

ACADEMIE DE LA RÉPUBLIQUE SOCIALISTE DE ROUMANIE

**REVUE ROUMAINE
DE MATHÉMATIQUES
PURÉS ET APPLIQUÉES**

TOME XXIX, No. 3

1984

TIRAGE À PART

EDITURA ACADEMIEI REPUBLICII SOCIALISTE ROMÂNIA

NEARLY-KAELER CURVATURE OPERATORS

BY

ANGEL FERRÁNDEZ

Let V be a hermitian complex vector space of real dimension $2n$ and let $NK(n)$ be the space of curvature operators on V verifying the second condition of curvature (3). It is shown that the representation of $U(n)$ on $NK(n)$ has four components and Nearly Kaehler curvature operators are determined by means of complex endomorphisms of V .

0. Introduction. In [9], I. M. Singer and J. A. Thorpe consider abstract curvature operators on a real metric vector space, V , of real dimension n , as symmetric endomorphisms, $R : \Lambda^2(V) \rightarrow \Lambda^2(V)$ and they give a natural decomposition of the space $N(n)$ of such operators. In particular, they obtain a canonical form for the curvature tensor of a 4-dimensional Einstein space. Using these results, D. L. Johnson [4] considers a hermitian complex vector space (V, g, J) of real dimension $2n$ and the subspace of $N(2n)$ of the Kaehler curvature operators, or K -curvature operators, that is, those satisfying $K(n) = \{R \in N(2n) | R \circ J = J \circ R\}$, getting a decomposition of $K(n)$ similar to the Singer-Thorpe's one. If V is the tangent space of a Kaehlerian manifold, $K(n)$ represents the space of the curvature operators satisfying the first condition of curvature, which are well known in the literature, for example Sitaramayya [10], Mori [6], Vanhecke [11] etc. Therefore, the abstract treatment of the curvature operators takes geometric meaning when, at each point, V is the tangent space of a Riemannian manifold. There exists a natural generalization when, at each point, V is the tangent space of a Nearly-Kaehler manifold.

The fundamental objective of this paper is to give a similar decomposition to the Johnson's one for a more general class of curvature operators; those satisfying the second condition of curvature. Using this decomposition, it is given also a theorem to obtain NK -curvature operators by means of complex symmetric endomorphisms of V . This is an interesting generalization, because there exists a remarkable almost complex manifold, S^6 , which is a Nearly-Kaehler manifold but a non-Kaehler; about it there is the following open problem: "prove that S^6 has no complex structure".

1. Algebraic preliminaries. Let $(M, \langle \cdot, \cdot \rangle)$ be an n -dimensional Riemannian manifold and let M_m denote the tangent space of M at any point m of M . For each $x, y \in M_m$ the Riemann curvature transformation at m is a linear operator $R_{xy} : M_m \rightarrow M_m$ such that 1) $R_{xy} = -R_{yx}$; 2) $\langle R_{xy}z, w \rangle = \langle R_{xz}x, y \rangle$; 3) $\langle R_{xy}z, w \rangle + \langle R_{yz}x, w \rangle + \langle R_{zx}y, w \rangle = 0$.

Note that any antisymmetric operator A on a vector space $V = M_m$ can be identified with an element of $\Lambda^2(V)$, via the isomorphism $\Lambda^2(V) \rightarrow \text{Antisim}(V)$ defined by $v \wedge w \mapsto A_{vw}$, where $A_{vw}x = (v \wedge w)x = \langle v, x \rangle w - \langle w, x \rangle v$; so, each element $v \wedge w \in \Lambda^2(V)$, can be considered either bivector or operator. Therefore, R can be defined as a linear application $R : V \otimes V \rightarrow \Lambda^2(V)$.

There are other identifications also used in the literature, usually differing only in sign, for example, Kobayashi-Namizu [5], Mori [6] etc. We choose the sign so that $R_{xyzw} = \langle R_{xy}z, w \rangle = \langle R(x \wedge y), z \wedge w \rangle$.

With respect to the induced metric on $\Lambda^2(V)$, $\langle A_{vw} x, y \rangle = \langle v \wedge w, x \wedge y \rangle$, condition (2) of the definition of R is equivalent to saying that R is an element of the symmetric tensor product $\Lambda^2(V) \circ \Lambda^2(V)$. This situation motivates the following

DEFINITION. An *algebraic curvature tensor* on a real metric vector space $(V, <, >)$ of dimension n is a symmetric linear operator on $\Lambda^2(V)$, that is, an element of the space $\Lambda^2(V) \circ \Lambda^2(V)$.

We shall denote by $N(n)$ the set of all algebraic curvature tensors on V , that also have a structure of real metric vector space with $\langle R, S \rangle = \text{trace}(R \cdot S)$. Moreover, $R \in N(n)$ is said Riemann if R satisfies the first Bianchi identity.

The Grassmann manifold $G(2, V)$ of the oriented vector subspaces of V can be identified with the space of the unitary decomposable bivectors $v \wedge w \in \Lambda^2(V)$; if P is a plane of V and $\{v, w\}$ an oriented orthonormal base of P , then P is identified with $v \wedge w \in \Lambda^2(V)$. The curvature function associated to R , $r_R : G(2, V) \rightarrow \mathbf{R}$ by $r_R(P) = \langle R(P), P \rangle$ is defined.

If $(V, <, >, J)$ is a hermitian complex vector space of real dimension $2n$, an element of $N(2n)$, also represented by J , is defined by $J(v \wedge w) = Jv \wedge Jw$. Since V is isomorphic to \mathbf{C}^n , by abuse of notation and when it were convenient, we shall write $\Lambda^2(\mathbf{C}^n)$ instead of $\Lambda^2(V)$. Also $G(2, V)$ will be denoted by $G(2, \mathbf{C}^n)$ or $G(2, 2n)$. An oriented plane $P \in G(2, 2n)$ is said holomorphic if $JP = P$. If $v \in P$ is a unitary vector, then $v \wedge Jv = \pm P$ is a representation of P as an element of $\Lambda^2(V)$. Moreover $\{P \in G(2, 2n) / JP = P\} \simeq \mathbf{CP}^{n-1} \times \mathbf{Z}_2 \equiv G(2, 2n)^J$ that is, by $G(2, 2n)^J$ we shall denote the holomorphic planes of $G(2, 2n)$. The holomorphic sectional curvature of R is the function $r_{R|G(2, 2n)^J}$.

We shall frequently identify $\Lambda^2(V)$ and $o(2n)$. Also $u(n) \subseteq o(2n)$ is the subspace of all elements $M \in o(2n) \equiv \Lambda^2(V)$, matrices or planes, such that $JM = M$, that is, $u(n)$ are the complex matrices of $o(2n)$. If $J : \mathbf{C}^n \rightarrow \mathbf{C}^n$, we put $J \in u(n)$ such that $JI = I$. Then, if $\{v_i, v_{i*}\}$, $i = 1, \dots, n$ is a unitary basis of \mathbf{C}^n , we have $I = \sum_{i=1}^n v_i \wedge v_{i*}$.

If $P \in G(2, 2n)$, choosing an orthonormal basis $\{v_\alpha\}$ of \mathbf{C}^n , $P = v \wedge w = (\sum_\alpha a_\alpha v_\alpha) \wedge (\sum_\beta b_\beta v_\beta)$. If v, w are orthonormal, $\langle P, JP \rangle = (-\langle P, I \rangle)^2$. Then P is holomorphic if and only if $\langle P, I \rangle = \pm 1$.

2. Nearly-Kaehler curvature operators.

DEFINITION. $R \in N(2n)$ will be called *Nearly-Kaehler curvature operator* or *NK-curvature operator* if it satisfies the second condition of curvature, that is, $R_{xyzw} = R_{JxJyzw} + R_{JxyJzw} + R_{xJyJzw}$. It will be denoted by $NK(n)$ the set of such curvature operators, which what the restriction of the inner product is a metric vectorial subspace of $N(2n)$.

Let λ^R denote the $(0,4)$ -tensor $R - R \circ J$, verifying the following properties (see [8])

- (a) $\lambda_{xyzw}^R = \lambda_{xzyw}^R = \lambda_{yxzw}^R = \lambda_{yxzw}^R$; (b) $\lambda_{xyzw}^R = -\lambda_{xyJzJw}^R = \lambda_{JxJyJzJw}^R$;
- (c) $\sigma \lambda_{xyzw}^R = \sigma \lambda_{xJyJzJw}^R$; (d) $\lambda_{xyzw}^R = \lambda_{xJyJzJw}^R$.

REMARKS. 1) If $R \in \mathbf{NK}(n)$, $R_{xyzw} = R_{JxJyJzJw}$ [3]. 2) If $\lambda^R = 0$, $\mathbf{NK}(n) = \mathbf{K}(n)$, that is, $\mathbf{NK}(n)$ is exactly the set of the Kähler curvature operators, [4].

3. Decomposition of the space $\mathbf{NK}(n)$. We define the following maps

- (i) $b : \mathbf{NK}(n) \rightarrow N(2n)$, by $b(R)_{xy} = \sigma_{xy} R_{xy}$;
- (ii) $a : \mathbf{NK}(n) \rightarrow V \cdot V$, given by $\langle a(R)v, w \rangle = \frac{1}{2} \left\{ \text{Trace } (v \rightarrow R_{vw} w) - \frac{1}{2} \text{Trace } (R_{vw} \circ J) + \frac{1}{2} \text{Trace } (u \rightarrow \lambda_{vw}^R w) \right\}$;
- (iii) $\text{tr} : \mathbf{NK}(n) \rightarrow R$, by $\text{tr}(R) = \frac{1}{2} \text{Trace } a(R)$.

REMARK. If $R \in \mathbf{NK}(n)$ and $b(R) = 0$, then $\text{tr}(R) = \sum_{i,j=1}^{2n} (R_{ii,jj} + \frac{1}{2} \lambda_{ijij}^R)$.

$U(n)$ acts on $\mathbf{NK}(n)$ by $A(R) = A^{-1}RA$, for $A \in U(n)$. It is known, [9], that b is equivariant under the natural action of $U(n)$. A direct computation shows that $a(A^{-1}RA) = A^{-1}a(R)A$, for all A in $U(n)$. Analogously for tr . So, the maps b , a and tr are equivariant under the action of $U(n)$ and the subspaces $\text{Ker } b$, $\text{Ker } a$, $\text{Ker } \text{tr}$, so as its complements and intersections, are invariant too. Then we define the following subspaces of $\mathbf{NK}(n)$

$$\begin{aligned} & A = (\text{Ker } b)^\perp; \quad C = (\text{Ker } a) \cap (\text{Ker } b); \\ & B = (\text{Ker } \text{tr})^\perp; \quad D = (\text{Ker } a)^\perp \cap (\text{Ker } \text{tr}). \end{aligned}$$

DEFINITION. For $x, y, z, w \in V$ and $R \in \mathbf{NK}(n)$ it is defined

$$\langle \Omega(y \wedge w^*), x \wedge z^* \rangle = -\frac{1}{2} \lambda_{xyzw}^R.$$

THEOREM. The space $\mathbf{NK}(n)$ is decomposed in an orthogonal direct sum $\mathbf{NK}(n) = A \oplus B \oplus C \oplus D$. Moreover

- 1). If $R \in A$, then $r_{R|G(2,2n)J} = 0$.
- 1'). If $R \in \mathbf{NK}(n)$ and $r_{R|G(2,2n)J} = 0$, then $R \in A \oplus C$.
- 2). $R \in B \oplus C \oplus D$ if and only if R is Riemann.
- 3). If $R \in A \oplus B$, then $r_{R|G(2,2n)J}$ is a constant.
- 3'). If $r_{R|G(2,2n)J}$ is a constant, then $R \in A \oplus B \oplus C$.
- 4). $R \in C$ if and only if $b(R) = 0$ and $a(R) = 0$; if and only if $b(R) = 0$ and $R(J) = \Omega(I)$.

5). $R \in B \oplus C$ if and only if $b(R) = 0$ and $a(R) = \xi(\text{id})$; if and only if $b(R) = 0$ and $R(I) = \Omega(I) + \xi(I)$.

6). $R \in C \oplus D$ if and only if $b(R) = 0$ and $\langle R(I), I \rangle = \langle \Omega(I), I \rangle$. The proof is a direct consequence of [4], [7] and of the following LEMMA. $A \subseteq \text{Ker } a \subseteq \text{Ker } \text{tr}$.

PROOF. If $R \in A$, take the family of holomorphic planes

$$P_t = (\cos t e_i + \sin t e_j) \wedge (\cos t e_{i^*} + \sin t e_{j^*}).$$

$$\text{From } r_R(P_t) = 0, \text{ we obtain } R_{ii^*jj^*} = -2R_{ij^*ij^*} + \lambda_{ijij}^R \text{ and } R_{ii^*jj^*} = -2R_{ijij} + \lambda_{ijij}^R.$$

If v_1 is an eigenvector of $a(R)$, $a(R)v_1 = \xi v_1$, by the above expressions, $\xi = 0$. So, $A \subseteq \text{Ker } a$.

4. An inversion theorem in $\text{NK}(n)$. Let S be a real symmetric linear operator on \mathbb{C}^n such that $SJ = JS$. Then, JS is antisymmetric and one can consider itself as an element of $\Lambda^2(\mathbb{C}^n)$. Next we shall work with the set $\mathbf{T} = \{S \in \mathbb{C}^n : \mathbb{C}^n / JS = SJ\}$.

DEFINITION. For each $S \in \mathbf{T}$ we define

$$(i) \alpha^S(x, y, z, w) = -\langle Sx \wedge y + x \wedge Sy - SJx \wedge Jy - Jx \wedge SJy, z \wedge w \rangle$$

$$(ii) \langle L^S x, y \rangle = \frac{1}{2} \sum_{j=1}^n \alpha^S(x, e_j, y, e_j).$$

where $x, y, z, w \in \mathbb{C}^n$ and $\{e_j\}$ is a unitary basis of \mathbb{C}^n .

LEMMA. 1). $\alpha^S(x, y, z, w) = -\alpha^S(y, x, z, w) = -\alpha^S(x, y, w, z) = -\alpha^S(z, w, x, y);$

$$2). \alpha^S(x, y, z, w) = -\alpha^S(x, y, Jz, Jw) = -\alpha^S(Jx, Jy, z, w) = -\alpha^S(Jx, Jy, Jz, Jw);$$

$$3). \alpha^S(x, y, z, w) = \alpha^S(x, Jy, z, Jw) = \alpha^S(Jx, y, Jz, w);$$

$$4). L^S \in \mathbf{T}.$$

The proof of this lemma is a straightforward verification.

DEFINITION [6], [11]. For $M, N \in \mathbf{T}$ and $v, w \in \mathbb{C}^n$ we define

$$\begin{aligned} \mathbf{L}_{MN}(v, w) = & Mv \wedge Nw + Nv \wedge Mw + JMv \wedge JNw + JNv \wedge JMw - \\ & - 2\langle Mv, Jw \rangle JN + 2\langle Jv, Nw \rangle JM. \end{aligned}$$

Note that the change of sign in the last two terms with respect to the above mentioned authors is coming from the definition adopted in the present note for the endomorphism $u \wedge v$.

THEOREM. Let $\sigma : \mathbf{T} \rightarrow \text{NK}(n)$ be the map given by

$$\begin{aligned} \sigma(T)(v, w) = & \frac{1}{2(n+2)} \mathbf{L}_{\tilde{T}T}(v, w) - \frac{(3n+5)t}{16(n+2)(n+1)} \mathbf{L}_H(v, w) + \\ & + \frac{(n+1)t}{8n(n-1)} (3v \wedge w - Jv \wedge Jw + 2\langle v, Jw \rangle J), \end{aligned}$$

where $\tilde{T} = T - 3LT$ and $t = \text{trace } (T)$. Then $\sigma(T) \in B \oplus D$ and

$$\langle a(\sigma(T - LT))v, w \rangle = \langle (T - LT)v, w \rangle$$

The proof is a laborious computation from the definitions and the above results.

REMARKS. 1). This theorem is a natural generalization of the corresponding theorem given by Johnson [4], since for a K -curvature operator we have $\lambda^{\sigma(T)} = 0$ and $\alpha^T = 0 = L^T$ for each $T \in \mathbf{T}$.

2) The expression of $\sigma(T)$ provides directly Nearly-Kaehler curvature operators by means of symmetric endomorphisms of \mathbb{C}^n , that is, it should be enough to choose the appropriate $T \in \mathbf{T}$ by obtaining $\sigma(T) \in \mathbf{NK}(n)$.

Received January 5, 1981

Departamento de Geometría y Topología
Facultad de Ciencias Matemáticas Bujasot,
Valencia, Espagne

REFERENCES

1. A. Ferrández, *Formas normales de los NK-operadores curvatura*. Tesis Doctoral. Publ. Depart. Geom. Topol. Univ. de Valencia 4, 1980.
2. A. Gray, *Nearly Kaehler manifolds*. J. Differential Geom. 4 (1970), 283—309.
3. A. Gray, *Curvature identities for Hermitian and almost Hermitian manifolds*. Tohoku Math. J. 28 (1976), 601—612.
4. D. L. Johnson, *Sectional curvature and curvature normal forms*. (To be published in Michigan Math. J.).
5. S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, I and II. Interscience, New York, 1963, 1969.
6. H. Mori, *On the decomposition of generalized K-curvature tensor fields*. Tohoku Math. J. 25 (1973), 225—235.
7. A. M. Naveira, and L. M. Hervella, *Schur's theorem for Nearly-Kaehler manifolds*. Proc. Amer. Math. Soc. 49 (1975), 421—425.
8. A. M. Naveira and L. Vanhecke, *Two problems for almost Hermitian manifolds*, Demonstratio Math. 10 (1977), 189—203.
9. I. M. Singer and J. A. Thorpe, *The curvature of 4-dimensional Einstein spaces*, In : Global Analysis (Papers in Honor of K. Kodaira), p. 355—365. Univ. of Tokyo Press, Tokyo, 1969.
10. M. Sitaramayya, *Curvature tensors in Kaehler manifolds*. Trans. Amer. Math. Soc. 183 (1973), 341—353.
11. L. Vanhecke, *On the decomposition of curvature vector fields on almost Hermitian manifolds*. In : Conf. on Diff. Geom. Proc., p. 17—32. Michigan State Univ., 1976.