Null scrolls, a wonderful source of examples

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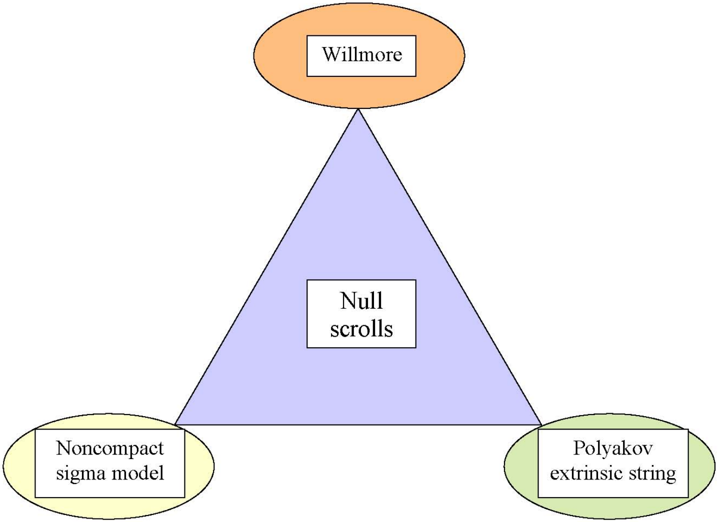
Resumen

Willmore, sigma model and Polyakov extrinsic string are three closely related energy actions, the latter two having important applications in Physics. We exhibit solutions of all of them, in a Lorentzian atmosphere, by means of null scrolls, a sort of timelike ruled surfaces with no counterpart in a Riemannian background.

1. Aims

Imagine three energy actions (Willmore, noncompact sigma model and Polyakov extrinsic string) located at the vertices of a triangle. Then we wish to explain how and why *null scrolls* are placed at the center of the triangle:



2. Null scrolls

$$\gamma : I \subset \mathbb{R} \to \mathbb{L}^3, \qquad \text{regular curve}$$

$$B : I \subset \mathbb{R} \to \mathbb{L}^3, \qquad \text{a vector field along } \gamma$$

$$X(s,t) = \gamma(s) + tB(s), \qquad \text{ruled surface}$$

$$X_s := \frac{\partial X}{\partial s} = \gamma' + tB', \qquad X_t := \frac{\partial X}{\partial t} = B$$

$$(g_{ij}) = \begin{pmatrix} \langle \gamma', \gamma' \rangle + 2t\langle \gamma', B' \rangle + t^2 \langle B', B' \rangle & \langle \gamma', B \rangle + t\langle B', B \rangle \\ \langle \gamma', B \rangle + t\langle B', B \rangle & \langle B, B \rangle \end{pmatrix}$$

According to the causal character of γ' and B, there are four possibilities:

Case	γ'	B	*Condition*
(1)	non-null	non-null	lin. indep.
(2)*	null	non-null	$\langle \gamma', B \rangle \neq 0$
(3)*	non-null	null	$\langle \gamma', B \rangle \neq 0$
(4)	null	null	$\langle \gamma', B \rangle \neq 0$

Definition. A null scroll $S(\gamma, B)$ will be a surface parametrized by

$$X(s,t) = \gamma(s) + t\,B(s)$$

such that

$$\langle \gamma'(s), \gamma'(s) \rangle = \langle B(s), B(s) \rangle = 0$$

and

$$\langle \gamma'(s), B(s) \rangle = -1 \quad (\textit{normalization condition}).$$

$$(g_{ij}) = \begin{pmatrix} 2t\langle \gamma', B' \rangle + t^2 \langle B', B' \rangle & -1 \\ -1 & 0 \end{pmatrix}.$$

Definition.

$$C(s) = \gamma'(s) \times B(s)$$

Gauss map.

$$N(s,t) = C(s) + t\,B'(s) \times B(s).$$

$$B'(s) \times B(s) = -f(s)\,B(s),$$

where $f(s) = \langle \gamma'(s), B'(s) \times B(s) \rangle = \det[\gamma'(s), B'(s), B(s)]$,

$$N(s,t) = -t\, f(s)\, B(s) + C(s).$$

$-f(s)$ is called the *parameter of distribution* of the ruled surface $\mathbf{S}(\gamma, B)$.

Shape operator.

$$dN \equiv \begin{pmatrix} f & t\, f' + \langle \gamma', \gamma'' \times B \rangle \\ 0 & f \end{pmatrix}.$$

Mean and Gauss curvature functions.

$$\boxed{H(s,t) = f(s)} \qquad \text{and} \qquad \boxed{K(s,t) = f(s)^2}$$

Laplacian.

$$\Delta = -2\frac{\partial^2}{\partial s \partial t} - 2\left(\langle \gamma', B' \rangle + t\, \langle B', B' \rangle\right) \frac{\partial}{\partial t} - \left(2t\langle \gamma', B' \rangle + t^2\, \langle B', B' \rangle\right) \frac{\partial^2}{\partial t^2}.$$

$$\boxed{\Delta H(s,t) = 0} \qquad \text{and} \qquad \boxed{H^2(s,t) - K(s,t) = 0}$$

Remark 2.1 *When $f(s) = const$, $X(\gamma, B)$ is called a B-scroll,* [L. K. Graves, *Codimension one isometric immersions between Lorentz spaces,* Trans. A.M.S. 252 (1979), 367-392.]

3. Constructing null scrolls

The metric:

$$dx^2 + dy^2 - dz^2$$

The light cone:

$$\mathbf{C} := \{x^2 + y^2 = z^2\}$$

Take a vector $\vec{u} \in \mathbf{C}$

$$\vec{u} = (x, y, z) = z\,(\cos\alpha, \sin\alpha, 1).$$

Then (z, α) parametrize $\mathbf{C}$.

$$\gamma'(s) = c(s)\,(\cos\omega(s), \sin\omega(s), 1), \qquad B(s) = r(s)\,(\cos\varphi(s), \sin\varphi(s), 1),$$

normalization condition:

$$\cos(\omega - \varphi) = 1 - \frac{1}{c(s)\,r(s)}.$$

The mean curvature of a null scroll:

$$\boxed{H(s,t) = f(s) = -\det[\gamma'(s), B(s), B'(s)] = -r(s)\,\varphi'(s)}$$

Remark 3.1 *The mean curvature of a null scroll only depends on the lightlike ruling flow.*

Algorithm for stationary ($H(s,t)=0$) null scrolls.

$H(s,t) = -r(s)\,\varphi'(s) = 0$ if and only if $\varphi'(s) = 0$, i. e., the lightlike ruling flow consists of parallel rulings.

In other words, the angular function associated with this flow is constant $\varphi(s) = \varphi_o \in \mathbb{R}$. Then

1. We first need a lightlike base curve, $\gamma(s)$, defined on a certain interval $I \subset \mathbb{R}$. To get it, we build its tangent vector field in the light cone and recuperate the curve, up to congruences, by quadratures. However

$$\gamma'(s) = c(s)\,(\cos\omega(s), \sin\omega(s), 1), \qquad s \in I,$$

therefore, to find the base curve we need two functions $c, \omega : I \subset \mathbb{R} \to \mathbb{R}$, defined in a suitable interval, one of them positive, $c(s) > 0$ anywhere.

2. To construct the ruling flow we need, a priori, a real number, say $\varphi_o \in \mathbb{R}$, and a function $r : I \subset \mathbb{R} \to \mathbb{R}$, $r(s) > 0$ anywhere. Then write

$$B(s) = r(s)\,(\cos\varphi_o, \sin\varphi_o, 1).$$

However, these data should obey the normalization condition, which allows one to determine the function $r(s)$ in terms of the remaining data by

$$r(s) = \frac{\csc^2\left(\frac{\omega(s)-\varphi_o}{2}\right)}{2\,c(s)}.$$

3. Therefore, to determine a stationary null scroll we need three parameters:

   (i) a positive function $c \in \mathcal{F}^+(I)$,

   (ii) a function $\omega \in \mathcal{F}(I)$, and

   (iii) a real number $\varphi_o \in \mathbb{R}$.

   With these data, the stationary null scroll $\mathbf{S}(c, \omega, \varphi_o)$ will be parametrized by $X(\gamma_{c\omega}, B_{\varphi_o})$, where

$$\gamma_{c\omega}(s) = \int_0^s c(t)\,(\cos\omega(t), \sin\omega(t), 1)\,dt$$

   and

$$B_{\varphi_o}(s) = \frac{\csc^2\left(\frac{\omega(s)-\varphi_o}{2}\right)}{2\,c(s)}\,(\cos\varphi_o, \sin\varphi_o, 1).$$

4. The nonlinear sigma model with boundary

Definition. Let Σ be an oriented riemannian manifold and let X be an oriented pseudo-riemannian manifold. Let us first assume that Σ has no boundary. The sigma model is a theory of maps $\varphi : \Sigma \to (X, g)$ governed by the following action

$$S[\varphi] := \frac{1}{2} \int_{\Sigma} ||d\varphi||^2 \mathrm{dvol}_{\Sigma}.$$

Σ is called the base space and its dimension is the dimension of the sigma model.

X is called the target space and its isometry group is the symmetry group of the sigma model, which gives its name to the model.

The solutions of the Euler-Lagrange equation associated to the above action are called the solutions of the sigma model.

For instance

(1) Σ and X are Riemannian, the solutions are harmonic maps.

(2) $O(3)$ 2-dimensional sigma model means that Σ is a surface and $X = \mathbb{S}^2$.

(3) $O(2,1)$ 2-dimensional sigma model means that Σ is a surface and $\mathrm{Iso}(X) = O(2,1)$, which has a twofold version:

(3.1) $X = \mathbb{S}^2_1$, (3.2) $X = \mathbb{H}^2$.

We consider the 2-dimensional $\mathbf{O}(2,1)$ nonlinear sigma model with boundary, as a natural continuation of the papers

M. Barros, *A geometric algorithm to construct new solitons in the* $O(3)$ *nonlinear sigma model*, Phys. Lett. B **553** (2003), 325-331,

M. Barros, M. Caballero and M. Ortega, *Rotational Surfaces in* $\mathbb{L}^3$ *and Solutions of the Nonlinear Sigma Model*, Comm. Math. Phys. **290** (2009), 437-477.

S a surface with boundary ∂S

$\phi : S \to \mathbb{L}^3$ timelike immersion

$N_\phi : \phi(S) \to \mathbb{S}^2_1$ its Gauss map

We will consider immersions fixing ∂S

Start with a set Γ of nonnull regular curves in $\mathbb{L}^3$ and a spacelike unit normal vector field N_o along Γ.

Then we consider the space $I_\Gamma(\mathbf{S}, \mathbb{L}^3)$ of timelike immersions satisfying the following first order boundary conditions

$$\phi(\partial S) = \Gamma, \qquad N_\phi/_\Gamma = N_o.$$

Roughly speaking, if we identify each immersion $\phi \in I_\Gamma(S, \mathbb{L}^3)$ with its graph, $\phi(S)$, viewed as a surface with boundary in $\mathbb{L}^3$, then $I_\Gamma(S, \mathbb{L}^3)$ can be viewed as the space of timelike surfaces in $\mathbb{L}^3$ having the same boundary and being tangent along the common boundary.

The energy governing the model $\mathcal{D} : I_\Gamma(S, \mathbb{L}^3) \to \mathbb{R}$ is now written as

$$\boxed{\mathcal{D}(\phi) = \int_S \|dN_\phi\|^2 dA_\phi}$$

where dA_ϕ stands for the element of area of $(S, \phi^*(\bar{g}))$

The solutions of these models are those satisfying the following vectorial field equation

$$\boxed{\Delta N_\phi - (N_\phi . \Delta N_\phi)\, N_\phi = 0} \tag{1}$$

which are the critical points of $(I_\Gamma(S, \mathbb{L}^3); \mathcal{D})$.

This way to see the model has several advantages, perhaps the more interesting is that it allows us to describe the solutions in terms of their geometrical invariants.

It will be convenient to introduce, on the same space of elementary fields, the Willmore problem with boundary (see [1, 2, 7, 13] and references therein), which is associated with the Willmore energy $\mathcal{W} : I_\Gamma(S, \mathbb{L}^3) \to \mathbb{R}$ defined by

$$\boxed{\mathcal{W}(\phi) = \int_S H_\phi^2\, dA_\phi - \int_{\partial S} \kappa_\phi}$$

The critical points, i. e., the solutions of this model are known as Willmore surfaces in $\mathbb{L}^3$.

This variational problem is invariant under conformal changes in $\mathbb{L}^3$, so it is actually stated in the conformal class, $[\bar{g}]$, of the Lorentz-Minkowski metric $\bar{g}$.

It will be convenient to remark that, in both cases, a critical point means a critical point for the corresponding induced problems on reasonable compact pieces or nonnull polygons. More precisely, a connected, simply connected, with nonempty interior, compact domain, $\Omega \subset S$, is said to be a *nonnull polygon* if it has a piecewise smooth boundary, $\partial\Omega$, which consists of a finite number of nonnull curves.

Both theories are equivalent. *The solutions of the 2-dimensional $O(2,1)$ nonlinear sigma model are just the Willmore surfaces. In particular, the nonlinear sigma model is invariant under conformal changes in $\mathbb{L}^3$.*

Proof.

$$\boxed{\|dN_\phi\|^2 = 4\,H_\phi^2 - 2\,K_\phi, \qquad \forall \phi \in I_\Gamma(S, \mathbb{L}^3)}$$

Apply the Gauss-Bonnet formula for general nonnull polygons (see [2] for details)

$$\boxed{-\int_P K_\psi\, dA_\psi + \int_{\partial P} \kappa(s)\, ds + \sum_{j=1}^{r} \theta_j = 0}$$

Then

$$\boxed{\mathcal{D}(P) = 4\mathcal{W}(P) + 2\int_{\partial P} \kappa(s)\, ds - 2\sum_{j=1}^{r} \theta_j}$$

The boundary conditions imply $\int_{\partial P} \kappa(s)\, ds - \sum_{j=1}^{r} \theta_j$ is constant under corresponding variations, which concludes the proof.

The two-dimensional $O(2,1)$ nonlinear sigma model with boundary is invariant under conformal changes in the Lorentz-Minkowski space.

The solutions of this model, that is, the solutions of the equation $\Delta N_\phi - (N_\phi.\Delta N_\phi)\, N_\phi = 0$ are Willmore surfaces with boundary.

Using the boundary conditions we get

$$\partial\mathcal{D}(\phi)[V] = 4\,\partial\mathcal{W}(\phi)[V].$$

The Euler-Lagrange equation is computed in [2]

$$\boxed{\partial\mathcal{W}(\phi)[V] = \int_S \left(\Delta_\phi H_\phi + 2H_\phi \left(H_\phi^2 - K_\phi\right)\right)\, \bar{g}(N_\phi, V)\, dA_\phi}$$

Threfore, equations $\Delta N_\phi - (N_\phi.\Delta N_\phi)\, N_\phi = 0$ and

$$\boxed{\Delta H_\phi + 2H_\phi \left(H_\phi^2 - K_\phi\right) = 0} \tag{2}$$

have the same solutions.

Null scrolls as solutions

Theorem 4.1 (1) *Every null scroll is a Willmore surface in* $\mathbb{L}^3$.

(2) *Null scrolls provide solutions of the two-dimensional* $O(2,1)$ *nonlinear sigma model.*

Corollary 4.2 *Let* S *be a Lorentzian surface, with constant mean curvature, in* $\mathbb{L}^3$. *Then, it is a solution of the two-dimensional* $O(2,1)$ *nonlinear sigma model if and only if either*

(1) S *is stationary; or*

(2) S *is a* B*-scroll.*

Corollary 4.3 *Let $\phi(S)$ be a ruled surface, with constant mean curvature in $\mathbb{L}^3$. Then, it is a solution of the $O(2,1)$ nonlinear sigma model if and only if one of the following statements holds:*

1. *$\phi(S)$ has nonnull ruling flow and then it is congruent to a surface in the following list*

   (1.1) *A Lorentzian plane;*

   (1.2) *A helicoid of the* 1*st kind;*

   (1.3) *A helicoid of the* 2*nd kind;*

   (1.4) *A helicoid of the* 3*rd kind;*

   (1.5) *The conjugate surface of Enneper of the* 2*nd kind.*

2. *$\phi(S)$ has null ruling flow and then it is congruent to a surface in the following list*

   (2.1) *A Lorentzian plane;*

   (2.2) *A B-scroll.*

5. Polyakov's extrinsic string action

The discussion of strings historically began with the Nambu-Goto action which is defined, on the space of immersions that fix the boundary but need not be tangent along the common boundary, by

$$\boxed{\mathcal{NG}(\phi) = c_o \int_S dA_\phi = c_o \int_S \sqrt{-\det \frac{\partial X^i}{\partial u^\alpha} \frac{\partial X^j}{\partial u^\beta} \eta_{ij}}\, d^2u}$$

Strings are curves that evolve in the target space generating surfaces that provide extremals of this energy action. This topic, from a geometric point of view, is well understood for a long time and the string solutions correspond with those surfaces with zero mean curvature $H_\phi = 0$.

The Nambu-Goto action is not easy to manage due to the presence of the square root.

Then, A. M. Polyakov [10] proposed to replace the area action by an *equivalent action* that involves an intrinsic $h_{\alpha\beta}$ metric besides the induced one from the ambient spacetime metric:

$$\boxed{\mathcal{P}(\phi) = \frac{c_o}{2} \int_S \sqrt{-\det h_{\alpha\beta}}\, h^{\alpha\beta} \frac{\partial X^i}{\partial u^\alpha} \frac{\partial X^j}{\partial u^\beta} \eta_{ij}\, d^2u}$$

Both theories provide the so called *classical string solutions* that correspond with *stationary surfaces* ($H = 0$). It should be noted that the new Polyakov action is still intrinsic from its own origin.

From a geometric point of view, this intrinsic-extrinsic disagreement between action and solutions is not satisfactory. If we wish to evolve curves in a target spacetime to generate surfaces being extremals of a certain action, it seems natural to *involve the extrinsic geometry of surfaces in the density of the action.*

This idea was materialized in 1986 independently by A. M. Polyakov [11] and H. Kleinert [9]. Both authors introduced the same new string action using different motivations and methods. Kleinert defined the action trying to imitate the elastic functional for membranes, obviously in a Eucliden context, introduced in 1973 by W. Helfrich (Z. Naturforsch 33a, 305). In this way the so called *Polyakov extrinsic action* was born as a string action. Then we call *Polyakov-Kleinert-Helfrich action.*

More precisely, this action is defined on $\mathbf{I}_\Gamma(S, \mathbb{L}^3)$ and it measures the total extrinsic curvature of the pair $(\phi(S), \phi(\partial S))$ in $\mathbb{L}^3$,

$$\mathcal{PKH} : \mathbf{I}_\Gamma(S, \mathbb{L}^3) \to \mathbb{R}, \qquad \boxed{\mathcal{PKH}(\phi) = \int_S H_\phi^2 \, dA_\phi - \int_{\partial S} \kappa_\phi \, ds}$$

where H_ϕ stands for the mean curvature of the immersion $\phi(S)$ and κ_ϕ is the geodesic curvature of $\phi(\partial S)$ in $\phi(S)$.

We deal with the dynamics associated with this Polyakov extrinsic string action, which is the flat version of the *Willmore functional*.

On the space of boundary immersed timelike surfaces, which are tangent along the common boundary, in a generic spacetime, say M, it works as

$$\boxed{\mathcal{W}(\phi) = \int_S \left(H_\phi^2 + R_\phi\right) dA_\phi - \int_{\partial S} \kappa_\phi \, ds}$$

Putting all together, we look for critical points of the $\mathcal{WPKH}$ action, which in differential geometry we know as Willmore surfaces for the prescribed boundary conditions.

In the context of string theories, they are worldsheets of the Polyakov extrinsic string action and so, bearing in mind the original extrinsic nature of the action, they will be called *extrinsic string solutions*.

However, the concept of critical point needs some extra technical considerations. A critical point of such a problem means a critical point of the induced problem on nonnull polygons. Now, $\phi \in \mathbf{I}_\Gamma(S, \mathbb{L}^3)$ provides a classical string solution if for any nonnull polygon $\Omega \subseteq S$, the restriction $\phi|_\Omega$ is a critical point of the Polyakov extrinsic action on $\mathbf{I}_{\phi(\partial\Omega)}(\Omega, \mathbb{L}^3)$.

The field equation associated with this variational problem, computed in [2], is

$$\Delta_\phi H_\phi + 2H_\phi \left(H_\phi^2 - K_\phi\right) = 0, \tag{3}$$

where K_ϕ denotes the Gaussian curvature of $\phi(S)$. In particular, every stationary surface ($H = 0$) is automatically Willmore and consequently the $\mathcal{NGP}$ string theory can be regarded as a sub-theory of the $\mathcal{WPKH}$ string theory. Said otherwise, *in the moduli space of the extrinsic string solutions one can find a sub-moduli space made up of the classical string solutions.*

It should be noted that, in particular, the surface S could be boundary free and in this case no boundary condition is needed. Let us give, as an illustration, a pair of explicit examples (see [6]):

Example 5.1 *A rotational stationary surface: The hyperbolic catenoid.*

Choose $S = \mathbb{R}^+ \times \mathbb{R}$ and define the timelike immersion $\phi \in \mathbf{I}(S, \mathbb{L}^3)$ by

$$\phi(s,t) = (\sinh s \ \sinh t, s, \sinh s \ \cosh t).$$

Then we obtain a stationary surface and consequently it is a classical string solution as well as an extrinsic string solution.

Example 5.2 *A ruled stationary surface: The helicoid of the third kind.*

Take $S = \mathbb{R}^2$ and the timelike immersion $\phi \in \mathbf{I}(S, \mathbb{L}^3)$ given by

$$\phi(s,t) = \left(\frac{t}{\cosh s}, -t \ \tanh s, \sinh s \right).$$

It is easy to check that its mean curvature function vanishes identically and so it provides a string solution in both theories.

Besides those classical string solutions that correspond with stationary surfaces, other extrinsic string solutions with non-zero constant mean curvature are known (see [12]).

Null scrolls as extrinsic string solutions

Remember that for a null scroll $\mathbf{S}(\gamma, B)$ we have got that

$$\Delta H(s,t) = 0, \quad \text{and} \quad H^2(s,t) - K(s,t) = 0.$$

Then

Theorem 5.3 *Every null scroll is an extrinsic string solution in the Lorentz-Minkowski conformal structure.*

The above result can be paraphrased as: *Whenever a curve propagates in $\mathbb{L}^3$ transversely through a geodesic null vector field, it is generating the worldsheet of an extrinsic string solution.*

An algorithm to build the big zoo of scroll solutions

We are going to provide a simple method to explicitly construct the scroll solutions for the $\mathcal{WPKH}$ string theory, as well as an algorithm to build as many extrinsic string solutions as we wish.

As we have seen

$$H(s,t) = f(s) = -r(s)\,\varphi'(s). \tag{4}$$

Remark 5.4 *Remember that the mean curvature of a null scroll only depends on the lightlike ruling flow. In particular, stationary null scrolls* $(H = 0)$ *correspond with parallel lightlike ruling flow, that is, ruling flow with* $\varphi(s)$ *constant. In this sense, they can be regarded as cylinders with lightlike generatrices. The moduli space of stationary null scrolls has been obtained in* [3]*. It can be viewed as a kind of circle bundle over the space of congruence classes of lightlike curves in* $\mathbb{L}^3$*. This result deeply contrast with the case of stationary cylinders with nonnull generatrices, where we only get a Lorentzian plane.*

The algorithm.

We can explicitly construct the complete class of extrinsic string solutions with prescribed Polyakov extrinsic density, say a function $h \in \mathcal{C}^\infty(I, R)$.

To do it, we first choose any positive function, $r(s)$, defined on the same interval and use (4) to compute a third function by

$$\varphi(s) = \int_s^0 \frac{h(s)}{r(s)}.$$

Then, we have the following lightlike flow

$$B(s) = r(s)\,(\cos\varphi(s), \sin\varphi(s), 1),$$

which can be used as the ruling flow to generate all of extrinsic solutions corresponding to scroll string solutions whose Polyakov extrinsic density is the given function $h(s)$.

Then, the profile strings of these solutions have an arbitrary positive time function $c(s)$ and an angular function which must be determined from

$$\omega(s) = \varphi(s) + \arccos\left(1 - \frac{1}{c(s)\,r(s)}\right).$$

Now, use quadratures to obtain the profile strings as

$$\gamma(s) = \int_0^s c(u)(\cos\omega(u), \sin\omega(u), 1)du.$$

In this way, we get that the scroll extrinsic string solution $S(\gamma, B)$ has mean curvature function $h(s)$.

To illustrate this algorithm we give the following

Example 5.5 *Suppose that we wish to obtain all scroll extrinsic string solutions, with constant mean curvature, say $h = 1$, which are generated when propagating, in $\mathbb{L}^3$, the lightlike helix $\gamma(s) = (\sin s, -\cos s, s)$.*

To solve this problem, we need to construct the lightlike ruling flows, that allow one to propagate the string in order to get the solutions. We put

$$B(s) = r(s)\,(\cos\varphi(s), \sin\varphi(s), 1),$$

which must satisfies the following two constraints

$$\begin{aligned} \cos\left(s - \varphi(s)\right) &= 1 - \frac{1}{r(s)} \qquad \text{(normalization condition)}, \\ \varphi'(s) &= -\frac{1}{r(s)} \qquad \text{(constant mean curvature condition)}. \end{aligned}$$

Consequently, the angular function $\varphi(s)$ must be a solution of the following differential equation

$$\frac{d\varphi(s)}{ds} = \cos\left[s - \varphi(s)\right] - 1.$$

We use the change $\psi(s) = s - \varphi(s)$ to reduce it to

$$\frac{d\psi(s)}{ds} = 2 - \cos\psi(s),$$

which can be easily solved by separation of variables

$$\frac{d\psi}{2-\cos\psi} = ds,$$

finding the following general solution

$$\psi(s) = 2\ \arctan\left\{\frac{\sqrt{3}}{3}\ \tan\left[\frac{\sqrt{3}}{2}\, s + C\right]\right\}, \qquad C \in \mathbb{R},$$

which provides the following parameters for the lightlike ruling flows

$$\varphi(s) = s - 2\ \arctan\left\{\frac{\sqrt{3}}{3}\ \tan\left[\frac{\sqrt{3}}{2}\, s + C\right]\right\},$$

$$r(s) = \frac{1}{1-\cos\left\{2\ \arctan\left\{\frac{\sqrt{3}}{3}\ \tan\left[\frac{\sqrt{3}}{2}\, s + C\right]\right\}\right\}}.$$

Consequently, there exists just a one-parameter class of lightlike flows that allow us to propagate the lightlike helix $\gamma(s) = (\sin s, -\cos s, s)$ to generate extrinsic string solutions with constant mean curvature $h = 1$.

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