Surfaces in Lorentzian space forms satisfying the condition $\Delta x = Ax + B$

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1. Introduction

In [**2**] the authors have obtained a classification of surfaces in the 3-dimensional Lorentz-Minkowski space $\mathbb{L}^3$ satisfying the condition $\Delta x = Ax + B$, where x stands for the isometric immersion, A is an endomorphism of $\mathbb{L}^3$ and B is a constant vector. That condition was originally introduced by Dillen, Pas and Verstraelen in [**5**] for surfaces in the 3-dimensional Euclidean space and it has been studied by several authors for hypersurfaces in Riemannian space forms, [**4**], [**6**] and [**8**], who have obtained some interesting classification theorems. It should be noticed that those results obtained in the Riemannian cases strongly depend on the diagonalizability of the shape operator.

However, a surface in a Lorentzian space can be endowed with a Riemannian or Lorentzian metric, and in the last case its shape operator does not need to be diagonalizable. Therefore, it is worth bringing that condition to the non-flat Lorentzian space forms, that is, the De Sitter space $\mathbb{S}^3_1 \subset \mathbb{R}^4_1$ and anti De Sitter space $\mathbb{H}^3_1 \subset \mathbb{R}^4_2$, and it seems natural to hope for finding new classes of examples having no Riemannian counterpart. Moreover, in this new situation the codimension of the surface in the corresponding pseudo-Euclidean space is two and the proofs given in [**2**] do not work here, even so we follow the techniques developed there.

In this paper we are going to classify the surfaces in $\mathbb{S}^3_1$ and $\mathbb{H}^3_1$ with isometric immersion x satisfying the condition $\Delta x = Ax + B$, where A is an endomorphism of the corresponding 4-dimensional pseudo-Euclidean space and B is a constant vector. The classification is given by showing that the asked condition is a constant mean curvature condition and, under non-minimality hypothesis, it yields a flat surface with parallel second fundamental form in the pseudo-Euclidean space. We point out that in contrast to the case of surfaces in $\mathbb{L}^3$, examples of surfaces in $\mathbb{H}^3_1$ satisfying that condition and having non-diagonalizable shape operator can be found (see Examples 5.1 and 5.2).

2. Preliminaries

Let us denote by $\bar{M}^3_1(c)$ the standard model of a 3-dimensional Lorentz space with constant curvature $c = 1, -1$, say the De Sitter space $\mathbb{S}^3_1 = \{x \in \mathbb{R}^4_1 : \langle x, x\rangle = 1\}$ and the anti De Sitter space $\mathbb{H}^3_1 = \{x \in \mathbb{R}^4_2 : \langle x, x\rangle = -1\}$, respectively, $\langle , \rangle$ standing for the indefinite inner product in the corresponding pseudo-Euclidean space $\mathbb{R}^4_q$, $q = 1, 2$, where $\bar{M}^3_1(c)$ is lying.

Let $x : M_s^2 \longrightarrow \bar{M}_1^3(c) \subset \mathbb{R}_q^4$ be a surface of index s ($s = 0, 1$) satisfying the condition

$$\Delta x = Ax + B,$$

where A is an endomorphism of $\mathbb{R}_q^4$ and B a constant vector in $\mathbb{R}_q^4$. Throughout this paper we will denote by H, N and α the mean curvature vector field of M_s^2 in $\mathbb{R}_q^4$, the unit normal vector field of M_s^2 in $\bar{M}_1^3(c)$ and the mean curvature in the direction of N, respectively. Thus we may write

$$H = \alpha N - cx.$$

From above equations, using the well known Laplace-Beltrami formula $\Delta x = -2H$, we easily deduce that

$$Ax = -2\alpha N + 2cx - B.$$

Taking covariant derivative in (1) and using the formula for ΔH given in [**3**, Lemma 3] we have the following equations

$$AX = 2(\alpha SX + cX) - 2X(\alpha)N,$$

for any vector field X tangent to M_s^2 and

$$\alpha AN = 2S(\nabla\alpha) + 2\varepsilon\alpha\nabla\alpha + \{\Delta\alpha + \varepsilon\alpha\,\mathrm{tr}(S^2)\}N - 2c\varepsilon\alpha^2 x - cB,$$

where S stands for the shape operator of M_s^2 in $\bar{M}_1^3(c)$, $\nabla\alpha$ is the gradient of α, $\varepsilon = \langle N, N\rangle$ and $\mathrm{tr}(S^2) = \mathrm{trace}(S^2)$.

For later use, we are going to deduce a couple of useful equations. The first one is a straight consequence of (4),

$$\langle AX, Y\rangle = \langle X, AY\rangle$$

for any tangent vector fields X and Y. The second one can be obtained by taking covariant derivative in (6),

$$\langle A\sigma(X,Z), Y\rangle - \langle A\sigma(Y,Z), X\rangle = \langle \sigma(X,Z), AY\rangle - \langle \sigma(Y,Z), AX\rangle,$$

where σ is the second fundamental form of M_s^2 in $\mathbb{R}_q^4$.

3. Some examples

Before going into the study of the condition $\Delta x = Ax + B$, let us see some examples of surfaces in $\bar{M}_1^3(c)$ satisfying that condition. They will be useful later in order to give the classification results.

Example 3.1 It is clear that every minimal surface M_s^2 in $\bar{M}_1^3(c)$ satisfies the condition $\Delta x = Ax + B$. In fact, $\alpha = 0$ implies $H = -cx$ in (2), which jointly with the Laplace-Beltrami equation gives $\Delta x = 2cx$. So, we have (1) with $A = 2cI_4$ and $B = 0$.

Example 3.2 Let M_s^2 be a totally umbilical surface in $\bar{M}_1^3(c)$. By using the classification theorem given by M.A. Magid in [**7**, Theorem 1.4] we get, according to $\langle H, H\rangle$ is positive, negative or zero, M_s^2 is an open piece of a pseudo-sphere $\mathbb{S}_s^2(r)$, a pseudo-hyperbolic space $\mathbb{H}_s^2(-r)$ or $\mathbb{R}_s^2$, respectively. Moreover, in the last case the isometric immersion is explicitly given by $x : \mathbb{R}_s^2 \longrightarrow$

$\bar{M}_1^3(c) \subset \mathbb{R}_{s+1}^4$, $x = f - x_0$, x_0 being a fixed vector and $f : \mathbb{R}_s^2 \longrightarrow \mathbb{R}_{s+1}^4$ the map defined by $f(u_1, u_2) = (q(u_1, u_2), u_1, u_2, q(u_1, u_2))$, where $q(u) = a_1\langle u, u\rangle + \langle v_0, u\rangle + a_0$, $a_0, a_1 \in \mathbb{R}$ with $a_1 \neq 0$ and $v_0 \in \mathbb{R}_s^2$.

It is not difficult to see that pseudo-spheres and pseudo-hyperbolic spaces both satisfy the condition (1). Indeed, if $\bar{x}$ is the standard immersion of $\mathbb{S}_s^2(r)$ or $\mathbb{H}_s^2(-r)$ in a hyperplane $\mathbb{R}_{s'}^3$ of $\mathbb{R}_q^4$, we know from [**1**] that $\Delta\bar{x} = \bar{A}\bar{x}$, $\bar{A}$ being an endomorphism of $\mathbb{R}_{s'}^3$. Now by embedding $\mathbb{R}_{s'}^3$ in $\mathbb{R}_q^4$, the immersion $\bar{x}$ becomes an immersion x from M_s^2 in $\bar{M}_1^3(c) \subset \mathbb{R}_q^4$ satisfying the condition $\Delta x = Ax$, where A is the 4×4 matrix obtained from $\bar{A}$ with zeros for each of the additional entries. Therefore the most interesting case arises when $\langle H, H\rangle = 0$. Now the Laplacian operator of the surface is given by

$$\Delta = \sum_{i=1}^{s} \frac{\partial^2}{\partial u_i^2} - \sum_{j=s+1}^{4} \frac{\partial^2}{\partial u_j^2}$$

and a simple computation shows that $\Delta x = -4a_1(1, 1, 1, 1)$. Thus this surface satisfies (1) with $A = 0$ and $B = -4a_1(1, 1, 1, 1)$. We will refer it as a *flat totally umbilical surface.*

Example 3.3 An easy computation shows that the following pseudo-Riemannian products are all non-minimal surfaces in $\bar{M}_1^3(c)$ satisfying the condition $\Delta x = Ax + B$ with $B = 0$ (see the attached table).

1) $\mathbb{S}_1^1(r) \times \mathbb{S}^1(\sqrt{1-r^2}) \subset \mathbb{S}_1^3$, with $0 < r < 1$ and $r \neq \sqrt{1/2}$, immersed by $x : \mathbb{R}_1^2 \longrightarrow \mathbb{S}_1^3 \subset \mathbb{R}_1^4$,

$$x(u_1, u_2) = (r \sinh\frac{u_1}{r}, r\cosh\frac{u_1}{r}, \sqrt{1-r^2}\cos\frac{u_2}{\sqrt{1-r^2}}, \sqrt{1-r^2}\sin\frac{u_2}{\sqrt{1-r^2}}),$$

2) $\mathbb{S}^1(r) \times \mathbb{H}^1(-\sqrt{r^2-1}) \subset \mathbb{S}_1^3$, with $r > 1$, and the immersion $x : \mathbb{R}^2 \longrightarrow \mathbb{S}_1^3 \subset \mathbb{R}_1^4$ is given by

$$x(u_1, u_2) = (r\cos\frac{u_2}{r}, r\sin\frac{u_2}{r}, \sqrt{r^2-1}\cosh\frac{u_1}{\sqrt{r^2-1}}, \sqrt{r^2-1}\sinh\frac{u_1}{\sqrt{r^2-1}}),$$

3) $\mathbb{S}^1(r) \times \mathbb{H}_1^1(-\sqrt{1+r^2}) \subset \mathbb{H}_1^3$, $r > 0$, with the usual parametrization $x : \mathbb{R}^2 \longrightarrow \mathbb{H}_1^3 \subset \mathbb{R}_2^4$ given by

$$x(u_1, u_2) = (r\cos\frac{u_2}{r}, r\sin\frac{u_2}{r}, \sqrt{1+r^2}\cos\frac{u_1}{\sqrt{1+r^2}}, \sqrt{1+r^2}\sin\frac{u_1}{\sqrt{1+r^2}}),$$

4) $\mathbb{S}_1^1(r) \times \mathbb{H}^1(-\sqrt{1+r^2}) \subset \mathbb{H}_1^3$, with $r > 0$, immersed by $x : \mathbb{R}^2 \longrightarrow \mathbb{H}_1^3 \subset \mathbb{R}_2^4$,

$$x(u_1, u_2) = (r\sinh\frac{u_1}{r}, \sqrt{1+r^2}\cosh\frac{u_2}{\sqrt{1+r^2}}, r\cosh\frac{u_1}{r}, \sqrt{1+r^2}\sinh\frac{u_2}{\sqrt{1+r^2}}),$$

5) $\mathbb{H}^1(-r) \times \mathbb{H}^1(-\sqrt{1-r^2}) \subset \mathbb{H}_1^3$, with $0 < r < 1$ and $r \neq \sqrt{1/2}$, parametrized by $x : \mathbb{R}^2 \longrightarrow \mathbb{H}_1^3 \subset \mathbb{R}_2^4$,

$$x(u_1, u_2) = (r\cosh\frac{u_1}{r}, \sqrt{1-r^2}\cosh\frac{u_2}{\sqrt{1-r^2}}, r\sinh\frac{u_1}{r}, \sqrt{1-r^2}\sinh\frac{u_2}{\sqrt{1-r^2}}),$$

We will refer them as the *non-minimal standard products*. Notice that all of them have diagonalizable shape operators.

r	Surface	A
$0<r<1$	$\mathbb{S}^1_1(r)\times\mathbb{S}^1(\sqrt{1-r^2})\subset\mathbb{S}^3_1$	$\begin{pmatrix} \frac{1}{r^2}I_2 & \mathbf{0} \\ \mathbf{0} & \frac{1}{1-r^2}I_2 \end{pmatrix}$
$r>1$	$\mathbb{S}^1(r)\times\mathbb{H}^1(-\sqrt{r^2-1})\subset\mathbb{S}^3_1$	$\begin{pmatrix} \frac{1}{r^2}I_2 & \mathbf{0} \\ \mathbf{0} & \frac{1}{r^2-1}I_2 \end{pmatrix}$
$r>0$	$\mathbb{S}^1(r)\times\mathbb{H}^1_1(-\sqrt{1+r^2})\subset\mathbb{H}^3_1$	$\begin{pmatrix} \frac{1}{r^2}I_2 & \mathbf{0} \\ \mathbf{0} & \frac{1}{1+r^2}I_2 \end{pmatrix}$
$r>0$	$\mathbb{S}^1_1(r)\times\mathbb{H}^1(-\sqrt{1+r^2})\subset\mathbb{H}^3_1$	$\begin{pmatrix} \frac{1}{r^2}I_2 & \mathbf{0} \\ \mathbf{0} & \frac{1}{1+r^2}I_2 \end{pmatrix}$
$0<r<1$	$\mathbb{H}^1(-r)\times\mathbb{H}^1(-\sqrt{1-r^2})\subset\mathbb{H}^3_1$	$\begin{pmatrix} \frac{1}{r^2}I_2 & \mathbf{0} \\ \mathbf{0} & \frac{1}{1-r^2}I_2 \end{pmatrix}$

4. First characterization results

The aim of this section is to show that the condition $\Delta x = Ax + B$ is a constant mean curvature condition and, under non-minimality hypothesis, it is also a flatness condition on the surface. First, let $x : M_s^2 \longrightarrow \bar{M}_1^3(c) \subset \mathbb{R}_q^4$ be a surface satisfying (1). From (4) we have $\langle AX, x\rangle = 0$ for any vector field tangent to M_s^2, and taking covariant derivative here we get

$$\langle A\sigma(X,Y), x\rangle = -\langle AX, Y\rangle$$

for any tangent vector fields X and Y. Now equation (1), jointly with (3), (4) and (5), implies that

$$\langle SX - \varepsilon\alpha X, Y\rangle\langle B, x\rangle = 0.$$

Let $\mathcal{U} = \{p \in M_s^2 : \nabla\alpha^2(p) \neq 0\}$ be the open set of regular points of α^2 and assume that it is not empty. If $\mathcal{W} = \{p \in \mathcal{U} : \langle B, x\rangle \neq 0\}$ is a non-empty set, then from (4) and (2), we have

$$AX = 2(c + \varepsilon\alpha^2)X - 2X(\alpha)N,$$

at the points of $\mathcal{W}$. Let us choose a tangent vector field X orthogonal to $\nabla\alpha$, that is $X(\alpha) = \langle X, \nabla\alpha\rangle = 0$. By using (3), we obtain that $2(c+\varepsilon\alpha^2)$ is an eigenvalue of A and therefore locally constant on $\mathcal{W}$, which is a contradiction. Hence $\mathcal{W} = \emptyset$ and $\langle B, x\rangle = 0$ on $\mathcal{U}$. Taking covariant derivative here we deduce that B has not tangent component and thus $B = \varepsilon\langle B, N\rangle N$. Finally, as $\langle B, N\rangle^2 = \varepsilon\langle B, B\rangle$ is constant we deduce that $\langle B, N\rangle = 0$, because $\mathcal{U}$ is not empty. Summing up, we have shown that if $x : M_s^2 \longrightarrow \bar{M}_1^3(c) \subset \mathbb{R}_q^4$ is an isometric immersion satisfying the condition $\Delta x = Ax + B$ and having non-constant mean curvature, then $B = 0$.

Now we are ready to prove the following result.

Proposition 4.1 *Let $x : M_s^2 \longrightarrow \bar{M}_1^3(c) \subset \mathbb{R}_q^4$ be an isometric immersion such that $\Delta x = Ax + B$. Then M_s^2 has constant mean curvature.*

Proof. If we assume that the mean curvature α is not constant, then we have just shown that $B = 0$. Using now equation (7), jointly with (3), (4) and (5), we obtain

$$TX(\alpha)SY = TY(\alpha)SX$$

on $\mathcal{U}$, for any tangent vector fields X and Y, where T denotes the self-adjoint operator given by $TX = 2\alpha X + \varepsilon SX$.

Case 1: $T(\nabla\alpha) \neq 0$ on $\mathcal{U}$. Then there exists a tangent vector field X such that $TX(\alpha) \neq 0$, which implies, by using (4), that S has rank one on $\mathcal{U}$. Thus we can choose a local orthonormal frame $\{E_1, E_2\}$ such that $SE_1 = 2\varepsilon\alpha E_1$, $SE_2 = 0$ and $\varepsilon_i = \langle E_i, E_i\rangle$. Also from (4) we have that $E_2(\alpha) = 0$ and E_1 is parallel to $\nabla\alpha$, and using again (3), (4) and (5) we obtain

$$\begin{aligned} AE_1 &= 2(c + 2\varepsilon\alpha^2)E_1 - 2E_1(\alpha)N, \\ AE_2 &= 2cE_2, \\ AN &= 6\varepsilon\varepsilon_1 E_1(\alpha)E_1 + \{\frac{\Delta\alpha}{\alpha} + 4\varepsilon\alpha^2\}N - 2c\varepsilon\alpha x, \\ Ax &= -2\alpha N + 2cx. \end{aligned}$$

Therefore, the associated matrix to the endomorphism $A^* = A|_{\mathrm{span}\{E_1,N,x\}}$ is given by

$$\begin{pmatrix} 2(c+2\varepsilon\alpha^2) & 6\varepsilon\varepsilon_1 E_1(\alpha) & 0 \\ -2E_1(\alpha) & \dfrac{\Delta\alpha}{\alpha} + 4\varepsilon\alpha^2 & -2\alpha \\ 0 & -2c\varepsilon\alpha & 2c \end{pmatrix},$$

whose invariants are

$$\begin{aligned} \lambda_1 &= \frac{\Delta\alpha}{\alpha} + 4(c + 2\varepsilon\alpha^2), \\ \lambda_2 &= 4(c+\varepsilon\alpha^2)(\frac{\Delta\alpha}{\alpha} + 4\varepsilon\alpha^2) + 12\varepsilon\varepsilon_1 E_1(\alpha)^2 + 4c(c + 2\varepsilon\alpha^2) - 4c\varepsilon\alpha^2, \\ \lambda_3 &= 4c(c+2\varepsilon\alpha^2)(\frac{\Delta\alpha}{\alpha} + 4\varepsilon\alpha^2) - 8c\varepsilon\alpha^2(c + 2\varepsilon\alpha^2) + 24\varepsilon\varepsilon_1 cE_1(\alpha)^2. \end{aligned}$$

Then we deduce that

$$2c\lambda_2 = \lambda_3 + 8(c + 2\varepsilon\alpha^2) + 4(\frac{\Delta\alpha}{\alpha} + 4\varepsilon\alpha^2) + 16c\alpha^4$$

and

$$\frac{\Delta\alpha}{\alpha} = \lambda_1 - 4(c + 2\varepsilon\alpha^2).$$

These two equations allow us to write

$$16\alpha^4 = 8 - 4c\lambda_1 + 2\lambda_2 - c\lambda_3,$$

and so α is locally constant on $\mathcal{U}$, which is a contradiction.

Case 2: There exits a point p in $\mathcal{U}$ such that $T(\nabla\alpha)(p) = 0$. Then from (4) and (5) we have

$$\langle AX, N\rangle(p) = -2\varepsilon X(\alpha)(p) = \langle X, AN\rangle(p).$$

Moreover, since $B = 0$ we also obtain from (3), (4) and (5) that

$$\begin{aligned} \langle AX, x\rangle &= \langle X, Ax\rangle, \\ \langle Ax, N\rangle &= \langle x, AN\rangle, \end{aligned}$$

which implies, jointly with (6) and (5), that A is a self-adjoint endomorphism of $\mathbb{R}^4_q$ and thus equation (5) remains valid at every point in $\mathcal{U}$. Therefore, $T(\nabla\alpha) = 0$ on $\mathcal{U}$ and $S(\nabla\alpha) = -2\varepsilon\alpha$.

Since $-2\varepsilon\alpha$ is an eigenvalue of S and $\mathrm{tr}(S) = 2\varepsilon\alpha$, then S is diagonalizable and we can choose a local orthonormal frame $\{E_1, E_2\}$ such that $SE_1 = -2\varepsilon\alpha E_1$, with E_1 parallel to $\nabla\alpha$, and $SE_2 = 4\varepsilon\alpha E_2$. Thus, from (4) we get

$$AE_2 = 2(c + 4\varepsilon\alpha^2)E_2.$$

Then $2(c + 4\varepsilon\alpha^2)$ is an eigenvalue of A and therefore α is locally constant on $\mathcal{U}$, which is a contradiction.

Anyway, we deduce that $\mathcal{U}$ is empty and then M^2_s has constant mean curvature. ■

Now, let $M^2_s \subset \bar{M}^3_1(c)$ be a surface satisfying the condition $\Delta x = Ax + B$ with non-zero constant mean curvature α in $\bar{M}^3_1(c)$. If M^2_s is not totally umbilical we obtain from (2) that $\langle B, x\rangle = 0$ and reasoning as we did at the beginning of this section we get $B = 0$. Therefore, equations (3), (4) and (5) are rewritten as follows

$$\begin{aligned} AX &= 2\alpha SX + 2cX, && (6) \\ AN &= \varepsilon \mathrm{tr}(S^2)N - 2c\varepsilon\alpha x, && (7) \\ Ax &= -2\alpha N + 2cx. && (8) \end{aligned}$$

Then the trace of A is given by

$$\mathrm{tr}(A) = 4\varepsilon\alpha^2 + \varepsilon\mathrm{tr}(S^2) + 6c,$$

which implies that $\mathrm{tr}(S^2)$ is also constant. Taking now covariant derivative in (7) we have

$$\tilde{\nabla}_X(AN) = -\varepsilon\mathrm{tr}(S^2)SX - 2c\varepsilon\alpha X,$$

and using (6) we obtain

$$\tilde{\nabla}_X(AN) = -A(SX) = -2\alpha S^2X - 2cSX.$$

Therefore the characteristic equation of the shape operator of M^2_s is given by

$$S^2 + \frac{2c - \varepsilon\mathrm{tr}(S^2)}{2\alpha}S - c\varepsilon I_2 = 0,$$

where I_2 stands for the identity operator on the tangent bundle of M^2_s. Thus, $\det(S) = -c\varepsilon$ and the Gaussian curvature of M^2_s is $K = c + \varepsilon\det(S) = 0$. Summing up, M^2_s is a flat isoparametric surface in $\bar{M}^3_1(c)$ with parallel second fundamental form in $\mathbb{R}^4_q$.

So, we have the following result.

Theorem 4.2 *Let $x : M^2_s \longrightarrow \bar{M}^3_1(c) \subset \mathbb{R}^4_q$ be a non-minimal isometric immersion satisfying $\Delta x = Ax + B$. Then M^2_s is totally umbilical in $\bar{M}^3_1(c)$ or M^2_s is a flat isoparametric surface in $\bar{M}^3_1(c)$ with parallel second fundamental form in $\mathbb{R}^4_q$.*

As a first interesting consequence of Theorem 4.2, we can give the following classification result for surfaces in $\bar{M}_1^3(c)$ with diagonalizable shape operator.

Corollary 4.3 *Let $x : M_s^2 \longrightarrow \bar{M}_1^3(c) \subset \mathbb{R}_q^4$ be a non-minimal isometric immersion with diagonalizable shape operator. Then M_s^2 satisfies $\Delta x = Ax + B$ if and only if M_s^2 is an open piece of one of the following surfaces:*
1) a totally umbilical surface in $\bar{M}_1^3(c)$.
2) a non-minimal standard product in $\bar{M}_1^3(c)$.

5. The classification theorem

All examples exhibited in Section 3, unless Example 3.1, have diagonalizable shape operator. However, it seems reasonable to look for Lorentzian surfaces in $\bar{M}_1^3(c)$ satisfying $\Delta x = Ax + B$ with non-diagonalizable shape operator. We find the two following examples.

Example 5.1 Let a and b be two real numbers such that $a^2 - b^2 = -1$ and $ab \neq 0$. Then the map $x : \mathbb{R}^2 \longrightarrow \mathbb{H}_1^3 \subset \mathbb{R}_2^4$, $x = (x^1, x^2, x^3, x^4)$, given by

$$\begin{aligned} x^1(u_1, u_2) &= b \cosh u_2 \cos u_1 - a \sinh u_2 \sin u_1, \\ x^2(u_1, u_2) &= a \sinh u_2 \cos u_1 + b \cosh u_2 \sin u_1, \\ x^3(u_1, u_2) &= a \cosh u_2 \cos u_1 + b \sinh u_2 \sin u_1, \\ x^4(u_1, u_2) &= a \cosh u_2 \sin u_1 - b \sinh u_2 \cos u_1, \end{aligned}$$

where (u_1, u_2) is the usual coordinate system in $\mathbb{R}^2$, parametrizes a non-minimal flat surface in $\mathbb{H}_1^3$ whose shape operator is given, in the usual frame $\left\{ \frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2} \right\}$, by

$$S = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix},$$

with $\alpha = \dfrac{2ab}{a^2 + b^2}$ and $\beta = \dfrac{-1}{a^2 + b^2}$. Magid, [**7**, Example 1.12], refers this surface as a *complex circle* of radius $a + bi$.

The Laplacian operator of a complex circle is given, in coordinates (u_1, u_2), by

$$\Delta = \frac{1}{(a^2 + b^2)^2} \left(\frac{\partial^2}{\partial u_1^2} + 4ab \frac{\partial^2}{\partial u_1 \partial u_2} - \frac{\partial^2}{\partial u_2^2} \right),$$

and it is easy to see that it satisfies $\Delta x = Ax$, where A is the following matrix

$$\frac{1}{(a^2 + b^2)^2} \begin{pmatrix} -2 & 0 & -4ab & 0 \\ 0 & -2 & 0 & -4ab \\ 4ab & 0 & -2 & 0 \\ 0 & 4ab & 0 & -2 \end{pmatrix}.$$

Example 5.2 Next example exhibits a surface whose shape operator is not diagonalizable but it has only a double real eigenvalue. Let $x : \mathbb{R}^2 \longrightarrow \mathbb{H}^3_1 \subset \mathbb{R}^4_2$ be the map given by $x = (x^1, x^2, x^3, x^4)$

$$\begin{aligned}
x^1(u_1, u_2) &= \frac{3}{2\sqrt{2}}\sin u_2 - \frac{1}{\sqrt{2}}(u_1 + \frac{u_2}{2})\cos u_2,\\
x^2(u_1, u_2) &= \frac{3}{2\sqrt{2}}\cos u_2 + \frac{1}{\sqrt{2}}(u_1 + \frac{u_2}{2})\sin u_2,\\
x^3(u_1, u_2) &= \frac{1}{2\sqrt{2}}\sin u_2 + \frac{1}{\sqrt{2}}(u_1 + \frac{u_2}{2})\cos u_2,\\
x^4(u_1, u_2) &= \frac{1}{2\sqrt{2}}\cos u_2 - \frac{1}{\sqrt{2}}(u_1 + \frac{u_2}{2})\sin u_2,
\end{aligned}$$

where (u_1, u_2) is the usual coordinate system in $\mathbb{R}^2$. Then x parametrizes a non-minimal flat surface in $\mathbb{H}^3_1$ whose shape operator is given by

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

(see [**7**, Example 1.13]).

It is not difficult to see that its Laplacian operator is given, in coordinates (u_1, u_2), by

$$\Delta = -2\frac{\partial^2}{\partial u_1 \partial u_2},$$

and this surface satisfies $\Delta x = Ax$, where A is the following matrix

$$\begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

Now, we are ready to show the main results of this paper.

Theorem 5.3 *Let $x : M^2_s \longrightarrow \mathbb{S}^3_1 \subset \mathbb{R}^4_1$ be an isometric immersion. Then $\Delta x = Ax + B$ if and only if one of the following statements holds true:*

1) M^2_s is a minimal surface in $\mathbb{S}^3_1$.

2) M^2_s is totally umbilical, and then it is an open piece of $\mathbb{H}^2(-r)$, $\mathbb{S}^2(r)$, $\mathbb{S}^2_1(r)$, $r > 0$, or a flat totally umbilical.

3) M^2_s is an open piece of one of the non-minimal standard products in $\mathbb{S}^3_1$: $\mathbb{S}^1(r)\times\mathbb{H}^1(-\sqrt{r^2-1})$, $r > 1$, and $\mathbb{S}^1_1(r) \times \mathbb{S}^1(\sqrt{1-r^2})$, $0 < r < 1$ and $r \neq \sqrt{1/2}$.

Theorem 5.4 *Let $x : M^2_s \longrightarrow \mathbb{H}^3_1 \subset \mathbb{R}^4_2$ be an isometric immersion. Then $\Delta x = Ax + B$ if and only if one of the following statements holds true:*

1) M^2_s is a minimal surface in $\mathbb{H}^3_1$.

2) M^2_s is totally umbilical, and then it is an open piece of $\mathbb{H}^2(-r)$, $\mathbb{H}^2_1(-r)$, $\mathbb{S}^2_1(r)$, $r > 0$, or a flat totally umbilical.

3) M^2_s is an open piece of one of the non-minimal standard products in $\mathbb{H}^3_1$: $\mathbb{S}^1(r)\times\mathbb{H}^1_1(-\sqrt{1+r^2})$ and $\mathbb{S}^1_1(r) \times \mathbb{H}^1(-\sqrt{1+r^2})$, $r > 0$, and $\mathbb{H}^1(-r) \times \mathbb{H}^1(-\sqrt{1-r^2})$, $0 < r < 1$ and $r \neq \sqrt{1/2}$.

4) M^2_1 is an open piece of the surface exhibited in Example 5.1.

5) M^2_1 is an open piece of the surface exhibited in Example 5.2.

Proof of Theorems 5.3 and 5.4. According to Corollary 4.3 it suffices to deal with non-diagonalizable case. Let $x : M_1^2 \longrightarrow \bar{M}_1^3(c) \subset \mathbb{R}_q^4$ be a non-minimal Lorentz surface in $\bar{M}_1^3(c)$ satisfying the condition $\Delta x = Ax + B$ with non-diagonalizable shape operator. By Theorem 4.2, we know that M_1^2 is a flat surface with parallel second fundamental form in $\mathbb{R}_q^4$, and the characteristic equation of its shape operator is given by

$$S^2 + \frac{2c - \mathrm{tr}(S^2)}{2\alpha} S - cI_2 = 0.$$

Therefore, the discriminant of its characteristic polynomial is written as

$$d_S = 4c + \left(\frac{2c - \mathrm{tr}(S^2)}{2\alpha}\right)^2,$$

which can be non-positive provided that $c = -1$. Then, by applying [**7**, Theorem 1.17] we get that M_1^2 is an open piece of a complex circle (Example 5.1) or the surface exhibited in Example 5.2. ∎

Bibliography

[1] L. J. Alías, A. Ferrández and P. Lucas. Submanifolds in pseudo-Euclidean spaces satisfying the condition $\Delta x = Ax + B$. *Geom. Dedicata*, **42** (1992), 345–354.

[2] L. J. Alías, A. Ferrández and P. Lucas. Surfaces in the 3-dimensional Lorentz-Minkowski space satisfying $\Delta x = Ax + B$. *Pacific J. Math.*, **156** (1992), 201–208.

[3] B. Y. Chen. Finite-type pseudo-Riemannian submanifolds. *Tamkang J. of Math.*, **17** (1986), 137–151.

[4] B. Y. Chen and M. Petrovic. On spectral decomposition of immersions of finite type. *Bull. Austral. Math. Soc.*, **44** (1991), 117–129.

[5] F. Dillen, J. Pas and L. Verstraelen. On surfaces of finite type in euclidean 3-space. *Kodai Math. J.*, **13** (1990), 10–21.

[6] T. Hasanis and T. Vlachos. Hypersurfaces of $\mathbb{E}^{n+1}$ satisfying $\Delta x = Ax + B$. *J. Austral. Math. Soc., Series A*, **53** (1992), 377–384.

[7] M. A. Magid. Isometric immersions of Lorentz space with parallel second fundamental forms. *Tsukuba J. Math.*, **8** (1984), 31–54.

[8] J. Park. Hypersurfaces satisfying the equation $\delta x = rx + b$. To appear in Proc. A.M.S.