# Relativistic particles with rigidity and torsion in $D=3$ spacetimes 

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#### Abstract

Models describing relativistic particles, where Lagrangian densities depend linearly on both the curvature and the torsion of the trajectories, are revisited in $D=3$ Lorentzian spacetimes with constant curvature. The moduli spaces of trajectories are completely and explicitly determined. Trajectories are Lancret curves including ordinary helices. To get the geometric integration of the solutions, we design algorithms that essentially involve the Lancret program as well as the notions of scrolls and Hopf tubes. The most interesting and consistent models appear in anti de Sitter spaces, where the Hopf mappings, both the standard and the Lorentzian ones, play an important role. The moduli subspaces of closed solitons in anti de Sitter settings are also obtained. Our main tool is the isoperimetric inequality in the hyperbolic plane.

The mass spectra of these models are also obtained. In anti de Sitter backgrounds, the characteristic feature is that the presence of real gravity makes that, under reasonable conditions, these physical spectra always present massive states. This fact has no equivalent in flat spaces where spectra necessarily present tachyonic sector. Furthermore, the existence of systems with only massive states, in anti de Sitter geometry, solves an early stated problem in spaces with a non trivial gravity.


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## 1 Introduction

The usual description of relativistic particles by Lagrangians is based on the consideration of some extra dimensions of the original spacetime. Recently however, a new approach appeared in the literature (see for example [1, 16, 21, 22, 23, 26, 27, 28] and references therein). In this setting, the particle systems are described by Lagrangians that, being formulated in the original spacetime (so they are intrinsic), depend on higher derivatives. Therefore, the attractive point in this new philosophy is that the spinning
degrees of freedom are assumed to be encoded in the geometry of the trajectories. Now, the Poincaré and invariance requirements imply that the admissible Lagrangian densities must depend on the extrinsic curvatures of the curves in the background gravitational field. In a very recent paper (see [4]), the first author, jointly with J. Arroyo and O. J. Garay, studied the same problem in 3-dimensional Riemannian space forms.

Most of published papers on this subject involve actions that only depend on the first curvature of trajectories (the curvature, which plays the role of proper acceleration of the particle). However, it seems important to investigate models of particles with curvature and torsion.

Throughout this paper, $M(C)$ will denote a three dimensional Lorentzian space with constant curvature $C$. Also, curves are assumed to be Frenet curves. So, in a suitable space $\Lambda$ of Frenet curves in $M(C)$ (for example, the space of closed curves or that of curves satisfying certain second order boundary data, such as clamped curves), we have a three-parameter family of actions, $\left\{\mathcal{F}_{m n p}: \Lambda \rightarrow \mathbb{R} \mid m, n, p \in \mathbb{R}\right\}$, defined by

$$
\begin{equation*}
\mathcal{F}_{m n p}(\gamma)=\int_{\gamma}(m+n \kappa+p \tau) d s \tag{1}
\end{equation*}
$$

where $s, \kappa$ and $\tau$ stand for the arclength parameter, curvature and torsion of $\gamma$, respectively, and the parameters $m, n$ and $p$ are not allowed to be zero simultaneously.

The main purpose of this paper is to determine, explicitly and completely, the moduli space of trajectories in the relativistic particle model $\left[M(C), \mathcal{F}_{m n p}\right]$. In particular, we provide algorithms to obtain the trajectories of a given model. The closed trajectories, when there exist, are also obtained from an interesting quantization principle.

It should be noticed that this problem was considered for flat spaces $(C=0)$ in [16]. There the authors showed that trajectories are helices (that is, curves with both curvature and torsion being constant) in $\mathbb{L}^{3}=M(0)$. However, this is not true. In fact, we prove here that trajectories in the model $\left[\mathbb{L}^{3}, \mathcal{F}_{0 n p}\right]$ are Lancret curves with slope determined by the values of $n$ and $p$. For a better understanding, in the next section we recall the nice geometry of the Lancret curves not only in classical setting but also in Lorentzian frameworks, the general references for more details will be [5, 9]. A Riemannian counterpart of this paper can be found in [4].

The mass spectra of these models in flat spaces were described in [16]. They obligatory contain tachyonic sector, while depending on the values of parameters, $m, n, p$, the model may or may not have massive sector and it may have two, one or no massless states. However, we think that the models $\left[\mathbb{L}^{3}, \mathcal{F}_{m 0 p}\right], p>0$, only has massive sectors for timelike trajectories (see Section 9).

On the other hand, in [17], the authors stated the following problem: Should the spectrum of a particle model, with Lagrangian density a function of curvature and torsion, obligatorily contain tachyonic states? In that paper, the authors give a negative answer to the above problem. They construct Lagrangian densities that give models, in $\mathbb{L}^{3}$, containing only massive states. However, the problem does not still have any answer for the particle models, $\mathcal{F}_{m n p}$, over backgrounds with non trivial gravity tensor field, say $M(C)$ with $C \neq 0$.

In this paper, we have obtained the mass spectra for helix solutions in $\left[\mathbf{A d S}_{3}, \mathcal{F}_{m n p}\right]$, of course, a similar study can be made in the de Sitter backgrounds. The existence
of non trivial gravity radically changes the nature of the physical spectra. The mass spectra of $\left[M(C), \mathcal{F}_{m n p}\right]$ depends strongly on the curvature, $C$. When $C<0$, (anti de Sitter geometry), they obligatory contain massive sector, under reasonable conditions, while depending of the parameters, they may also have tachyonic sector and two, one or non massless states. Furthermore, this behaviour of the mass spectrum is completely governed by the curvature, $C$. We exhibit some of these models in $\mathbf{A d S}_{3}$ containing only massive states in their physical spectra and so solving in the negative the above mentioned problem. The size of the corresponding massive sectors only depending on $C$.

The mass spectra for Lancret solutions in the models $\left[\mathbb{L}^{3}, \mathcal{F}_{0 n p}\right]$ and $\left[\mathbf{A d S}_{3}, \mathcal{F}_{m n p}\right]$ with $m n p \neq 0$ are completely unknown and their study is an interesting open problem.

## 2 The Lorentzian Lancret program

A Lancret curve (or general helix) in $\mathbb{L}^{3}$ is a Frenet curve whose tangent indicatrix is contained in some plane $\Pi \subset \mathbb{L}^{3}$. It will be called degenerate or nondegenerate according to the causal character of such a plane. As in the Euclidean setting, the Lancret curves in $\mathbb{L}^{3}$ correspond with those for which the ratio of curvature to torsion is constant. However, whereas the Lancret curves of $\mathbb{R}^{3}$ are geodesics of right general cylinders, these curves in $\mathbb{L}^{3}$ are geodesics of either right general cylinders (when those are nondegenerate) or flat scrolls over null curves (when those are degenerate); see [12] and Appendix A for details on scrolls.

In $[5,9]$ the notion of Lancret curve was extended to real space forms and spacetimes $M(C)$, with $C \neq 0$, respectively, where the notion of Killing vector field along a curve played an important role. We will consider the class of Lancret curves including not only those curves with torsion vanishing identically, but also the ordinary helices (or simply helices), whose curvature and torsion are both nonzero constants. We will refer these two cases as trivial Lancret curves. The solving natural equations problem is completely solved for Lancret curves in these backgrounds. It is a bit more subtle than one might suppose a priori, as evidenced by the difference between the spherical and the hyperbolic cases. As a resume we have:

1. A curve in the de Sitter space $\mathbf{d S}_{3}$ is a Lancret curve if and only if either its torsion vanishes identically or it is an ordinary helix. That is, the class of Lancret curves in the de Sitter geometry is just reduced to that of ordinary Lancret curves.
2. A curve in the anti de Sitter space, $\mathbf{A d S}_{3}$ with constant curvature $C, C<0$, is a Lancret curve if and only if either its torsion vanishes identically or the curvature $\kappa$ and the torsion $\tau$ are related by $\tau=\omega \kappa \pm \sqrt{-C}$, where $\omega$ is a certain constant which will be interpreted as a kind of slope.
3. A curve in $\mathbf{A d S}_{3}$ is a nondegenerate Lancret curve if and only if it is a geodesic of either a Hopf tube or a hyperbolic Hopf tube. That is a surface obtained when one makes the complete lifting, via the corresponding Hopf map, of a curve in either the hyperbolic plane or the anti de Sitter plane.
4. A curve in $\mathbf{A d S}_{3}$ is a degenerate Lancret curve if and only if it is a geodesic of a flat scroll over a null curve.

However, the closed curve problem for Lancret curves in $\mathbf{A d S}_{3}$ can be also obtained by taking advantage from the well known isometry type of the Hopf tori obtained from closed curves in the hyperbolic plane (see Appendix B), remarking a very deep difference with respect to the $\mathbb{L}^{3}$ classical setting.

The main result in the first part of this paper can be stated as follows (compare with [4]):

A curve $\gamma \in \Lambda$ is a critical point of $\mathcal{F}_{m n p}$ if and only if $\gamma$ is a Lancret curve in $M(C)$. In other words, the spinning relativistic particles in the model [ $\left.M(C), \mathcal{F}_{m n p}\right]$ evolve along Lancret curves of $M(C)$.

## 3 The Euler-Lagrange equations

The metric of $M(C)$ will be denoted by $g=\langle$,$\rangle and its Levi-Civita connection by \nabla$. Let $\gamma=\gamma(t): I \subset \mathbb{R} \rightarrow M(C)$ be an immersed curve with speed $v(t)=\left|\gamma^{\prime}(t)\right|$, curvature $\kappa$, torsion $\tau$ and Frenet frame $\{T, N, B\}$. Then, one can write the Frenet equations of $\gamma$ as

$$
\begin{aligned}
\nabla_{T} T & =\varepsilon_{2} \kappa N, \\
\nabla_{T} N & =-\varepsilon_{1} \kappa T+\varepsilon_{3} \tau B, \\
\nabla_{T} B & =-\varepsilon_{2} \tau N,
\end{aligned}
$$

where $\varepsilon_{i}, 1 \leq i \leq 3$, denotes the causal character of $T, N$ and $B$, respectively.
In order to derive first variation formulas for $\mathcal{F}_{m n p}$, we will use the following standard terminology (see [18] for details). For a curve $\gamma:[0, L] \rightarrow M$, we take a variation, $\Gamma=\Gamma(t, r):[0, L] \times(-\varepsilon, \varepsilon) \rightarrow M$ with $\Gamma(t, 0)=\gamma(t)$. Associated with this variation we have the vector field $W=W(t)=\frac{\partial \Gamma}{\partial r}(t, 0)$ along the curve $\gamma(t)$. We also write $V=V(t, r)=\frac{\partial \Gamma}{\partial t}(t, r), W=W(t, r), v=v(t, r), T=T(t, r), N=N(t, r), B=B(t, r)$, etc., with the obvious meanings. Let $s$ denote the arclength, and put $V(s, r), W(s, r)$ etc., for the corresponding reparametrizations. To obtain the formulas avoiding tedious computations, we give general formulas for the variations of $v, \kappa$ and $\tau$ in $\gamma$ in the direction of $W$. These are obtained using standard computations that involve the Frenet equations:

$$
\begin{aligned}
& W(v)=\varepsilon_{1} v\left\langle\nabla_{T} W, T\right\rangle, \\
& W(\kappa)=\left\langle\nabla_{T}^{2} W, N\right\rangle-2 \varepsilon_{1} \kappa\left\langle\nabla_{T} W, T\right\rangle+\varepsilon_{1} C\langle W, N\rangle, \\
& W(\tau)=\varepsilon_{2}\left(\frac{1}{\kappa}\left\langle\nabla_{T}^{2} W+\varepsilon_{1} C W, B\right\rangle\right)_{s}-\varepsilon_{1} \tau\left\langle\nabla_{T} W, T\right\rangle+\varepsilon_{1} \kappa\left\langle\nabla_{T} W, B\right\rangle,
\end{aligned}
$$

where the subscript $s$ denotes differentiation with respect to the arclength.
Now, we use a standard argument which involves the above formulas and some integrations by parts to get the variation of $\mathcal{F}_{m n p}$ along $\gamma$ in the direction of $W$

$$
\begin{equation*}
\delta \mathcal{F}_{m n p}(\gamma)[W]=\int_{\gamma}\langle\Omega(\gamma), W\rangle d s+[\mathcal{B}(\gamma, W)]_{0}^{L}, \tag{2}
\end{equation*}
$$

where $\Omega(\gamma)$ and $\mathcal{B}(\gamma, W)$ stand for the Euler-Lagrange and Boundary operators, respectively, which are given by

$$
\begin{aligned}
& \Omega(\gamma)=\left(-\varepsilon_{1} \varepsilon_{2} m \kappa+\varepsilon_{1} \varepsilon_{2} p \kappa \tau-\varepsilon_{2} \varepsilon_{3} n \tau^{2}+\varepsilon_{1} n C\right) N+\left(-\varepsilon_{1} p \kappa_{s}+\varepsilon_{3} n \tau_{s}\right) B \\
& \begin{aligned}
\mathcal{B}(\gamma, W) & =\varepsilon_{2} \frac{p}{\kappa}\left\langle\nabla_{T}^{2} W, B\right\rangle+n\left\langle\nabla_{T} W, N\right\rangle \\
& +\varepsilon_{1} m\langle W, T\rangle+\left(-\varepsilon_{3} n \tau+\varepsilon_{1} \varepsilon_{2} \frac{p C}{\kappa}+\varepsilon_{1} p \kappa\right)\langle W, B\rangle
\end{aligned}
\end{aligned}
$$

Proposition 1 (Second order boundary conditions) Given $q_{1}, q_{2} \in M$ and $\left\{x_{1}, y_{1}\right\}$, $\left\{x_{2}, y_{2}\right\}$ orthonormal vectors in $T_{q_{1}} M$ and $T_{q_{2}} M$, respectively, define the space of curves

$$
\Lambda=\left\{\gamma:\left[t_{1}, t_{2}\right] \rightarrow M \mid \gamma\left(t_{i}\right)=q_{i}, T\left(t_{i}\right)=x_{i}, N\left(t_{i}\right)=y_{i}, 1 \leq i \leq 2\right\}
$$

Then the critical points of the variational problem $\mathcal{F}_{m n p}: \Lambda \rightarrow \mathbb{R}$ are characterized by the following Euler-Lagrange equations

$$
\begin{align*}
\varepsilon_{3} m \kappa-\varepsilon_{3} p \kappa \tau+\varepsilon_{1} n \tau^{2}+\varepsilon_{1} n C & =0  \tag{3}\\
-\varepsilon_{1} p \kappa_{s}+\varepsilon_{3} n \tau_{s} & =0 \tag{4}
\end{align*}
$$

Proof. Let $\gamma \in \Lambda$ and $W \in T_{\gamma} \Lambda$, then $W$ defines a curve in $\Lambda$ associated with a variation $\Gamma=\Gamma(t, r):[0, L] \times(-\epsilon, \epsilon) \rightarrow M$ of $\gamma, \Gamma(t, 0)=\gamma(t)$. Therefore, we can make the following computations along $\Gamma$

$$
\begin{aligned}
W & =d \Gamma\left(\partial_{r}\right) \\
\nabla_{T} W & =f T+d \Gamma\left(\partial_{r} T\right) \\
\nabla_{T}^{2} W & =\left(\partial_{s} f\right) T+\left(\varepsilon_{2} f\left(\varepsilon_{1}+\kappa\right)+\varepsilon_{2} \partial_{r} \kappa\right) N+\varepsilon_{2} \kappa d \Gamma\left(\partial_{r} N\right)+R(T, W) T
\end{aligned}
$$

here $f=\partial_{r}(\log v)$. Then, we evaluate these formulas along the curve $\gamma$ by making $r=0$ and use the second order boundary conditions to obtain the following values at the endpoints

$$
\begin{aligned}
W\left(t_{i}\right) & =0 \\
\nabla_{T} W\left(t_{i}\right) & =f\left(t_{i}\right) x_{i} \\
\nabla_{T}^{2}\left(t_{i}\right) W & =\left(\partial_{s} f\right)\left(t_{i}\right) x_{i}+\left(\varepsilon_{2} f\left(\varepsilon_{1}+\kappa\right)+\varepsilon_{2} \partial_{r} \kappa\right)\left(t_{i}\right) y_{i}
\end{aligned}
$$

As a consequence,

$$
[\mathcal{B}(\gamma, W)]_{t_{1}}^{t_{2}}=0
$$

Then, $\gamma$ is a critical point of the variational problem $\mathcal{F}_{m n p}: \Lambda \rightarrow R$, that is, $\delta \mathcal{F}_{m n p}(\gamma)[W]=$ 0 , for any $W \in T_{\gamma} \Lambda$, if and only if $\Omega(\gamma)=0$, which gives (3) and (4).

## 4 The moduli spaces of trajectories

The field equations (3) and (4) can be nicely integrated. First, notice that they can be written as

$$
\begin{align*}
\varepsilon_{1} p \kappa-\varepsilon_{3} n \tau & =a  \tag{5}\\
-\varepsilon_{1} m \kappa+a \tau & =\varepsilon_{3} n C \tag{6}
\end{align*}
$$

where $a$ denotes an undetermined integration constant. Then we have

- If $\varepsilon_{1} p a+\varepsilon_{2} m n \neq 0$, then the solutions are helices (ordinary helices or trivial Lancret curves) with curvature and torsion given by

$$
\kappa=\frac{a^{2}+n^{2} C}{\varepsilon_{1} p a+\varepsilon_{2} m n}, \quad \tau=\frac{m a+\varepsilon_{3} n p C}{p a-\varepsilon_{3} n m} .
$$

- Otherwise, the existence of solutions is equivalent to $n^{2} C+a^{2}=m a+\varepsilon_{3} p n C=0$. A first consequence is that $C \leq 0$. Therefore, the trajectories in the relativistic particle model $\left[\mathbf{d S}_{3}, \mathcal{F}_{m n p}\right]$ are helices in the de Sitter space with curvature and torsion given as above. This seems reasonable since the de Sitter space is free of non-trivial Lancret curves.
- Now, in the above setting, we may assume that $n \neq 0$, otherwise we have the free fall particle model. Thus, the field equations reduces to equation (5). If $m=p=0$, then $\tau^{2}=-C$, so that $\gamma$ is a plane curve (when $C=0$ ) or the horizontal lift, via the Hopf map $\pi_{-}$or $\lambda$, of a curve in either $\mathbb{H}^{2}(4 C)$ or $\mathbf{A d S}_{2}(4 C)$ (when $C<0$ ). Otherwise, the trajectories are curves whose curvatures satisfy

$$
\begin{equation*}
\tau=-\varepsilon_{2} \omega \kappa \pm \frac{m}{p}, \quad \text { with } \quad \omega=\frac{p}{n} \quad \text { and } \quad-C=\frac{m^{2}}{p^{2}} . \tag{7}
\end{equation*}
$$

Then they are Lancret curves in either $\mathbb{L}^{3}$ or in $\mathbf{A d S}_{3}$. The slope in both cases is $p / n$.

The moduli space of trajectories is summarized in the following tables which correspond with Lorentz-Minkowski, de Sitter and anti de Sitter spaces, respectively. All solutions are Lancret curves. Similarly to the classical Euclidean case, helices are considered as special cases of Lancret curves (trivial Lancret curves). For simplicity of interpretation, we have represented different cases according to the values of the parameters defining the action.

| $m$ | $n$ | $p$ | Solutions in $\mathbb{L}^{3}, C=0$ |
| :--- | :--- | :--- | :--- |
| $\neq 0$ | $=0$ | $=0$ | Geodesics $(\kappa=0)$ |
| $=0$ | $=0$ | $\neq 0$ | Circles $\kappa$ constant and $\tau=0)$ |
| $=0$ | $\neq 0$ | $=0$ | Plane curves $(\tau=0)$ |
| $\neq 0$ | $\neq 0$ | $=0$ | Helices with arbitrary $\tau$ and $\kappa=\varepsilon_{2} \frac{n \tau^{2}}{m}$ |
| $\neq 0$ | $=0$ | $\neq 0$ | Helices with arbitrary $\kappa$ and $\tau=\frac{m}{p}$ |
| $=0$ | $\neq 0$ | $\neq 0$ | Circles and Lancret curves with $\tau=-\varepsilon_{2} \frac{p}{n} \kappa$ |
| $\neq 0$ | $\neq 0$ | $\neq 0$ | Helices with $\kappa=\frac{\varepsilon_{1} a^{2}}{a p-\varepsilon_{3} n m}$ and $\tau=\frac{m a}{a p-\varepsilon_{3} n m}, a \in \mathbb{R}-\left\{\frac{\varepsilon_{3} n m}{p}\right\}$ |


| $m$ | $n$ | $p$ | Solutions in $\mathbf{d S}_{3}, C=c^{2}$ |
| :--- | :--- | :--- | :--- |
| $\neq 0$ | $=0$ | $=0$ | Geodesics $(\kappa=0)$ |
| $=0$ | $=0$ | $\neq 0$ | Circles $(\kappa$ constant and $\tau=0)$ |
| $=0$ | $\neq 0$ | $=0$ | Do not exist |
| $\neq 0$ | $\neq 0$ | $=0$ | Helices with arbitrary $\tau$ and $\kappa=\varepsilon_{2} \frac{n\left(c^{2}+\tau^{2}\right)}{m}$ |
| $\neq 0$ | $=0$ | $\neq 0$ | Helices with arbitrary $\kappa$ and $\tau=\frac{m}{p}$ |
| $=0$ | $\neq 0$ | $\neq 0$ | Helices with $\kappa=\varepsilon_{1} \frac{n^{2} c^{2}+a^{2}}{a p}$ and $\tau=\varepsilon_{3} \frac{n c^{2}}{a}, a \in \mathbb{R}-\{0\}$ |
| $\neq 0$ | $\neq 0$ | $\neq 0$ | Helices with $\kappa=\frac{n^{2} c^{2}+a^{2}}{\varepsilon_{1} p a+\varepsilon_{2} m n}$ and $\tau=\frac{m a+\varepsilon_{3} n p c^{2}}{p a-\varepsilon_{3} m n}, a \in \mathbb{R}-\left\{\varepsilon_{3} \frac{m n}{p}\right\}$ |


| $m$ | $n$ | $p$ | Solutions in $\mathbf{A d S}_{3}, C=-c^{2}$ |
| :--- | :--- | :--- | :--- |
| $\neq 0$ | $=0$ | $=0$ | Geodesics $(\kappa=0)$ |
| $=0$ | $=0$ | $\neq 0$ | Circles ( $\kappa$ constant and $\tau=0)$ |
| $=0$ | $\neq 0$ | $=0$ | Horizontal lifts, via a Hopf map $\pi_{-}$or $\lambda$, of curves in either |
| $\neq 0$ | $\neq 0$ | $=0$ | Helices with arbitrary $\tau$ and $\kappa=\varepsilon_{2} \frac{n\left(\tau^{2}-c^{2}\right)}{m}$ |
| $\neq 0$ | $=0$ | $\neq 0$ | Helices with arbitrary $\kappa$ and $\tau=\frac{m}{p}$ |
| $=0$ | $\neq 0$ | $\neq 0$ | Helices with $\kappa=\varepsilon_{1} \frac{a^{2}-n^{2} c^{2}}{a p}$ and $\tau=-\varepsilon_{3} \frac{n c^{2}}{a}, a \in \mathbb{R}-\{0\}$ |
| $\neq 0$ | $\neq 0$ | $\neq 0$ | Helices with $\kappa=\frac{a^{2}-n^{2} c^{2}}{\varepsilon_{1} p a+\varepsilon_{2} m n}$ and $\tau=\frac{m a-\varepsilon_{3} n p c^{2}}{p a-\varepsilon_{3} m n}, a \in \mathbb{R}-\left\{\varepsilon_{3} \frac{m n}{p}\right\}$ |
| $\neq 0$ | $\neq 0$ | $\neq 0$ | Lancret curves with $\tau=-\varepsilon_{2} \frac{p}{n} \kappa \pm \frac{m}{p}$ and $c= \pm \frac{m}{p}$ |

## 5 Geometric algorithms to get the moduli spaces of trajectories in the models $\left[\mathbb{L}^{3}, \mathcal{F}_{\text {mnp }}\right]$

As we mentioned in the introduction, the particle models $\left[\mathbb{L}^{3}, \mathcal{F}_{m n p}\right]$ were considered in [16]. In that paper, the authors showed that trajectories are ordinary helices (that is, trivial Lancret curves or curves with both curvature and torsion being constant) in $\mathbb{L}^{3}$. However, this is not true. Actually, we have proved that trajectories in the model $\left[\mathbb{L}^{3}, \mathcal{F}_{0 n p}\right]$ are Lancret curves. From the corresponding table to $\mathbb{L}^{3}$, we see that this is the more interesting model. The trajectories in the other ones are just certain classes of trivial Lancret curves which are completely determined, up to motions in $\mathbb{L}^{3}$, from a couple of constants giving curvature and torsion. However, a geometric algorithm to obtain explicitly, up to motions in $\mathbb{L}^{3}$, all trajectories of the model $\left[\mathbb{L}^{3}, \mathcal{F}_{0 n p}\right]$ should be convenient. The existence of degenerate Lancret curves, with no Riemannian counterpart, increases the convenience of this algorithm. That will be determined by the values of $n$ and $p$, that is, by the difference weights of $\kappa$ and $\tau$ in the Lagrangian density. The key point to get degenerate Lancret trajectories is the Theorem 3 of [9]. In fact, the Lancret curves in $\mathbb{L}^{3}$ are characterized by the equation $\tau=r \kappa$, where $r$ is a certain constant. Now, degenerate Lancret curves are determined by $r= \pm 1$ and spacelike acceleration ( $\varepsilon_{2}=1$ ).

### 5.1 The model $\left[\mathbb{L}^{3}, \mathcal{F}_{0 n p}\right]$ with $n^{2} \neq p^{2}$

We will see that any trajectory $\beta$ is a geodesic, up to congruences, of a certain right cylinder. To do that we proceed as follows.

1. Let $\beta$ be a trajectory of $\left[\mathbb{L}^{3}, \mathcal{F}_{0 n p}\right]$. Then its curvature $\kappa$ and its torsion $\tau$ satisfy $\tau=-\varepsilon_{2} \frac{p}{n} \kappa$. Define a vector field, $V$, along $\beta$ by $V=\varepsilon_{2} p T-n B$. Since $\nabla_{T} V=0$, it is the restriction to $\beta$ of a translation vector field on $\mathbb{L}^{3}$. Thus, there exists $v \in \mathbb{L}^{3}$ such that $V(s)=v$. Furthermore $\langle v, v\rangle=\delta \neq 0$, since $n^{2} \neq p^{2}$.
2. Choose a nondegenerate plane, $P$ in $\mathbb{L}^{3}$ which is orthogonal to $v$. In this plane, up to congruences in $P$, there exists a unique curve, say $\alpha$ with curvature function $\bar{\kappa}=\left|\beta^{\prime}\right| \kappa$ and acceleration with causal character $\delta_{2}=\varepsilon_{2}$, under these conditions, it should be noticed that its velocity causal character is determined from $\delta_{2}$ and the causal character of the axis $v$.
3. Next, we consider the right cylinder, $\mathbf{C}_{\alpha, v}$, in $\mathbb{L}^{3}$ with directrix $\alpha$ and generatrix $v$. It can be parametrized by $X(s, t)=\alpha(s)+t v$.
4. In $\mathbf{C}_{\alpha, v}$, we choose the geodesic $\gamma(s)=\alpha(s)+m s v$, where $m=\delta_{1} \varepsilon_{1} \frac{p}{n}$. A direct computation shows that $\beta$ and $\gamma$ have the same curvature, the same torsion as well as their Frenet frame have the same causal character. Consequently, they are congruent in $\mathbb{L}^{3}$.

The converse also holds, so all trajectories are completely determined by this algorithm.

### 5.2 The model $\left[\mathbb{L}^{3}, \mathcal{F}_{0 n p}\right]$ with $n^{2}=p^{2}$

Given a path $\beta$ of $\left[\mathbb{L}^{3}, \mathcal{F}_{0 n p}\right]$, we define, as above, the vector field $V=\varepsilon_{2} p T-n B$. This is parallel and so it comes from a vector $v \in \mathbb{L}^{3}$. However, $\langle v, v\rangle=\varepsilon_{1} p^{2}+$ $\varepsilon_{3} n^{2}$, and consequently we have two types of trajectories. First, there exists a class of nondegenerate Lancret curves with timelike acceleration $\left(\varepsilon_{2}=-1\right)$. These solutions can be geometrically obtained as geodesics, with slope $m= \pm \delta_{1} \varepsilon_{1}$, in right cylinders $\mathbf{C}_{\alpha, v}$ with generatrix $v$ just as in the above considered models. However, there exists another family of degenerate Lancret trajectories which correspond with those with spacelike acceleration $\left(\varepsilon_{2}=1\right)$. Just now we list the steps providing the corresponding algorithm to describe geometrically these solutions.

1. Parametrize the path $\beta$ with constant speed, say $\left\langle\beta^{\prime}, \beta^{\prime}\right\rangle=\varepsilon_{1} \ell^{2}$ constant, $\ell>0$. As $n^{2}=p^{2}$ and $\varepsilon_{2}=1$, we find $\tau= \pm \kappa$. We may assume, without loss of generality, that $\tau=\kappa$, changing orientation if necessary.
2. Define the following vector fields along $\beta$

$$
\begin{aligned}
A(s) & =\frac{\ell}{2}(T(s)+B(s)) \\
D(s) & =-\frac{\varepsilon_{1}}{\ell}(T(s)-B(s)) \\
F(s) & =N(s)
\end{aligned}
$$

3. Let $\alpha(s)$ be a curve in $\mathbb{L}^{3}$ with tangent vector field $\alpha^{\prime}(s)=A(s)$. Then $\alpha$ is a null curve and $\{A(s), D(s), F(s)\}$ is a Cartan frame along $\alpha(s)$ with $\mu=0$ and $\rho=\ell \kappa$.
4. Let $\mathbf{S}_{\alpha, D}$ be the associated scroll parametrized by $\Phi(s, t)=\alpha(s)+t D(s)$. It is obvious that $\mathbf{S}_{\alpha, D}$ is a flat surface in $\mathbb{L}^{3}$.
5. Choose the geodesic in $\mathbf{S}_{\alpha, D}$ defined by $\gamma(s)=\alpha(s)-\frac{\varepsilon_{1} \ell^{2}}{2} s D(s)$. It is not difficult to see that $\beta(s)$ and $\gamma(s)$ have the same curvature and torsion functions. Moreover, the causal characters of their Frenet frames also agree. Consequently, they are congruent in $\mathbb{L}^{3}$.

As the converse also holds, then all trajectories are completely determined by these algorithms.

## 6 Moduli spaces of trajectories in non flat backgrounds

The uninteresting case corresponds to relativistic particles evolving in the de Sitter background. This is due in part to the absence of non trivial Lancret curves in $\mathbf{d S}_{3}$. Therefore, most of the models $\left[\mathbf{d S}_{3}\left(c^{2}\right), \mathcal{F}_{m n p}\right]$ admit a one-parameter family of trajectories which are trivial Lancret curves. The exception to this rule is the model $\left[\mathbf{d} \mathbf{S}_{3}\left(c^{2}\right), \mathcal{F}_{0 n 0}\right]$, which is associated to the action measuring the total curvature of trajectories (known as the Plyushchay model for a massless relativistic particle, $[26,27]$ ) and it does not provide any consistent dynamics (see [7] for more details).

## Particles evolving in anti de Sitter backgrounds

The most interesting models in the anti de Sitter space $\mathbf{A d S} \mathbf{S}_{3}$ are $\left[\mathbf{A d S}_{3}\left(-c^{2}\right), \mathcal{F}_{0 n 0}\right]$ and $\left[\boldsymbol{A d S}_{3}\left(-c^{2}\right), \mathcal{F}_{m n p}\right]$ with $m n p \neq 0$. The former corresponds again with the action giving the total curvature, that we have called the Plyushchay model describing a massless relativistic particle. In [7] it is shown that the three-dimensional anti de Sitter space is the only spacetime (no matter the dimension) with constant curvature providing a consistent dynamics for this action. More precisely, the trajectories of this model are nothing but the horizontal lifts, via either the usual Hopf map $\pi_{-}$or the Lorentzian Hopf map $\lambda$, of arbitrary curves in either the hyperbolic plane or the anti de Sitter plane, respectively (see Appendix B). It should be noticed that those horizontal curves are Lancret ones, where the curvature is an arbitrary function and the torsion is nicely determined by the radius of the anti de Sitter space (for instance, $\tau= \pm 1$ if $C=-1$ ).

However, the latter provides a model very rich in solutions. We are going to describe explicitly the trajectories for a better understanding of their nice dynamics. First, the model admits a one-parameter class $\mathcal{T}$ of trajectories which are ordinary helices (see Table 3). They can be geometrically obtained as geodesics of either a Hopf tube over a curve with constant curvature in the corresponding hyperbolic plane or a hyperbolic Hopf tube over a curve with constant curvature in the anti de Sitter plane (see [8] and Appendix B for more details). The dynamics are completed with classes of non trivial Lancret paths whose existence is related to the values of the parameters defining the action. First of all, notice that the ratio $\frac{m}{p}$ and the curvature $C$ of $\mathbf{A d S}_{3}(C)$ should
satisfy $\frac{m}{p}= \pm \sqrt{-C}$. Therefore, without loss of generality, we may assume that $C=-1$ and $m= \pm p$, so we will put $m=p$ in the discussion. On the other hand, the non trivial Lancret curves in the anti de Sitter space (with $C=-1$ ) are characterized by the following constraint between curvature and torsion

$$
\tau=b \kappa \pm 1, \quad \text { for a certain constant } b \in \mathbb{R}
$$

Furthermore, as in the flat case, degenerate Lancret curves correspond with $b= \pm 1$ and spacelike acceleration, $\varepsilon_{2}=1,[9]$. Consequently, we have to distinguish two cases.

### 6.1 The dynamics in $\left[\mathrm{AdS}_{3}, \mathcal{F}_{m n p}\right]$ with $n^{2} \neq p^{2}$

Besides the above mentioned class $\mathcal{T}$ of ordinary helices, this model has a second class, $\mathcal{T}_{n^{2} \neq p^{2}}$, of trajectories which, according to Table 3, are nondegenerate Lancret curves (because $n^{2} \neq p^{2}$ ) satisfying

$$
\tau=-\varepsilon_{2} \frac{p}{n} \kappa \pm 1
$$

This class of solutions is made up of curves that are geodesics in Hopf tubes over curves either in the hyperbolic plane or in the anti de Sitter plane. In both cases the slope is determined by $n$ and $p$. The steps will be sketched as follows.

## Trajectories being geodesics of Hopf tubes

The algorithm to get the solutions of this subfamily runs as follows.

1. Take a unit speed curve $\gamma(s)$ in the hyperbolic plane $\mathbb{H}^{2}(-4)$ and consider its Hopf tube $\pi_{-}^{-1}(\gamma)$ in $\mathbf{A d S}_{3}$ (see Appendix B).
2. This is a Lorentzian flat surface that can be parametrized with coordinate curves being, respectively, the fibers and the horizontal lifts of $\gamma$, in the following way

$$
\Phi(s, t)=\cos (t) \bar{\gamma}(s)+\sin (t) i \bar{\gamma}(s)
$$

3. Choose now the arclength parametrized geodesic of $\pi_{-}^{-1}(\gamma)$ defined by

$$
\gamma_{n p}(u)=\Phi(a u, b u), \quad a^{2}-b^{2}=\varepsilon_{1}, \quad \frac{b^{2}}{a^{2}}=\frac{p^{2}}{n^{2}}
$$

4. Let $\rho$ be the curvature function of $\gamma$ into $\mathbb{H}^{2}(-4)$. Then a direct computation gives the curvature $\kappa$ and the torsion $\tau$ of $\gamma_{n p}$ in $\mathbf{A d S}_{3}$

$$
\begin{aligned}
\kappa & =a^{2} \rho+2 a b \\
\tau^{2} & =\kappa^{2}-\varepsilon_{1} \kappa \rho+1
\end{aligned}
$$

From these equations, we obtain that $\tau=-\varepsilon_{2} \frac{p}{n} \kappa \pm 1$. Therefore, $\gamma_{n p}$ is a path in $\left[\mathbf{A d S}_{3}, \mathcal{F}_{m n p}\right]$ with $n^{2} \neq p^{2}$.
5. Finally, notice that all of solutions $\gamma_{n p}$ of this kind are either spacelike or timelike, according to $n^{2}>p^{2}$ or $n^{2}<p^{2}$, respectively.

## Trajectories being geodesics of hyperbolic Hopf tubes

In this case, the algorithm is as follows.

1. Choose a unit speed curve $\sigma(s)$ in the anti de Sitter plane $\mathbf{A d S}_{2}(-4)$ with curvature function $\rho$ and causal character $\delta_{1}$. Now, we consider its hyperbolic Hopf tube $\lambda^{-1}(\sigma)$ in $\mathbf{A d S}_{3}$ (see Appendix B).
2. This is a flat surface which is either Riemannian or Lorentzian, according to $\sigma$ is spacelike or timelike, respectively. It can be parametrized with coordinate curves being, respectively, the fibers and the horizontal lifts of $\sigma$, in the following way

$$
\Psi(s, t)=\cosh (t) \bar{\sigma}(s)+\sinh (t) i \bar{\sigma}(s)
$$

3. Choose now the arclength parametrized geodesic of $\lambda^{-1}(\sigma)$ defined by

$$
\sigma_{n p}(u)=\Psi(a u, b u), \quad \delta_{1} a^{2}+b^{2}=\varepsilon_{1}, \quad \frac{b^{2}}{a^{2}}=\frac{p^{2}}{n^{2}}
$$

4. Let $\rho$ be the curvature function of $\sigma$ into $\mathbf{A d S}_{2}(-4)$. A direct computation gives the curvature $\kappa$ and the torsion $\tau$ of $\sigma_{n p}$ in $\mathbf{A d S}_{3}$

$$
\begin{aligned}
\kappa & =a^{2} \rho+2 a b \\
\tau^{2} & =\kappa^{2}-\varepsilon_{1} \delta_{1} \kappa \rho+1
\end{aligned}
$$

From here we get $\tau=-\varepsilon_{2} \frac{p}{n} \kappa \pm 1$. Therefore, $\sigma_{n p}$ is a trajectory of $\left[\mathbf{A d S} \mathbf{S}_{3}, \mathcal{F}_{m n p}\right]$ with $n^{2} \neq p^{2}$.
5. When $\sigma$ is chosen to be timelike in $\mathbf{A d S} \mathbf{S}_{2}(-4)$, then the solutions $\sigma_{n p}$ are either spacelike or timelike according to $n^{2}<p^{2}$ or $n^{2}>p^{2}$, respectively.

Furthermore, the converse of these algorithms also hold. That is, all trajectories of these relativistic particle models are obtained according to them.

### 6.2 The dynamics in $\left[\mathbf{A d S}_{3}, \mathcal{F}_{m n p}\right]$ with $n^{2}=p^{2}$

Notice that, in this case, no solutions are obtained as geodesics in Hopf tubes. In addition, no solutions are obtained in Lorentzian hyperbolic Hopf tubes. Therefore, besides the one-parameter class of ordinary helices, the model presents the following soliton families.

## Trajectories being geodesics of Riemannian hyperbolic Hopf tubes

These solutions are obtained by means of the following algorithm.

1. Choose a spacelike unit speed curve $\sigma(s)$ in the anti de Sitter plane $\mathbf{A d S}_{2}(-4)$ with curvature function $\rho$. Consider its hyperbolic Hopf tube $\lambda^{-1}(\sigma)$, which is a Riemannian flat surface in $\mathbf{A d S}_{3}$ (see Appendix B). As above, it can be parametrized
with coordinate curves being, respectively, the fibers and the horizontal lifts of $\sigma$, in the following way

$$
\Psi(s, t)=\cosh (t) \bar{\sigma}(s)+\sinh (t) i \bar{\sigma}(s) .
$$

2. Take now the arclength parametrized geodesics of $\lambda^{-1}(\sigma)$ defined by

$$
\sigma_{ \pm p p}(u)=\Psi( \pm p u, p u) .
$$

3. Let $\rho$ be the curvature function of $\sigma$ into $\mathbf{A d S}_{2}(-4)$. The curvature $\kappa$ and the torsion $\tau$ of $\sigma_{ \pm p p}$ in $\mathbf{A d S}_{3}$ are

$$
\begin{aligned}
\kappa & =-\frac{1}{2} \rho+1 \\
\tau^{2} & =-\kappa^{2}-\kappa \rho+1
\end{aligned}
$$

Then $\tau=\kappa-1$, so that $\sigma_{ \pm p p}$ are trajectories of $\left[\mathbf{A d S}_{3}, \mathcal{F}_{m n p}\right]$ with $n^{2}=p^{2}$.
As the converse in this algorithm also holds, all nondegenerate Lancret solitons of the model are obtained following this method.

## Trajectories being geodesics of scrolls over null curves

These solitons are degenerate Lancret curves obtained from the following algorithm.

1. Take a null curve $\alpha(s), s \in I \subset \mathbb{R}$, in $\mathbf{A d S}_{3}$. Given a Cartan frame $\{A(s), D(s), F(s)\}$ along $\alpha$, consider the flat scroll (notice that $\mu= \pm 1$, see Appendix A) $\mathbf{S}_{\alpha D}$, which can be parametrized by

$$
\Phi(s, t)=\alpha(s)+t D(s), \quad(s, t) \in I \times \mathbb{R}
$$

2. For an arclength parametrized geodesic $\beta$ of $\mathbf{S}_{\alpha D}$, one can see that its acceleration is spacelike. Furthermore, its curvature and torsion functions are computed to satisfy $\tau= \pm \varepsilon_{1} \kappa \pm 1$ and so it is a trajectory of this model.
3. The converse also holds. Indeed, for a degenerate Lancret path $\beta$ of $\left[\mathbf{A d S}_{3}, \mathcal{F}_{m n p}\right]$ with $n^{2}=p^{2}$, its acceleration is spacelike and we may assume that $\tau=\kappa+\varepsilon_{1}$. The remaining cases can be handled likewise.
4. Define a null curve $\alpha$ in $\mathbf{A d S}_{3}$ by

$$
\alpha(s)=\beta(s)-\frac{1}{2} s(T(s)-B(s)) .
$$

5. Take the following vector fields along $\alpha$

$$
\begin{aligned}
A(s) & =-\frac{\varepsilon_{1}}{2} s \beta(s)+\frac{1}{2}(T(s)+B(s))+\frac{\varepsilon_{1}}{2} s N(s), \\
D(s) & =-\varepsilon_{1}(T(s)-B(s)) \\
F(s) & =-\frac{1}{2} s(T(s)-B(s))+N(s) .
\end{aligned}
$$

It is not difficult to see that $\{A(s), D(s), F(s)\}$ is a Cartan frame along $\alpha$ with $\mu=1$ and $\rho=\tau$.
6. Consider the scroll $\mathbf{S}_{\alpha D}$, which is a flat surface in $\mathbf{A d} \mathbf{S}_{3}$ and can be parametrized by $\Phi(s, t)=\alpha(s)+t D(s)$.
7. Finally, notice that $\beta$ can be viewed as a geodesic in this scroll, because $\beta(s)=$ $\Phi\left(s, \frac{\varepsilon_{1}}{2} s\right)$.

## 7 A summary of trajectories

We can describe the dynamics of the relativistic particle models $\left[\mathbf{A d S}_{3}, \mathcal{F}_{m n p}\right]$ by applying the above algorithms. We then summarize the corresponding moduli space of trajectories as follows.
(A) The model $\left[\operatorname{AdS}_{3}, \mathcal{F}_{m n p}\right]$ with $n^{2}>p^{2}$

The moduli space of solitons is made up of the following classes of trajectories:

- A one-parameter class of ordinary helices.
- A class of spacelike nondegenerate Lancret curves obtained as geodesics of Hopf tubes

$$
\Gamma_{\left(n^{2}>p^{2}\right)}=\left\{\gamma_{n p} \mid \gamma \text { is a curve in } \mathbb{H}^{2}(-4)\right\} .
$$

- A class of spacelike nondegenerate Lancret curves obtained as geodesics of Riemannian hyperbolic Hopf tubes

$$
\Sigma_{\left(n^{2}>p^{2}\right)}^{+}=\left\{\sigma_{n p} \mid \sigma \text { is a spacelike curve in } \mathbf{A d S}_{2}(-4)\right\}
$$

- A class of timelike nondegenerate Lancret curves obtained as geodesics of Lorentzian hyperbolic Hopf tubes

$$
\Sigma_{\left(n^{2}>p^{2}\right)}^{-}=\left\{\sigma_{n p} \mid \sigma \text { is a timelike curve in } \mathbf{A} \mathbf{d} \mathbf{S}_{2}(-4)\right\}
$$

(B) The model $\left[\operatorname{AdS}_{3}, \mathcal{F}_{m n p}\right]$ with $n^{2}<p^{2}$

The moduli space of solitons is made up of the following classes of trajectories.

- A one-parameter class of ordinary helices.
- A class of timelike nondegenerate Lancret curves obtained as geodesics of Hopf tubes

$$
\Gamma_{\left(n^{2}<p^{2}\right)}=\left\{\gamma_{n p} \mid \gamma \text { is a curve in } \mathbb{H}^{2}(-4)\right\} .
$$

- A class of spacelike nondegenerate Lancret curves obtained as geodesics of Riemannian hyperbolic Hopf tubes

$$
\Sigma_{\left(n^{2}<p^{2}\right)}^{+}=\left\{\sigma_{n p} \mid \sigma \text { is a spacelike curve in } \mathbf{A d S}_{2}(-4)\right\}
$$

- A class of spacelike nondegenerate Lancret curves obtained as geodesics of Lorentzian hyperbolic Hopf tubes

$$
\Sigma_{\left(n^{2}<p^{2}\right)}^{-}=\left\{\sigma_{n p} \mid \sigma \text { is a timelike curve in } \mathbf{A d S}_{2}(-4)\right\}
$$

(C) The model $\left[\mathbf{A d S}_{3}, \mathcal{F}_{m n p}\right]$ with $n^{2}=p^{2}$

The moduli space of solitons is made up of the following classes of trajectories.

- A one-parameter class of ordinary helices.
- A class of spacelike nondegenerate Lancet curves obtained as geodesics of Riemannian hyperbolic Hopf tubes

$$
\Sigma_{\left(n^{2}=p^{2}\right)}^{+}=\left\{\sigma_{n p} \mid \sigma \text { is a spacelike curve in } \mathbf{A d S}_{2}(-4)\right\} .
$$

- A class of degenerate Lancret curves obtained as geodesics of scrolls over null curves

$$
\Upsilon=\left\{\left.\beta_{\alpha D}(s)=\alpha(s) \pm \frac{s}{2} D(s) \right\rvert\, \alpha \text { is a null curve in } \mathbf{A d S}_{3}\right\} .
$$

Remark. It should be noticed that a Lancret curve in $\mathbf{A d S}_{3}$ with Lorentzian rectifying plane at any point is simultaneously degenerate and nondegenerate, because it admits both null and non null axes. Consequently, it can be viewed as a geodesic of a Hopf tube but also as one in a flat scroll over a null curve. Therefore, a Lancret curve can be regarded as a trajectory in different models.

## 8 Closed trajectories

To study closed trajectories, we will modify a little bit the model $\left[M(C), \mathcal{F}_{m n p}\right]$ in the sense that the action $\mathcal{F}_{m n p}$ is now assumed to be defined on the space of closed curves in $M(C)$. Then no boundary conditions are dropped. For obvious reasons, we will restrict ourselves to anti de Sitter backgrounds, and without loss of generality we will consider the case where $C=-1$. In this setting, we obtain similar field equations which we gather in the following.

Proposition 2. Let $\mathcal{C}$ be the space of immersed closed curves in $\mathbf{A d S}_{3}$. The critical points of the variational problem associated with the action $\mathcal{F}_{\text {mnp }}: \mathcal{C} \rightarrow \mathbb{R}$ are those closed curves which are solutions of the following Euler-Lagrange equations

$$
\begin{aligned}
\varepsilon_{3}(m-p \tau) \kappa+\varepsilon_{1} n\left(\tau^{2}-1\right) & =0, \\
\varepsilon_{1} p \kappa_{s}-\varepsilon_{3} n \tau_{s} & =0 .
\end{aligned}
$$

It is clear that the solutions of the above field equations are Lancret curves in $\mathbf{A d S}_{3}$. Consequently, we have to determine the closed Lancret curves in $\mathbf{A d S}_{3}$. These paths can be characterized according to the following program (see Appendix B for other details).

1. Choose a closed curve $\gamma(u), u \in \mathbb{R}$, in $\mathbb{H}^{2}(-4)$. Then, its Hopf tube $\mathbf{S}_{\gamma}=\pi_{-}^{-1}(\gamma)$ becomes a Lorentzian flat torus of $\mathbf{A d S}_{3}$. The isometry type of a Hopf torus is determined by the length $L$ of $\gamma$ in $\mathbb{H}^{2}(-4)$ and its enclosed area $A$. In this sense, $\mathbf{S}_{\gamma}=\pi_{-}^{-1}(\gamma)$ is isometric to $\mathbb{L}^{2} / \Lambda$, where $\Lambda$ is the lattice in $\mathbb{L}^{2}$ generated by $(2 A, L)$ and $(2 \pi, 0)$.
2. Consequently, given a Lancret curve of $\mathbf{A d S}_{3}$, then $\tau=b \kappa \pm 1$. Now it is closed if and only if its inverse slope $b$ satisfies

$$
b=\frac{1}{L}(2 A+q \pi), \quad q \in \mathbb{Q} .
$$

3. According to Table 3, the most interesting models of particles evolving in anti de Sitter backgrounds are those with $n \neq 0$ and either $m=p=0$ or $m p \neq 0$. Otherwise, trajectories are ordinary helices and their closedness conditions have been considered in [8]. In particular, for these models with paths being ordinary helices one can get a rational one-parameter class of closed solitons.
4. The first interesting case corresponds with the so called Plyushchay model and its moduli space of closed solitons was obtained in [7] according to the following result: Closed solitons of the Plyushchay relativistic particle model in anti de Sitter backgrounds are obtained just lifting, via the standard Hopf mapping, the fold covers of closed curves in hyperbolic plane that bounds an area which is a rational multiple of $\pi$.
5. The second case, with $m n p \neq 0$, is a bit more subtle. Besides the rational oneparameter family of closed helices obtained inside the class of solutions $\mathcal{T}$, the model also admits closed solitons which are non trivial Lancret curves. To get them, we proceed as follows. Take an embedded closed curve $\gamma$ in $\mathbb{H}^{2}(-4)$. Notice that the only restriction on $L$ and $A$ to define an embedded closed curve in the hyperbolic plane comes from the isoperimetric inequality in $\mathbb{H}^{2}(-4)$,

$$
L^{2} \geq 4 \pi A+4 A^{2}
$$

equality holding just on the geodesic circles.
6. It is clear that, in terms of $(2 A, L)$, the above inequality writes as $(2 A+\pi)^{2}-L^{2} \leq$ $\pi^{2}$. Therefore, in the $(2 A, L)$-plane, we define the following region

$$
\Delta=\left\{(2 A, L) \mid(2 A+\pi)^{2}-L^{2} \leq \pi^{2}, \quad 0<A, 0<L\right\} .
$$

Now, for each point $z=(2 A, L) \in \Delta$, there is an embedded closed curve in $\mathbb{H}^{2}(-4)$ with length $L$ and enclosed area $A$.
7. Let $\gamma^{z}$ be an embedded closed curve in $\mathbb{H}^{2}(-4)$ and let $(2 A, L) \in \Delta$ be the corresponding point in $\Delta$. Let $\mathbf{S}_{\gamma}=\pi_{-}^{-1}\left(\gamma^{z}\right)$ be its Lorentzian Hopf torus parametrized, as usual, by $\Phi(s, t)=e^{i t} \bar{\gamma}^{z}(s)$. Now, let $\beta_{b}^{z}$ be a non null geodesic of this torus with slope $b$ (measured with respect to fibres). Then $\beta_{b}^{z}$ is closed if and only if the point $z=(2 A, L)$ in the region $\Delta$ of the $(2 A, L)$-plane also lies in a straight line with slope $\frac{1}{b}$, which cuts the $L$-axis at the height $\frac{b L-2 A}{\pi} \in \mathbb{Q}$.

In particular, we have proved the following existence result for non trivial closed solitons in the models $\left[\mathbf{A d S}_{3}, \mathcal{F}_{\text {mnp }}\right]$.

Theorem. For any couple of parameters $n$ and $p$, with $n p \neq 0$, there exists an infinite class of closed trajectories relative to the model $\left[\mathbf{A d S}_{3}, \mathcal{F}_{\text {mnp }}\right]$ with $m^{2}=p^{2}$. This class

includes all geodesics $\beta_{b}^{z}$ in $\mathbf{S}_{\gamma^{z}}=\pi_{-}^{-1}\left(\gamma^{z}\right)$ with slope $b=\frac{p}{n}$ and $\gamma^{z}$ determined as above by $z=(2 A, L)$ in the following region

$$
\Delta \cap\left(\bigcup_{q \in \mathbb{Q}} S_{q}\right)
$$

where $S_{q}=\left\{(2 A, L) \left\lvert\, \frac{p}{n} L-2 A=q \pi\right.\right\}$.
The converse of the above statement also holds, so we get the following quantization principle to describe the moduli space of non trivial closed solitons in anti de Sitter settings.

Corollary. The essential moduli space of closed trajectories in $\left[\mathbf{A d S}_{3}, \mathcal{F}_{m n p}\right]$ can be identified with the region in the $(x, y)$-plane defined by

$$
\mathrm{R}_{b}=\Delta \cap\left(\bigcup_{q \in \mathbb{Q}} S_{q}\right), \quad b=\frac{p}{n}
$$

where

$$
\begin{aligned}
\Delta & =\left\{(x, y) \mid(x+\pi)^{2}-y^{2} \leq \pi^{2}, \quad x>0, y>0\right\} \\
S_{q} & =\left\{(x, y) \left\lvert\, \frac{p}{n} y-x=q \pi\right.\right\}
\end{aligned}
$$

### 8.1 An example

The key point in this algorithm is the searching for curves with prescribed area $A$ and length $L$ in $\mathbb{H}^{2}(-4)$. To do that we will use the hyperbolic Lambert isoareal map. Let us consider

$$
\mathbb{H}^{2}(-4)=\left\{(x, y, z) \in \mathbb{R}^{3} \left\lvert\, x^{2}+y^{2}-z^{2}=-\frac{1}{4}\right., z>0\right\}
$$

with coordinates $(\varphi, \theta)$ given by

$$
X(\varphi, \theta)=-\frac{1}{2}(\cos \varphi \sinh \theta, \sin \varphi \sinh \theta, \cosh \theta)
$$

Let $\Psi: \mathbb{R}^{2} \rightarrow V=\left\{(x, y) \in \mathbb{R}^{2} \left\lvert\, y \geq \frac{1}{4}\right.\right\}$ be the map given by

$$
\Psi(\varphi, \theta)=\left(\varphi, \frac{1}{4} \cosh \theta\right) .
$$

Then the hyperbolic Lambert isoareal map $L: \mathbb{H}^{2}(-4) \rightarrow V$ is defined by $L=\Psi \circ X^{-1}$, with inverse map given by

$$
L^{-1}(x, y)=X(x, \operatorname{arccosh}(4 y)) .
$$

Therefore, given a curve $\nu: I=\left[x_{0}, x_{1}\right] \rightarrow V, \nu(t)=\left(\nu_{1}(t), \nu_{2}(t)\right)$, the curve $\beta=L^{-1} \circ \nu$ will enclose the same area as $\nu$. It is easy to see that

$$
\beta^{\prime}(t)=\nu_{1}^{\prime}(t) X_{\varphi}+\frac{4 \nu_{2}^{\prime}(t)}{\sqrt{16 \nu_{2}(t)^{2}-1}} X_{\theta}
$$

and then

$$
\begin{equation*}
\operatorname{Length}(\beta)=\int_{x_{0}}^{x_{1}} \frac{1}{2} \sqrt{\left(\varphi(t)^{2}-1\right) \nu_{1}^{\prime}(t)^{2}+\frac{\varphi^{\prime}(t)^{2}}{\varphi(t)^{2}-1}} d t \tag{8}
\end{equation*}
$$

where $\varphi(t)=4 \nu_{2}(t)$.
The following ellipses

$$
a^{2} x^{2}+b^{2}\left(y-1-\frac{1}{b}\right)^{2}=1, \quad a, b>0
$$

live in $V$ and the enclosed area is $\frac{\pi}{a b}$. By parametrizing them by

$$
\left(\nu_{1}(t), \nu_{2}(t)\right)=\left(\frac{1}{a} \sin t, \frac{1}{b}(\cos t+1)+1\right),
$$

we can apply (8) to find their lengths. In general, it is not possible to get the exact value, so we must use numerical methods to estimate them. For example, if we take $a=1$ and $b=2$, then Length $(\beta) \cong 11.8955$, so the slopes of closed curves are given by

$$
b \cong \frac{1}{11.8955}(q+1),
$$

$q$ being a rational number.

## 9 Dynamical parameters of the trajectories: mass and spin

We have include this section to be compared with [16]. In this sense, we discuss just the main differences that the dynamical parameters present when particles evolve in non flat backgrounds (anti de Sitter spaces after the above discussion) in contrast with the evolution in flat spaces that was considered in [16]. Also, along the section we will restrict ourselves to trajectories being helices. The main reason to do so is that in this case, the geometry of the solutions is determined by two integrals of motion, the curvature and the torsion. Therefore in this context, the geometry of solutions turn out to be equivalent to the dynamics (mass and spin) of particles. Moreover, there exists
a dependence between curvature and torsion that in the above equivalence gives the classical mass spectra of the models. In this approach, we can determine the curvature of trajectories in terms of the torsion and then to express the dynamical parameters (mass and spin) of those trajectories as functions of the torsion. However, this does not work for the remaining solutions, i.e. for Lancret trajectories in the models $\left[\mathbb{L}^{3}, \mathcal{F}_{0 n p}\right]$ and $\left[\mathbf{A d S}_{3}, \mathcal{F}_{m n p}\right]$ with $m n p \neq 0$. In these cases, an alternative way to determine mass and spin as functions of some geometric invariant should be given. Perhaps, the slope of Lancret trajectories could be used to study the corresponding mass spectra of these models, however this remains as an open problem.

Let us start by introducing squared mass and spin in terms of the integrals of motion and associated infinitesimal symmetries.

The models describing relativistic particles with Lagrangian density being a function, $F(\kappa, \tau)$, of both the curvature and the torsion of the trajectories, present nice symmetries. The trajectories or critical curves admit a pair of Killing vector fields, say $P$ and $J$, along them. This means that one can deform trajectories in the direction of $P$ (or $J$ ) without changing shape, only position, [15]. In particular, when the space has constant curvature, then $P$ and $J$ come from Killing vector fields on the whole space. If $F$ is lineal, i.e. $F(\kappa, \tau)=m+n \kappa+p \tau$, then

$$
P=\varepsilon_{1} m T+\left(\varepsilon_{1} p \kappa-\varepsilon_{3} n \tau\right) B, \quad J=-\varepsilon_{1} p T-\varepsilon_{3} n B .
$$

These infinitesimal symmetries can be used to get the integrals of motion. Therefore, in $M(C)$, the trajectories are characterized as those curves that satisfy the following integral equations

$$
\begin{aligned}
<P, P>-C<J, J> & =d, \\
<P, J> & =e,
\end{aligned}
$$

where $d$ and $e$ are constants related with the dynamical parameters, mass, $M$, and spin, $S$, of particles according to

$$
\begin{align*}
-M^{2} & =<P, P>-C<J, J>  \tag{9}\\
M S & =<P, P>-C<J, J>. \tag{10}
\end{align*}
$$

The field equations, (5-6), can be used to get the following dependence between the curvature and torsion of trajectories

$$
\varepsilon_{1} \kappa(p \tau-m)=\varepsilon_{3} n\left(\tau^{2}+C\right),
$$

Therefore, in the models with $n=0$, i.e. when curvature does not appear in the Lagrangian density, the squared mass can be obtained as a function of the curvature of trajectories

$$
M^{2}(\kappa)=\varepsilon_{1}\left(-p \kappa^{2}+C p^{2}-m^{2}\right) .
$$

It should be noticed that at any backgrounds (for any value of $C$ ) the trajectories of these models are helices with arbitrary $\kappa>0$ and fixed torsion $m / p$ (see the earlier
tables). Consequently, on one hand, one can describe the mass spectra of those models in terms of the curvature of solutions and on the other hand, one observes that the curvature of the space, $C$, plays an important role that description. Since the sign of $m$ does not participate in that function, one can assume, without loss of generality, that $p>0$ and consequently, when $C \geq 0, M^{2}(\kappa)>0$ for timelike trajectories while $M^{2}(\kappa)<0$ for spacelike ones. In other words, the models $\mathcal{F}_{m 0 p}$ present only a massive sector of timelike solutions and a tachyonic sector of spacelike solutions in both $\mathbb{L}^{3}$ and $\mathbf{A d S}_{3}$ while in $\mathbf{d S}_{3}$ they could present massive and tachyonic sectors connected by a massive state for any kind of solutions. This is an important difference between, on one hand, the de Sitter geometry and, on the other hand, the Lorentz-Minkoski and the anti de Sitter geometries.

To compare with the mass spectra in $\mathbb{L}^{3}$, that were considered in [16], from now on we focus on timelike curves in the anti de Sitter space, i.e., $\varepsilon_{1}=-1, \varepsilon_{2}=1, \varepsilon_{3}=1$, other cases can be similarly treated. In this framework, the squared mass is given as a function of the torsion by

$$
M^{2}(\tau)=m^{2}+C\left(n^{2}-p^{2}\right)-n^{2}\left(\frac{m \tau+p C}{p \tau-m}\right)^{2}
$$

To analyze the massive sector, the massless states and the tachyonic sector of these models, we need to obtain qualitative graphs of $M^{2}(\tau)$. It is clear that, a priori, there are a high number of cases to be considered. However, it can be simplified if we consider $M^{2}(\tau)$ as a function of four parameters and notice the following symmetries

$$
\begin{aligned}
M^{2}(\tau, m, n, p) & =M^{2}(-\tau,-m, n, p) \\
M^{2}(\tau, m, n, p) & =M^{2}(\tau, m,-n, p)
\end{aligned}
$$

To obtain the complete graph of $M^{2}(\tau)$, we need the following considerations

1. The function $M^{2}(\tau)$ has a critical point at $\tau=-C p / m$ because

$$
\left(M^{2}\right)^{\prime}(\tau)=-\frac{2 n^{2}\left(m^{2}+C p^{2}\right)(C p+m \tau)}{(m-p \tau)^{3}}
$$

and the critical value is $M^{2}(-C p / m)=m^{2}+C\left(n^{2}-p^{2}\right)$.
2. This critical point is a maximum if $m \neq 0$ because

$$
\left(M^{2}\right)^{\prime \prime}(-C p / m)=-\frac{2 m^{4} n^{2}}{\left(m^{2}+C p^{2}\right)^{2}}
$$

and so we can reasonably assume that $m^{2}+C\left(n^{2}-p^{2}\right)>0$.
3. The function $M^{2}(\tau)$ has a vertical asymptotic direction at $\tau_{o}=m / p$. Moreover

$$
\begin{cases}\lim _{\tau \rightarrow \pm \infty} M^{2}(\tau)=\frac{\left(C p^{2}-m^{2}\right)\left(n^{2}-p^{2}\right)}{p^{2}} & \text { if } p \neq 0 \\ \lim _{\tau \rightarrow \pm \infty} M^{2}(\tau)=-\infty & \text { if } p=0\end{cases}
$$

Consequently, the graph is free of asymptotic directions if $p=0$, i.e., the torsion of trajectories does not appear in the Lagrangian density.
4. The squared mass, $M^{2}(\tau)$, has a pair of zeroes, i.e., there exist at most two massless states in the mass spectrum of the system, which are obtained for trajectories whose torsions are

$$
\tau^{1}=\frac{-\xi m-p C}{m-\xi p}, \quad \tau^{2}=\frac{\xi m-p C}{m+\xi p}
$$

where $\xi=\sqrt{\frac{C\left(n^{2}-p^{2}\right)+m^{2}}{n^{2}}}$. We put $\tau_{\min }=\min \left\{\tau^{1}, \tau^{2}\right\}$ and $\tau_{\max }=\max \left\{\tau^{1}, \tau^{2}\right\}$.
5. The squared mass function also satisfies $M^{2}(-\sqrt{-C})=M^{2}(+\sqrt{-C})=m^{2}+$ $C\left(2 n^{2}-p^{2}\right)$.

Therefore, the four essentially different cases of complete graphs with exactly two zeroes of $M^{2}(\tau)$ are
(i) $m^{2}+p^{2} C>0$ and $m p>0$ :

(iii) $m=0$ and $p \neq 0$ :

(ii) $m^{2}+p^{2} C<0$ and $m p>0$ :

(iv) $m \neq 0$ and $p=0$ :


To obtain the mass spectra of these models, we need to solve the obvious constraint $\kappa>0$, which using the field equations (5-6), can be written

$$
\kappa=-\frac{n\left(\tau^{2}+C\right)}{p \tau-m}>0
$$

Therefore, the mass spectra of models associated with these graphs are the following.
(i) The mass spectrum of $\left[\mathbf{A d S}_{3}, \mathcal{F}_{m n p}\right]$ with $m^{2}+p^{2} C>0$ and $m p>0$

It is necessary to distinguish two cases:
(i1) If $n p>0$, then the system has

- A massive sector $-\infty<\tau<-\sqrt{-C}$.
- A massive sector $+\sqrt{-C}<\tau<\tau_{\text {min }}$.
- A massless state at $\tau=\tau_{\text {min }}$.
- A tachyonic sector $\tau_{\min }<\tau<\frac{m}{p}$.
(i2) If $n p<0$, then the system has
- A massive sector $-\sqrt{-C}<\tau<+\sqrt{C}$.
- A tachyonic sector $\frac{m}{p}<\tau<\tau_{\text {max }}$.
- A massless state at $\tau=\tau_{\text {max }}$.
- A massive sector $\tau_{\max }<\tau<+\infty$.
(ii) The mass spectrum of $\left[\mathbf{A d S}_{3}, \mathcal{F}_{m n p}\right]$ with $m^{2}+p^{2} C<0$ and $m p>0$.

It is necessary to distinguish two cases:
(ii1) If $n p>0$, then the system has

- A massive sector $-\infty<\tau<-\sqrt{-C}$.
- A tachyonic sector $\frac{m}{p}<\tau<\tau_{\text {max }}$.
- A massless state at $\tau=\tau_{\text {max }}$.
- A massive sector $\tau_{\text {max }}<\tau<+\sqrt{-C}$.
(ii2) If $n p<0$, then the system has
- A massive sector $-\sqrt{-C}<\tau<\tau_{\text {min }}$.
- A massless state at $\tau=\tau_{\text {min }}$.
- A tachyonic sector $\tau_{\text {min }}<\tau<\frac{m}{p}$.
- A massive sector $+\sqrt{-C}<\tau<+\infty$.
(iii) The mass spectrum of $\left[\operatorname{AdS}_{3}, \mathcal{F}_{m n p}\right]$ with $m=0$ and $p \neq 0$.

It is necessary to distinguish two cases:
(iii1) If $n p>0$, then the system has

- A massive sector $-\infty<\tau<-\sqrt{-C}$.
- A tachyonic sector $0<\tau<\tau_{\text {max }}$.
- A massless state at $\tau=\tau_{\text {max }}$.
- A massive sector $\tau_{\max }<\tau<+\sqrt{-C}$.
(iii2) If $n p<0$, then the system has
- A massive sector $-\sqrt{-C}<\tau<\tau_{\text {min }}$.
- A massless state at $\tau=\tau_{\text {min }}$.
- A tachyonic sector $\tau_{\min }<\tau<0$.
- A massive sector $+\sqrt{-C}<\tau<+\infty$.
(iv) The mass spectrum of $\left[\mathbf{A d S}_{3}, \mathcal{F}_{m n p}\right]$ with $m \neq 0$ and $p=0$.

It is necessary to distinguish two cases:
(iv1) If $n m>0$, then the system has

- A tachyonic sector $-\infty<\tau<\tau_{\text {min }}$.
- A massless state at $\tau=\tau_{\text {min }}$.
- A massive sector $\tau_{\text {min }}<\tau<-\sqrt{-C}$.
- A massive sector $+\sqrt{-C}<\tau<\tau_{\max }$.
- A massless state at $\tau=\tau_{\max }$.
- A tachyonic sector $\tau_{\max }<\tau<+\infty$.
(iv2) If $n m<0$, then the system has only massive solutions. It has just a massive sector which is made up of helices with torsion satisfying $-\sqrt{-C}<\tau<+\sqrt{-C}$.

Remark. It should be noticed that the mass spectrum of the system $\left[\mathbf{A d S} \mathbf{S}_{3}, \mathcal{F}_{m n 0}\right]$ with $m n<0$ and $m^{2}+2 C n^{2} \geq 0$ has only massive solutions. In fact, it has just a massive sector which is made up of helices with torsion satisfying $-\sqrt{-C}<\tau<+\sqrt{-C}$. This has no equivalent in absence of gravity.

Furthermore, in massive, $M^{2}>0$, and tachyonic, $M^{2}<0$, sectors the relativistic spin in these models, gives the classical dependence of mass versus spin, (2),

$$
S(\tau)=\frac{m\left(p^{2}-n^{2}\right) \tau-p\left(m^{2}+n^{2} C\right)}{(p-\tau m) M(\tau)}
$$

This formula shows that one can recuperate the mass of the system as a singled-valued spin function like in the flat case, [16].

The following symmetries are useful to reduce the cases and then, we proceed as above to get the corresponding spin pictures

$$
\begin{aligned}
& S(\tau, m, n, p)=-S(-\tau,-m, n, p) \\
& S(\tau, m, n, p)=S(\tau, m,-n, p)
\end{aligned}
$$


(iii) $n^{2}<p^{2} / 2$ and $m=0$ :

(iv) $m^{2}>-2 C n^{2}$ and $p=0$ :


## 10 Final discussions and conclusions

In three dimensional spacetimes with constant curvature, $M(C)$, we have considered models for relativistic particles, $\mathcal{F}_{m n p}$, where the Lagrangian densities depend linearly on both the curvature and the torsion of the trajectories.

These particle models have been extensively considered in the literature. For example, in [16], the authors studied these models in flat backgrounds, $C=0$. However, they did not use a sufficiently powerful mathematical machinery to completely describe the dynamics of particles, even in the more simplest case, $\left[\mathbb{L}^{3}, \mathcal{F}_{m n p}\right]$ with $m \neq 0$, where the trajectories are helices. The matter is that the complete description of the dynamics is a very subtle problem that, in particular, includes the resolution of the so called solving natural equations one. Now, in $\mathbb{L}^{3}$ there are degenerate helices, that is helices with null axis, and to geometrically describe these helices we need the notion of scroll. In addition, the model $\left[\mathbb{L}^{3}, \mathcal{F}_{0 n p}\right]$ has a more complicate dynamics where classes of, both non-degenerate and degenerate, Lancret trajectories appear. Certainly, complications increase when one considers non-flat gravitational fields, in particular anti de Sitter backgrounds. In this case, the use of the different Hopf mappings constitutes an essential ingredient to solve the dynamics.

In this paper, we have employed a powerful mathematical machinery to explicitly obtain the complete moduli spaces of trajectories that describe the dynamics of these models. That machinery includes the geometry of the scrolls as well as the construction of flat surfaces in anti de Sitter backgrounds using the Hopf mappings and the knowledge of the Lancret curves geometry. Those moduli spaces of solutions are formally collected in three tables where one can appreciate the essential differences, depending on the different backgrounds. Furthermore, we have designed explicit algorithms providing, step by step, the geometric integrations to get the trajectories. As far as we know, this is the first time that all the trajectories that constitute the dynamics of these models are exhibited in the literature.

The interest of the anti de Sitter geometry increased greatly due in part to the Maldacena conjecture [20] and the holography hypothesis. At the same time it plays an important role in, for instance, the theory of higher spin gauge fields. The group manifold case, $\mathbf{A d S}_{3}$ is special and interesting in many respects, as it can be checked along a wide literature on different topics that involves this background. In this paper, once more, we show this interest with respect to the dynamics of the particle models $\mathcal{F}_{\text {mnp }}$. In
fact, the most interesting models $\left[M(C), \mathcal{F}_{m n p}\right]$ appear just when $M(C)=\mathbf{A d S}_{3}$. This conclusion is supported, among other arguments, in the following

1. Let us consider the model $\mathcal{F}_{0 n 0}$, known as the Plyushchay one, in $n+1$ dimensions, $M^{n+1}(C)$ with, a priori, arbitrary $n$. To study the dynamics of this model, it is convenient to use a Lorentzian version of a codimension reduction technique, which in Riemannian geometry was given by J. Erbacher [11]. That will prove that the particle dynamics takes place in Lorentzian submanifolds with dimension at most three that, in addition, are totally geodesic, i.e. they have trivial extrinsic geometry in $M^{n+1}(C)$. Consequently, the, a priori arbitrary, chosen $n$ must be at most 2 , that is the dynamics takes place in $2+1$ dimensions, $M^{2+1}(C)$. For example, the dynamics associated with the Plyushchay model does not make sense in $\mathbb{L}^{4}$ which strongly contrast with results obtained in $[26,27]$ where this setting is considered. Once reduced the general case to $2+1$ dimensions, we look at the tables of this paper to get the following possibilities
$\mathbb{L}^{3}$ : When $C=0$, we can continue reducing dimension. The particles evolve throughout plane trajectories. Furthermore, any plane curve is a trajectory of the model. The action is constant on each regular homotopy class of curves and this fact makes trivial the dynamics.
$\mathbf{d S}_{3}$ : In the de Sitter backgrounds, we have no solutions of the corresponding field equations and so no dynamics.
$\mathbf{A d S}_{3}$ : Finally, in anti de Sitter backgrounds, the dynamics runs throughout trajectories that are obtained as horizontal lifts, via Hopf mappings, of curves in either a hyperbolic plane or an anti de Sitter plane.

The conclusion is that $\mathbf{A d S}_{3}$ is the only spacetime with constant curvature, no matter the dimension, where the Plyushchay model makes sense.
2. The existence, or not, of closed trajectories of a dynamics system is a quite important and non trivial fact in many fields. In the context of these models, it is strongly related with the so called closed curve problem. This problem is completely solved in this paper. The answer also points to the interest of $\mathbf{A d S}_{3}$. In fact, the only models admitting closed trajectories are formulated in $\mathbf{A d S}_{3}$. Furthermore, in this paper, we have obtained a Dirac quantization type result and an elegant algorithm to obtain the essential moduli space of closed solitons in models working in anti de Sitter spaces. The isoperimetric inequality in the hyperbolic plane plays an important role in searching for this moduli space of closed trajectories which can be reflected in the picture.
3. The mass spectra of the models $\left[\mathbf{A d S}_{3}, \mathcal{F}_{m n p}\right]$ have been studied in the paper. They are completely different to those in $\left[\mathbb{L}^{3}, \mathcal{F}_{m n p}\right]$ that where considered in [16]. In fact, the existence of a non trivial gravity field changes the nature of physical spectra. Hence, in anti de Sitter backgrounds these system always have massive sector, under reasonable conditions, while they may have tachyonic sector and two, one or none massless states. Furthermore, we exhibit some that only have massive sector in the mass spectrum. This solve, for models with linear Lagrangian density, a problem posed in [17]. In the study of this behaviour for mass spectra, the curvature $C<0$, of $\mathbf{A d S}_{3}$, plays an important role. For example, the model
$\left[\mathbf{A d S}_{3}, \mathcal{F}_{m n 0}\right]$ with $m n<0$ and $m^{2}+2 C n^{2} \geq 0$ has only massive states. The massive sector is made up of those helices in $\mathbf{A d S}_{3}$ whose torsion, $\tau$ lies in the interval $(-\sqrt{-C},+\sqrt{-C})$. Therefore, we can change the size of this sector, just scaling the non trivial gravity tensor field.

To conclude this section, let us include a final comment on the physical relevance of the results obtained. In fact, the gravity strongly govern the models that are considered from the mass spectra point of view. This claim is supported by the following results on the mass spectra of timelike solutions
(A) How distinguish de Sitter geometry from both Lorentz-Minkoski and anti de Sitter geometries. Consider models with no curvature in the Lagrangian density, then

- $\left[\mathbb{L}^{3}, \mathcal{F}_{m 0 p}\right]$ and $\left[\mathbf{A d S}_{3}, \mathcal{F}_{m 0 p}\right]$ with $p>0$ have mass spectra of timelike made up of just a massive sector.
- $\left[\mathbf{d S}_{3}, \mathcal{F}_{m 0 p}\right]$ with $p>0$ has a massive physical spectrum of timelike trajectories if $C p^{2}-m^{2}<0$ while that spectrum has:

1. A tachyonic sector of helices with curvature satisfying $0<\kappa<+\sqrt{\frac{C p^{2}-m^{2}}{p}}$ and torsion $\tau=\frac{m}{p}$.
2. A massless state corresponding to the helix with curvature $\kappa=+\sqrt{\frac{C p^{2}-m^{2}}{p}}$ and torsion $\tau=\frac{m}{p}$.
3. A massive sector of solutions being helices with curvature $+\sqrt{\frac{C p^{2}-m^{2}}{p}}<\kappa<$ $+\infty$ and torsion $\tau=\frac{m}{p}$
(B) How distinguish among Lorentz-Minkoski and anti de Sitter geometries. Consider models with no torsion in the Lagrangian density, then

- $\left[\mathbb{L}^{3}, \mathcal{F}_{m n 0}\right]$ with $n m<0$ has no timelike trajectories and so no mass spectrum.
- $\left[\mathbf{A d S}_{3}, \mathcal{F}_{m n 0}\right]$ with $n m<0$ has mass spectrum according to the following cases

1. If $m^{2}+C n^{2}<0$, then its mass spectrum of timelike trajectories has a tachyonic sector of helices with torsion satisfying $-\sqrt{-C}<\tau<+\sqrt{-C}$.
2. If $m^{2}+C n^{2}=0$, then its physical spectrum of timelike solutions has a pair of tachyonic sectors of helices with torsion satisfying $-\sqrt{-C}<\tau<0$ and $0<\tau<+\sqrt{-C}$, respectively which are connected by a massless state corresponding to a circle.
3. If $m^{2}+2 C n^{2}<0<m^{2}+C n^{2}$, the the physical mass spectrum of timelike trajectories has tachyonic sector, helices with torsion in the intervals $\left(-\sqrt{-C},-\frac{1}{n} \sqrt{m^{2}+C n^{2}}\right) \bigcup\left(+\frac{1}{n} \sqrt{m^{2}+C n^{2}},+\sqrt{-C}\right)$, a massive sector, of helices with torsion in the interval $\left(-\frac{1}{n} \sqrt{m^{2}+C n^{2}},+\frac{1}{n} \sqrt{m^{2}+C n^{2}}\right)$, and a pair of massless states corresponding to helices with torsion $\tau= \pm \frac{1}{n} \sqrt{m^{2}+C n^{2}}$.
4. If $m^{2}+2 C n^{2} \geq 0$ then the mass spectrum of timelike solutions presents only a massive sector of helices with torsion $-\sqrt{-C}<\tau<+\sqrt{-C}$. Notice that this physical spectrum is impossible in absence of gravity, i.e., in LorentzMinkowski geometry.

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## Appendix A: Scrolls over null curves

Let $M(C)$ be a three dimensional Lorentzian space form with constant curvature $C$, i.e., $\mathbf{d S}_{3}, \mathbb{L}^{3}$ or $\mathbf{A d S}_{3}$, according to the sign of $C$. These models can be regarded (see [24]) as totally umbilical hypersurfaces in certain pseudo-Euclidean spaces, say $\mathbf{E}$, so that any point of $M(C)$ can be identified with its position vector in $\mathbf{E}$. As it is usual, an over point means the derivative in $\mathbf{E}$. Let $\alpha(s), s \in I \subset \mathbb{R}$, be a null curve in $M(C)$ and let $\{A, D, F\}$ be a set of vector fields along $\alpha$ satisfying

$$
\begin{aligned}
& \langle A, A\rangle=\langle D, D\rangle=0, \quad\langle A, D\rangle=-1, \\
& \langle A, F\rangle=\langle D, F\rangle=0, \quad\langle F, F\rangle=1,
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{\alpha} & =A, \\
\dot{A} & =\rho F, \quad \rho=\rho(s) \neq 0, \\
\dot{D} & =C \alpha+\mu F, \quad \mu \text { being a constant }, \\
\dot{F} & =\mu A+\rho D .
\end{aligned}
$$

In this setting, we will say that $\{A(s), D(s), F(s)\}$ is a Cartan frame along $\alpha(s)$. Let us define the immersion $\Phi: I \times \mathbb{R} \rightarrow \mathbf{E}$ by

$$
\Phi(s, t)=\alpha(s)+t D(s) .
$$

Observe that the position vector of $p \in M(C)$, in $\mathbf{E}$, is normal to $M(C)$ in $\mathbf{E}$. This can be combined with the lightlike nature of the vector field $D(s)$ to see that the above immersion defines a Lorentzian surface in $M(C)$, which will be called a scroll (see [12]). The Gaussian curvature of a scroll is given by $C+\mu^{2}$ and its unit normal vector field in $M(C)$ is the spacelike one given by $\xi(s, t)=\mu t D(s)+F(s)$.

## Appendix B: The Hopf mappings

The classical Hopf mapping, from the three sphere onto the two sphere, is more than 70 years old. It was invented by H.Hopf, [14], to show that the third homotopy group of the two sphere is not trivial. Since then, it has been extensively used and applied in a lot of different problems not only in Mathematics but also in Physics (see [32] for a recent survey on the subject). In particular, the Hopf mapping was used in [5] to extend the Lancret program to the three sphere including both the solving natural equations and the closed curve problems. The Hopf mapping has a nice mate in the Lorentzian setting. It is a map, from the three anti de Sitter space onto the hyperbolic plane, which can be also regarded as a circle principal bundle. However, the special Lorentzian framework and the richness of the three dimensional anti de Sitter geometry both make possible the existence of another Hopf mapping, that we will call Lorentzian Hopf mapping, which applies the anti de Sitter 3 -space onto the pseudo-hyperbolic plane. It can be viewed as a principal bundle with spacelike fiber being the unit circle in a Lorentzian plane. It is clear that can not occur in the Riemannian setting. The purpose of this section is to recall some of the main properties that we have used along this paper.

## B.1: The standard Hopf mappings

Let us consider the maps $\pi_{\varepsilon}: \mathbb{C}^{2} \rightarrow \mathbb{C}^{2}$, defined by

$$
\pi_{\varepsilon}\left(z_{1}, z_{2}\right)=\frac{1}{2 r}\left(\bar{z}_{1} z_{1}-\varepsilon \bar{z}_{2} z_{2}, 2 z_{2} \bar{z}_{1}\right)
$$

where $\varepsilon= \pm 1$ and $r \in \mathbb{R}$.
Endow $\mathbb{C}^{2}$ with the semi-Riemannian metric $\langle z, w\rangle=\operatorname{Real}\left(z_{1} \bar{w}_{1}+\varepsilon z_{2} \bar{w}_{2}\right)$. Then, the hyperquadric $\langle z, z\rangle=\varepsilon r^{2}$ can be naturally identified with either the three sphere $\mathbb{S}^{3}\left(1 / r^{2}\right)$ of curvature $1 / r^{2}$, or the anti de Sitter space $\mathbf{A d S}_{3}\left(-1 / r^{2}\right)$ of curvature $-1 / r^{2}$, according to $\varepsilon$ is +1 or -1 , respectively. Moreover, the image of these hyperquadrics under $\pi_{\varepsilon}$ can be identified with either $\mathbb{S}^{2}\left(4 / r^{2}\right)$, if $\varepsilon=1$, or the hyperbolic plane $\mathbb{H}^{2}\left(-4 / r^{2}\right)$, if $\varepsilon=-1$. Consequently, we can consider a couple of restriction maps

$$
\pi_{+}: \mathbb{S}^{3}\left(\frac{1}{r^{2}}\right) \rightarrow \mathbb{S}^{2}\left(\frac{4}{r^{2}}\right) \quad \text { and } \quad \pi_{-}: \mathbf{A d S}_{3}\left(-\frac{1}{r^{2}}\right) \rightarrow \mathbb{H}^{2}\left(-\frac{4}{r^{2}}\right)
$$

which are known as Hopf mappings. They are semi-Riemannian submersions and the O'Neill formulae relating the corresponding connections $\nabla$, in the base space, and $\bar{\nabla}$, in the total space, are

$$
\begin{aligned}
\bar{\nabla}_{\bar{X}} \bar{Y} & =\overline{\nabla_{X} Y}-\frac{\varepsilon}{r}(\langle i X, Y\rangle \circ \pi) V, \\
\bar{\nabla}_{\bar{X}} V & =\bar{\nabla}_{V} \bar{X}=\frac{1}{r} i \bar{X}, \\
\bar{\nabla}_{V} V & =0,
\end{aligned}
$$

where $V$ is the unit vertical vector field defined by $V(z)=i z$ and overbars, as usually, denote horizontal lifts of corresponding objects in the base space. Notice that the $V$ flow is made up of either geodesics in $\mathbb{S}^{3}\left(1 / r^{2}\right)$, when $\varepsilon=+1$, or timelike geodesics in $\boldsymbol{A d S}_{3}\left(-1 / r^{2}\right)$, when $\varepsilon=-1$.

These maps can be also regarded as principal fibre bundles with structure group $\mathbb{S}^{1}$ in both cases. The action is defined by

$$
e^{i \theta}\left(z_{1}, z_{2}\right)=\left(e^{i \theta} z_{1}, e^{i \theta} z_{2}\right) .
$$

Therefore, the trajectories of $V$ are the fibers and a natural connection can be defined. In fact, just attach to each point $z$ the horizontal plane $\mathbf{H}_{z}=\operatorname{span}\{i z\}^{\perp}$. Let $\omega$ be the gauge potential associated with this principal connection. The strength field $\Omega$ can be computed using the existence of a two-form $\Theta$ on the base space (two-sphere or hyperbolic plane) such that $\Omega=\pi_{ \pm}^{*}(\Theta)$. Now the structure equation gives

$$
\Theta=\frac{2 \varepsilon}{r} d A,
$$

where $d A$ stands for the canonical area form of either $\mathbb{S}^{2}\left(4 / r^{2}\right)$, with $\varepsilon=+1$, or $\mathbb{H}^{2}\left(-4 / r^{2}\right)$, with $\varepsilon=-1$.

## B.2: The Lorentzian Hopf mapping

From many points of view, the role played by the anti de Sitter space in Lorentzian geometry corresponds with that played by the three sphere in Riemannian context. The above recalled Hopf fibrations provide an example. However, the three anti de Sitter geometry allows one to build a new Hopf fibration as follows. In the above construction, we replace $\mathbb{C}$ by

$$
\mathbb{E}=\left\{a+b \xi \mid a, b \in \mathbb{R}, \xi \notin \mathbb{R}, \xi^{2}=1\right\}
$$

We can define the usual conjugation in $\mathbb{E}$ by

$$
m=a+b \xi \mapsto \bar{m}=a-b \xi .
$$

Then consider the map $\lambda: \mathbb{E}^{2} \rightarrow \mathbb{E}^{2}$ given by

$$
\lambda(m, n)=\frac{1}{2 r}(\bar{m} m-\bar{n} n, 2 n \bar{m}) .
$$

On the other hand, equip $\mathbb{E}^{2}$ with the semi-Riemannian metric defined by $\langle p, q\rangle=$ $\operatorname{Real}\left(p_{1} \bar{q}_{1}+p_{2} \bar{q}_{2}\right)$. It is not difficult to see that the hyperquadric $\langle p, p\rangle=-r^{2}$, with the induced metric, can be identified with $\mathbf{A d S}_{3}\left(-1 / r^{2}\right)$. Moreover, its image under the map $\lambda$ is nothing but the pseudo-hyperbolic plane with curvature $-4 / r^{2}$, that is, $\mathbb{H}_{1}^{2}=\mathbf{A d S}_{2}\left(-4 / r^{2}\right)$. The restriction of this map provides a Lorentzian summersion which we denote by

$$
\lambda: \boldsymbol{A d S}_{3}\left(-\frac{1}{r^{2}}\right) \longrightarrow \mathbf{A d S}_{2}\left(-\frac{4}{r^{2}}\right) .
$$

Let $\mathbb{H}^{1}$ be the unit circle in the Lorentzian plane. Then, a natural action of $\mathbb{H}^{1}$ on $\boldsymbol{A d S}_{3}\left(-1 / r^{2}\right)$ yields to see the above map as an $\mathbb{H}^{1}$-principal bundle.

## B.3: Some properties

Let $\phi: \mathbf{M} \rightarrow \mathbf{B}$ be one of the above three Hopf mappings. Then, the complete lift $\mathbf{S}_{\gamma}=\phi^{-1}(\gamma)$ of any Frenet curve $\gamma$ in the base space $\mathbf{B}$ is a flat surface. This is clear because it can be parametrized with coordinate curves being the fibers and the class of horizontal lifts of $\gamma$, respectively. Consequently, the first fundamental form is locally constant and so the Gaussian curvature vanishes identically. Furthermore, these surfaces are embedded in $\mathbf{M}$ if and only if the corresponding curves are free of self-intersections in $B$.

About the causal character of these flat surfaces, it is worth doing the following considerations.

1. $\mathbf{S}_{\gamma}=\pi_{+}^{-1}(\gamma)$ is a Riemannian surface which is called the Hopf tube on $\gamma$. If $\gamma$ is closed, then $\mathbf{S}_{\gamma}=\pi_{+}^{-1}(\gamma)$ is a Riemannian flat torus, which is called the Hopf torus on $\gamma$.
2. $\mathbf{S}_{\gamma}=\pi_{-}^{-1}(\gamma)$ is a Lorentzian surface, which is called the Hopf tube on $\gamma$. If $\gamma$ is closed, then $\mathbf{S}_{\gamma}=\pi_{-}^{-1}(\gamma)$ is a Lorentzian flat torus.
3. $\mathbf{S}_{\gamma}=\lambda^{-1}(\gamma)$ is Riemannian or Lorentzian, according to the causal character of $\gamma$. If $\gamma$ is chosen to be spacelike, then the surface is Riemannian, while it is Lorentzian if the curve is chosen to be timelike. In any case, $\mathbf{S}_{\gamma}=\lambda^{-1}(\gamma)$ is a flat surface which is called the hyperbolic Hopf tube of $\gamma$.

The standard Hopf mappings $\pi_{ \pm}$provide Hopf tori when one lifts an immersed closed curve either in the two sphere by $\pi_{+}^{-1}$ or in the hyperbolic plane by $\pi_{-}^{-1}$. These tori are embedded when the associated curves are free of self-intersections. They are flat, with the induce metric, whose isometry type can be computed in terms of not only the length of the curve $\gamma$, but also the enclosed area by $\gamma$ in either the two sphere or the hyperbolic plane. To be more precise we have

- Let $\gamma$ be an immersed closed curve in $\mathbb{S}^{2}\left(4 / r^{2}\right)$ of length $L$ and enclosing an area $A$. Then its Hopf torus $\mathbf{S}_{\gamma}=\pi_{+}^{-1}(\gamma)$ is isometric to the Riemannian flat torus $\mathbb{R}^{2} / \Gamma$, where $\Gamma$ is the lattice in $\mathbb{R}^{2}$ generated by $(2 \pi r, 0)$ and $(2 A, L)$.
- Let $\gamma$ be an immersed closed curve in $\mathbb{H}^{2}\left(-4 / r^{2}\right)$ of length $L$ and enclosing an area $A$. Then its Hopf torus $\mathbf{S}_{\gamma}=\pi_{-}^{-1}(\gamma)$ is isometric to the Lorentzian flat torus $\mathbb{L}^{2} / \Gamma$, where $\Gamma$ is the lattice in $\mathbb{L}^{2}$ generated by $(2 \pi r, 0)$ and $(2 A, L)$.

We will sketch the proof in the Lorentzian case and will assume, without loss of generality, that $r=1$ (see [6]). First, we choose any horizontal lift $\bar{\gamma}$ of $\gamma$. The map $\Psi: \mathbb{L}^{2} \rightarrow \mathbf{S}_{\gamma}$ defined by

$$
\Psi(t, s)=e^{i t} \bar{\gamma}(s),
$$

is a Lorentzian covering. The lines parallel to the $t$-axis in $\mathbb{L}^{2}$ are mapped by $\Psi$ onto the fibres of $\pi_{-}$, while those parallel to the $s$-axis are mapped into the horizontal lifts of $\gamma$. The former are closed, but the latter are not, because the non-trivial holonomy of the involved gauge potential. However, there exists a smooth function, say $\delta(s)$, satisfying

$$
\bar{\gamma}(s)=e^{i \delta(s)} \bar{\gamma}(s+L) .
$$

Take derivative in the last equation and use the horizontality of $\bar{\gamma}^{\prime}$ and $i \bar{\gamma}^{\prime}$ to conclude that $\delta(s)$ is actually a constant, the holonomy number of $\omega$. Furthermore, it can be computed from the strength field $\Omega=\pi_{-}^{*}(\Theta)$, by $\delta=-\int_{c} \Theta$, where $c$ is any 2-chain in $\mathbb{H}^{2}(-4)$ with $\partial c=\gamma$ (see [13]). As $\Theta=-2 d A$, we find that $\delta=2 A$ and the proof finishes.

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