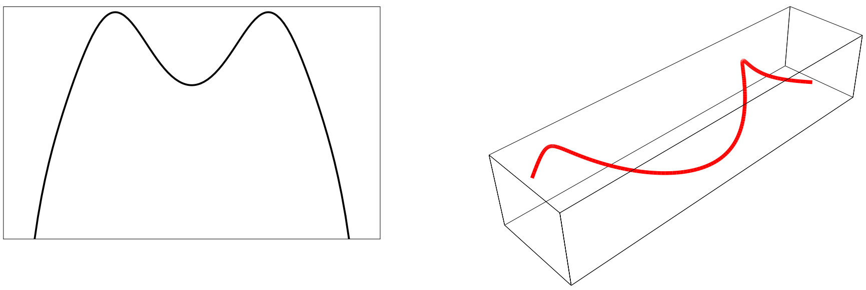
$$\frac{1}{16}(k')^2\left(-4(k'')^2-2(2k+\mu)(k')^2+5k^4+2\mu k^3-2\lambda k^2+4ck+2c\mu+\lambda^2+\omega\right)^2 \\ +(2k''+3k^2+\mu k-\lambda)(4kk''-2(k')^2+4k^3+\mu k^2+2c)-\varepsilon p^2 \quad = \quad 0.$$

The integration of this equation is very complicated, but we can use computing methods to make us an idea of their solutions (see Figures 1, 2 and 3).



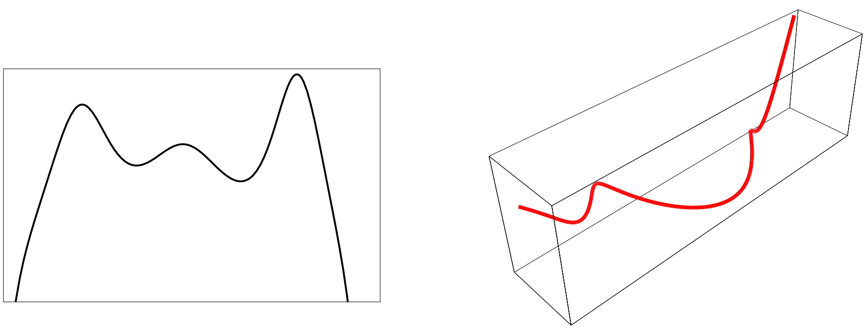
$k(t)$ $\gamma(t)$

Figure 1: Caso cuadrático y J espacial.



$k(t)$ $\gamma(t)$

Figure 2: Caso cuadrático y J temporal.



$k(t)$ $\gamma(t)$

Figure 3: Caso cuadrático y J nulo.

5 Discussion and outlook

We have studied actions in $(2+1)$-dimensions whose Lagrangians are arbitrary functions f on the curvature of the particle path, completing previous works [20, 21, 22]. We have shown that null helices, [23, 24], are always possible trajectories of the particles for every Lagrangian function f. Otherwise the non-zero vector field P, obtained from the Euler-Lagrange equation, possesses a non-vanishing space-like component orthogonal to the light-like particle trajectory, which seems to be a manifestation of a generic feature of higher-derivative theories. This vector field can be interpreted as the linear momentum of the particle since it is constant along the curve, which agrees with the conserved linear momentum law.

When f is a quadratic function, $f(k) = \rho k^2 + \mu k + \lambda$, we go further. In the simplest geometrical particle model (the constant case) we show that the worldline of the particle is a Cartan helix with axis given by the vector P. This was already shown by Nersessian and Ramos using a Hamiltonian formulation, but here we offer an alternative proof which exploits the geometry of the particle path. In the linear case we completely solve the Euler-Lagrange equation and integrate the Cartan equations in cylindrical coordinates around the linear momentum P. Finally, in the proper quadratic case we obtain that the curvature of the particle path should fulfill an ordinary differential equation of degree two.

To conclude, let us indicate some problems that deserve further attention.

First, it is necessary to study more deeply the proper quadratic case in order to integrate completely the Cartan equations of the worldlines.

Secondly, even though we have got an explicit description of the motion equation at $D = (2+1)$, we note that a priori there is no restriction to apply these ideas in other background gravitational fields of greater dimension. In particular we can consider actions in $D = d+1$ dimensions ($d \geq 3$) whose Lagrangians depend linearly on the curvature and study what are the trajectories of the relativistic particles in this model.

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