Addendum to "New classical string solutions in AdS$_3$ trough null scrolls

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Abstract

In [2] we showed that null scrolls with zero mean curvature provide a nice class of classical string solutions in $\mathbf{AdS}_3$. In this note we prove that the whole class of null scrolls give a huge family of solutions to the extrinsic Polyakov string theory, in that background, which contains the classical theory as a natural submodel . Then we use this result to exhibit the solitonic nature of the field equation associated to the Polyakov action.

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If one wishes to evolve curves in a target spacetime to generate surfaces being extremals of a certain action, it seems natural to involve the extrinsic geometry of surfaces in the density of the action. This idea was materialized by A. M. Polyakov [3] by introducing the so called extrinsic Polyakov action. In the anti de Sitter world $\mathbf{AdS}_3$, which we take of curvature -1, that action is given by

$$\mathcal{P}(S) = \int_S \left(H^2 - 1\right)\, dA - \int_{\partial S} \kappa\, ds, \tag{1}$$

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where it runs through the space of timelike surfaces S with the same boundary ∂S and tangent along the common boundary. Here H means the mean curvature of S and κ the curvature of ∂S in S.

The field equation providing the solutions of this string theory, extrinsic Polyakov string solutions, was computed in [1] as

$$\Delta H + 2H(H^2 - K - 1) = 0, \tag{2}$$

where K and Δ denote, respectively, the Gaussian curvature and the Laplacian of S endowed with the induced metric. Obviously, stationary surfaces are solutions of (2). Therefore, every classical string solution is an extrinsic Polyakov string solution.

A null scroll $S(\gamma, B)$ in $\mathbf{AdS}_3 \subset \mathbb{R}^{2,2}$ (see [2]) is a ruled surface naturally parameterized by

$$X(s,t) = \gamma(s) + t\, B(s), \tag{3}$$

where the directrix $\gamma(s)$ is a null curve in $\mathbf{AdS}_3$, and the ruling flow $B(s)$ is a null vector field tangent to $\mathbf{AdS}_3$ at $\gamma(S)$. The Laplacian of a null scroll associated to the induced metric is given by

$$\Delta = -2\frac{\partial^2}{\partial s \partial t} - 2\left(\langle \gamma', B'\rangle + t\,\langle B', B'\rangle\right)\frac{\partial}{\partial t} - \left(2t\langle \gamma', B'\rangle + t^2\,\langle B', B'\rangle\right)\frac{\partial^2}{\partial t^2}. \tag{4}$$

The mean curvature H of $S(\gamma, B)$ is given by

$$H(s,t) = \det[\gamma', B, B'](s),$$

while the Gaussian curvature is computed to be

$$K(s,t) = -1 + H^2(s,t). \tag{5}$$

These two functions are invariant along the ruling flow, so we use (4) to get $\Delta H(s,t) = \Delta K(s,t) = 0$, so that they are harmonic functions. We combine this information with (2) and (5) to conclude the following statement:

Every null scroll in $\mathbf{AdS}_3$ *provides a solution of* (2) *and so an extrinsic Polyakov string configuration.*

To compute the charges that they could carry, we check the critical values of (1) on a non-null polygon, say Ω. If $\partial\Omega$ is made up of n smooth pieces and θ_j, $1 \le j \le n$, denote the exterior hyperbolic angles, then we use the Gauss-Bonnet formula (see [1]) to get

$$\mathcal{P}(\Omega) = \int_\Omega (H^2 - 1)\, dA - \int_{\partial\Omega} \kappa\, ds = \int_\Omega K\, dA - \int_{\partial\Omega} \kappa\, ds = \sum_{j=1}^{n} \theta_j.$$

Then we have proved that *the charges are encoded in the boundaries, namely in the corners along the boundaries. This shows a holographic principle for the* $\mathcal{P}$ *critical values, on scroll Polyakov string solutions*

Now let $\mathcal{F}_1$ be the class of null scrolls in $\mathbf{AdS}_3$. To any $S \in \mathcal{F}_1$ we can associate a wave function ϕ_S which appears as a solution of the generalized Liouville equation (see [2], Theorem 6.1). Therefore, if $\mathcal{F}_2$ denotes the class of solutions of the generalized Liouville equation, we can define a map

$$\Phi : \mathcal{F}_1 \to \mathcal{F}_2, \quad \Phi(S) = \phi_S,$$

which is surjective (inverse scattering). Moreover, $\Phi^{-1}(\phi)$ is made up of a $C^\infty(\mathbb{R})$-family of isometric solutions of (2) which are not congruent in $\mathbf{AdS}_3$. This result shows the solitonic nature of equation (2), the field equation of the extrinsic Polyakov string theory in $\mathbf{AdS}_3$.

Other consequences could be obtained by using this approach that allows us to see the generalized Liouville equation as a submodel of the extrinsic Polyakov string theory in $\mathbf{AdS}_3$. For instance, *every null scroll in* $\mathbf{AdS}_3$ *can be seen as a soliton of* (2) *wich carries charges that can be holographically computed in the conformal boundary of* $\mathbf{AdS}_3$.

It should be noted that the map Φ provides a one-to-one correspondence between the submodel made up of classical string solutions and the space of solutions of the Liouville equation (compare with [2], Corollary 6.2).

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