### Principles governing helices in Nature

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This is a survey of a joint work with M. Barros





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#### Preface

Introduction: Helices in Nature Generalized helices A geometric way to see helices But ... still more Turning over the classic idea of a helix



In 1939 Paul Dirac wrote: The research worker, in his effort to express the fundamental laws of Nature in mathematical form, should strive mainly for mathematical beauty. It often happens that the requirements of simplicity and beauty are the same, but where they clash the latter must take precedence.

An intuitive vision of curvature and torsion

## Helices everywhere

Helical configurations are structures commonly found in Nature. They appear in microscopic systems (biomolecules, bacterial fibers, nanosprings, protein chains in particular DNA,...) as well as in macroscopic phenomena (strings, ropes, climbing plants, coiled springs, horns of mountain goats, vortices,...).















An intuitive vision of curvature and torsion

## Helices everywhere

 Famous helices are, for example, those of Leonardo da Vinci, as a forerunner of the helicopter.



An intuitive vision of curvature and torsion

## Helices everywhere

 In architecture: Gaudi's Güell park and tree-shaped columns of the Sagrada Familia); Frank Lloyd's Guggenheim museum in New York, etc.





An intuitive vision of curvature and torsion

## Helices everywhere

- In technology: screws, corkscrews, coils, turbins, etc.
- Anywhere, even macaroni or stirring soup or coffee up!

An intuitive vision of curvature and torsion

## Helices in Biology

In particular, they are very important and ubiquitous in Biology as a consequence of the following known, in the biological community since the work of Pauling, **theorem**: *Identical objects, regularly assembled, form a helix.* 

It seems that the success of helical configurations, seen as a popular shape of molecules, it is because Nature tries to adapt to the external conditions. The spiral shape of the DNA obeys to the available space into the cell, as well as the shape of a spiral staircase obeys to the size of the apartment.

An intuitive vision of curvature and torsion

## Preliminaries: curvature and torsion of a helix

#### Curvature

By squashing a helix on the plane perpendicular to its axis, we know that its **curvature**  $\kappa$  is a function measuring at a what extent the plane curve moves away, at each point, from its tangent line.

#### Torsion

Looking at the helix, the torsión  $\tau$  is a function measuring, at a point, the extent of the lifting of the curve regarding that plane.

### Example of a straight circular helix

It is well known that for the helix  $f(t) = (a \cos t, a \sin t, b t)$ , we get

$$\kappa = rac{a^2}{a^2 + b^2}$$
  $au = rac{ab}{a^2 + b^2}$ 

## Looking for the model

In order to build our model, we will *identify a helicoidal structure with its central line* to see it as a one-dimensional object or curve. Perhaps, for simplicity, helical structures are usually identified, in the literature, with the simplest idea of *circular helices*.

However, that does not fit the real world. Nobody can believe that squirrels chasing one another up and around tree trunks follow a path of circular helix. First because the cross section of a tree trunk is not circular, but also because the axis of a tree trunk is not exactly a straight line.

As another example, we find many types of bacteria, such as certain strains of *Escherichia coli* or *Salmonella typhimorium* swim by rotating flagellar filaments. These are polymers which are flexible enough to switch among different helical forms, which are really far from circular helices.

Therefore, the question states as follows:

### The characterization

# How to build a mathematical model to describe helicoidal configurations in Nature?

A generalized or Lancret helix (hereafter *helix*) is defined as a curve  $\gamma(s)$  whose tangent vector  $\gamma'(s)$  forms a constant angle with a fixed direction  $\vec{v}$ , the axis of the helix, and  $\|\vec{v}\| = \text{const.}$ 

Lancret's theorem: A curve is a (generalized) helix if, and only if, the ratio of curvature to torsion be constant

$$\frac{\tau}{\kappa} = \text{const.}$$

### Under construction

Let  $\alpha : [0, L] \subset \mathbb{R} \to \mathbb{E}^2$  be a planar regular curve, parametrized by the arc-length, lying in the plane  $\Pi$  orthogonal to a unit vector  $\vec{x} \in \mathbb{E}^3$ . The right circular cylinder  $C_{\alpha}$ , whose generatrices are parallel to  $\vec{x}$  and cross section  $\alpha$ , can be parametrized by

$$\phi: I \times \mathbb{R} \to \mathbb{E}^3, \qquad \phi(s, u) = \alpha(s) + u \vec{x}.$$

It is well known that  $C_{\alpha}$  is a flat surface whose geometry is encoded in the geometry of  $\alpha$ . In particular, the geodesics of  $C_{\alpha}$  are the images by  $\phi$  of straight lines, i. e.,

$$\gamma(t) = \phi(at, bt) = \alpha(at) + bt \vec{x},$$

where b/a is the slope of the corresponding straight line.

A straightforward computation yields

$$\kappa_{\gamma} = \frac{a^2}{a^2 + b^2} \kappa_{\alpha}$$
 and  $\tau_{\gamma} = \frac{ab}{a^2 + b^2} \kappa_{\alpha}$ .

Then

$$rac{ au_{\gamma}}{\kappa_{\gamma}} = rac{b}{a}.$$

As a consequence Any geodesic of a right cylinder over a planar curve is automatically a helix whose axis is that of the cylinder. Therefore,

Given regular curve  $\gamma$  in  $\mathbb{E}^3$ , the following statements are equivalent:

- ${\bf 0}~\gamma$  is a helix, i. e., it makes a constant angle with a fixed direction.
- **②** The ratio of curvature to torsion is constant, the slope.
- $\bullet$   $\gamma$  is a geodesic of a right cylinder over a planar curve.

## A variational problem

Lorsqu'il arrive quelque changement dans la Nature la quantité d'action, nécessaire pour ce changement, est la plus petite possible (Pierre Louis Moreau de Maupertuis, Lyon 1756, Vol IV, page 36).

#### Problem

Look for functions  $F(\kappa, \tau)$  so that the critical points of the energy functional, over a suitable space  $\Lambda$  of curves,

$$\mathcal{F}: \Lambda o \mathbb{R}, \quad \mathcal{F}(\gamma) = \int_{\gamma} F(\kappa, \tau) \, ds,$$

be helices.

### A simple case

$$\mathcal{F}_{mnp}(\gamma) = \int_{\gamma} (m + n\kappa + p\tau) ds,$$

The critical points for that action when m = 0, are just helices with slope h = p/n.

## Main aim: explaining closed helices

It is clear that there exist no closed helices  $\mathbb{E}^3$ , because its tangent field always has a positive component, precisely in the axis direction.

Now, there are known many cases of circular proteins in bacteria, plants and animals whose components form closed helices.

Then we have to propose a model to describe either circular proteins or any other more complicated configuration involving protein chains.

### Closed helices: Weiner's curves

Given  $n \in \mathbb{N}$ , the curve

$$\gamma_n(t) = \left( \left( \frac{n - \cos t}{n} \right) \cos \left( \frac{t}{n} \right); \left( \frac{n - \cos t}{n} \right) \sin \left( \frac{t}{n} \right); \frac{\sin t}{n} \right),$$

is closed of period  $2\pi n$ , and gets wound n times around the revolution torus obtained when rotating the circle centred at (1,0,0) and radius 1/n around the z-axis.

Those closed curves look like helices which can be found in Nature.

### **Closed** helices

Given a curve  $\gamma : [0, L] \to \mathbb{E}^3$ , let us define the function  $\mu : [0, L] \to \mathbb{R}$  by  $\mu(t) = \tau(t)/\kappa(t)$ . Then consider the width of its graphic

$$\mathsf{B}[\mu] = \max_{[0,L]} \mu - \min_{[0,L]} \mu.$$

By computing the width  $B[\mu_n]$  of the function  $\mu_n$  corresponding to the curve  $\gamma_n$  in a turn, say  $[0, 2\pi n]$ , then

$$\lim_{n\to\infty} \{B[\mu_n]\} = 0.$$

So, those curves can be seen as a kind of helices at the infinity.







 $\gamma_1$ 

 $\gamma_3$ 

 $\gamma_{10}$ 





### The axis, the key element.

We have to change the classic concept of helix in a obvious way. To do that we will relax the condition that the axis should be a straight line.

### Could the axis be a vector field?

**A crucial detail.** The above curves  $\gamma_n$  form a constant angle with a rotational field, i. e., with a vector field whose integral curves are circles.

#### Definition

Given a regular vector field X, the helices with axis X (also called X-helices) will be the curves making a constant angle with the integral curves of X.

### Closed helices: some remarks



non-closed spherical helix



closed toric helix

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## Closed helices: going from $\mathbb{E}^3$ to $\mathbb{S}^3$

### Program

To describe closed helices we will try to make use of our knowledge in  $\mathbb{E}^3$  to modify and improve that we deem appropriate.

 $(\underline{\text{Step 1}})$  The axis: the vector field whose integral curves are parallel straight lines (a Killing vector field) will be replaced by a vector field whose integral curves are parallel circles (a rotational or conformal Killing vector field).

### Questions to be solved:

- (1) Going from straight lines to circles; and
- (2) Going from parallel straight lines to "parallel circles".

 $\frac{(\text{Step 2}) \text{ Cylinders in } \mathbb{E}^3 \text{ will be replaced by Hopf tubes in } \mathbb{S}^3. }{\text{Problem: Building Hopf tubes throughout Hopf map.} }$ 

 $(\underline{Step\ 3})$  The results obtained in  $\mathbb{S}^3$  will be brought to  $\mathbb{E}^3$  via the stereographic projection.

### Closed helices: more remarks



cutting torus off by bitangent planes



### First characterization of helices in $\mathbb{S}^3$

### Helices in $\mathbb{S}^3$ (curvature version)

First of all, M. Barros [General helices and a theorem of Lancret. *Proc. AMS*, **125** (1997), 1503-1509] proved that a curve  $\gamma$  en  $\mathbb{S}^3$  is a helix if and only if either (i)  $\tau = 0$  and so  $\gamma$  lies in  $\mathbb{S}^2$ , totally geodesic in  $\mathbb{S}^3$ ; or (ii) The curvatures of  $\gamma$  satisfy

 $\tau = \lambda \kappa \pm 1$ ,  $\lambda$  being a constant.

The Hopf fibration:  $\mathbb{S}^3 \to \mathbb{S}^2$ 

We will see the 3-dimensional sphere  $\mathbb{S}^3$  of radius 1 in  $\mathbb{R}^4$  as

$$\mathbb{S}^3 = \{(x_1, y_1, x_2, y_2) \in \mathbb{R}^4 \ : \ x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1\}.$$

It is better thinking of  $(x_k, y_k)$  as a complex number  $z_k = x_k + iy_k$ , so that  $\mathbb{S}^3$  in  $\mathbb{R}^4$  can be seen as

$$\mathbb{S}^3 = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} \ : \ |z_1|^2 + |z_2|^2 = 1\}.$$



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The Hopf fibration:  $\mathbb{S}^3 \to \mathbb{S}^2$ 

We sketch  $\mathbb{S}^3$  as a planar circle in a complex plane. Then the  $z_1$ -axis, for instance, is a complex line, i. e., a real plane which intersects the sphere  $\mathbb{S}^3$  in the circle  $\{(z_1, 0) : |z_1|^2 = 1\} = \mathbb{S}^1$ . The same happens for the  $z_2$ -axis, as well as for any other straight line throughout the origin, i. e., for any straight line of the form  $z_2 = \lambda z_1$ ,  $\lambda$  being a complex number.

Then, any number  $\lambda \in \mathbb{C}$  defines a complex line  $z_2 = \lambda z_1$  cutting  $\mathbb{S}^3$  off in a circle. Indeed, there exists a circle in  $\mathbb{S}^3$  for any complex number  $\lambda$ . Furthermore, though the the equation of the  $z_2$ -axis is not as above, it can be viewed as the corresponding  $\lambda = \infty$ , because the slope of the vertical axis is  $\infty$ .

The Hopf fibration:  $\mathbb{S}^3 \to \mathbb{S}^2$ 

The Hopf map is defined as follows

$$h: \mathbb{S}^3 \subset \mathbb{C} \times \mathbb{C} \to \mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$$
  
 $(z_1, z_2) \mapsto \frac{z_2}{z_1}$ 

For each point  $\lambda \in \mathbb{S}^2$ , the fibre  $h^{-1}(\lambda)$ , which is a great circle in  $\mathbb{S}^3$ , is called a **Hopf circle**.

## The Hopf fibration: $\mathbb{S}^3 \to \mathbb{S}^2$

Therefore,  $\mathbb{S}^3$  is made of circles, so that we have a circle for each point of  $\mathbb{S}^2$ . Any two of them, for different values of  $\lambda$ , do not meet each other. This splitting of the sphere  $\mathbb{S}^3$  in circles is known as *the Hopf fibration*.





Any two projected fibres are linked circles, except  $s \circ h^{-1}(0,0,1)$  is a line

## The Hopf fibration: $\mathbb{S}^3 \to \mathbb{S}^2$

**Remark.** Fixing a parallel  $\wp$  in  $\mathbb{S}^2$  is the same as fixing the modulus of a complex number, so the preimage of a parallel is described by an equation of the form  $|z_2/z_1| = \text{constant}$ . For example, let us choose 1 for this constant so that  $z_1$  and  $z_2$  have the same modulus. But don't forget that  $|z_1|^2 + |z_2|^2 = 1$ , so the modulus of  $z_1$  and of  $z_2$  are both equal to  $\sqrt{2}/2$ . Therefore,  $h^{-1}(\wp)$  consists of the  $(z_1, z_2)$  where  $z_1$  and  $z_2$  are chosen arbitrarily on the circle centered at the origin with radius  $\sqrt{2}/2$ . Thus we see that the preimage of the parallel is a torus of revolution.



A practical vision of the 3-sphere

## The Hopf fibration: $\mathbb{S}^3 \to \mathbb{S}^2$

✓ The "parallel" passing through  $(z_1, z_2)$  is the set of points of the form  $(\lambda z_1, z_2)$ , where  $\lambda$  belongs to the circle of complex numbers of modulus 1.

✓ The "meridian" passing through  $(z_1, z_2)$  is the set of points of the form  $(z_1, \lambda z_2)$ .

✓ The Hopf circle passing through  $(z_1, z_2)$  is the set of points of the form  $(\lambda z_1, \lambda z_2)$ .

 $\checkmark$  We don't stop here either; through each point  $(z_1, z_2)$  we can also consider the "symmetric" circle of points of the form  $(\lambda z_1, \lambda^{-1} z_2)$  which gives us a fourth circle traced on the torus of revolution.

We have just shown that through each point of a torus of revolution one can draw four circles: a "meridian", a "parallel", a Hopf circle and the symmetric circle of a Hopf circle.

Compare with a torus in  $\mathbb{E}^3$ .



$$\delta: I \subset \mathbb{R} \to \mathbb{S}^2$$

$$\mathbf{T}_{\delta} = h^{-1}(\delta)$$

is a flat surface in  $\mathbb{S}^3$ , which we call **the Hopf tube over**  $\delta$ . It can be parametrized by  $X(s, t) = e^{it} \overline{\delta}(s)$ , where  $\overline{\delta}$  is a horizontal lifting of  $\delta$ .

### Helices in $\mathbb{S}^3$ (geodesic version)

A curve  $\gamma$  in  $\mathbb{S}^3$  is a helix if, and only if, up to congruences in  $\mathbb{S}^3,$  is a geodesic of a Hopf tube.

## Hopf vector field

$$S^{3} = \{z = (z_{1}, z_{2}) \in \mathbb{C} : |z|^{2} = |z_{1}|^{2} + |z_{2}|^{2} = 1\}$$

$$S^{1} \times S^{3} \rightarrow S^{3}$$

$$e^{it}z = (e^{it}z_{1}, e^{it}z_{2})$$

$$\frac{d}{dt}(e^{it}z)_{|t=0} = iz$$

$$X \in \mathfrak{X}(S^{3}), \quad X(z) = iz$$

## Clifford paralellism

**Def 1.** Let *d* be the metric on  $\mathbb{S}^3$  defined as

$$d(u,v)=\cos^{-1}(u,v),$$

i. e., d(u, v) is the length of the shorter arc, between u and v, of the great circle of  $\mathbb{S}^3$  through u and v. **Def 2.** Let C be a great circle of  $\mathbb{S}^3$ . Define

$$d(u, C) = \inf\{d(u, v), v \in C\}.$$

**Def 3.** For any two circles  $C_1$  and  $C_2$  of  $\mathbb{S}^3$ , define

$$d(C_1, C_2) = \inf\{d(u, C_2), v \in C_1\}.$$

**Def 4.** Two great circles  $C_1$  and  $C_2$  of  $\mathbb{S}^3$ , are **Clifford parallel** if

$$d(u, C_2) = d(v, C_2), \quad \forall u, v \in C_1.$$

Then we will write C||C'.

## Clifford paralellism

Two Hopf circles C and C' are Clifford parallel.

This follows from the fact that the action of  $\mathbb{S}^1$  on  $\mathbb{S}^3$ ,  $\lambda \cdot (u, v) = (\lambda u, \lambda v), \ \lambda \in \mathbb{S}^1$ , is isometric and transitive.

Given a Hopf circle  $C = h^{-1}(a)$  in  $\mathbb{S}^3$  and  $\theta \in [0, \pi]$ , we define

$$C_{\theta} = \{ x \in \mathbb{S}^3 : d(z, C) = \theta \}.$$



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#### Remarks.

(i)  $C_0 = C$ ,  $C_{\frac{\pi}{2}} = C^{\perp}$ ,  $C_{\pi-\theta} = C_{\theta}$ ,  $C_{\frac{\pi}{2}-\theta} = C_{\theta}^{\perp}$ , where  $C^{\perp}$  states for the great circle obtained when cutting  $\mathbb{S}^3$  off by the plane which is orthogonal to the plane containing C.

(ii)  $C_{ heta}=\{z=(z_1,z_2)\,:\,|z_1|=\cos heta,|z_2|=\sin heta\}$ , that is, is a torus.

(iii) Given a great circle C in  $\mathbb{S}^3$  and  $\theta \in [0, \pi]$ , for each  $z \in C_{\theta}$ , there exist exactly two great circles C' (of the first kind) and C'' (of the second kind) through z which are Clifford parallel to C. Furthermore,  $C' \neq C''$  if  $z \in \mathbb{S}^3 - \{C \cup C^{\perp}\}$ .

(iv) Clifford parallelism is not an equivalence relation, however it can be split in two of them.

## Villarceau circles

The flow of great circles Clifford parallel to C can be obtained from the action of an isometry group of  $\mathbb{S}^3$ . That is just that getting the Hopf fibration. Therefore, that flow is generated by the Hopf vector field (infinitesimal translation of the 3-sphere)

$$H(z) = i z.$$

Clifford parallelism points out two families (first and second kind) of circles, the so-called **Villarceau circles**.

It is enough to consider one of them, that of the first kind, for instance, which generates the Villarceau field V, which we have got by stereographic projection of the Hopf vector field H.

## Villarceau flow

Given  $z_0 \in \mathbb{S}^3$ , take a great circle C through  $z_0$  and consider the stereographic projection

$$E_0: \mathbb{S}^3 - \{z_0\} \rightarrow \mathbb{R}^3.$$

In  $\mathbb{E}^3$  we will take a coordinate system such that the Z-axis of  $E_0(C)$ .

Then,  $\forall \theta \in (0, \frac{\pi}{2})$ ,  $E_0(C_{\theta}) = T_{\theta}$  is a revolution torus around Z-axis in  $\mathbb{E}^3$  with radii  $\frac{1}{\cos \theta}$  and  $\frac{\sin \theta}{\cos \theta}$ , so that

$$\{T_{\theta} : \theta \in (0,\frac{\pi}{2})\}$$

is a foliation of  $\mathbb{E}^3 - \{Z - axis\}.$ 

### Summing up

When considering the Villarceau flow to study helices in  $\mathbb{E}^3$  we really are changing the Euclidean parallelism of parallel lines flow (generatrices of cylinders) by the conformal projection (angle preserving) of Clifford parallelism.

The advantage is clear, because the flow lines are circles, which is the prelude to get closed helices.

In this context, the two families of Villarceau circles in  $T_{\theta}$  are obtained as the images by  $E_0$  of the two families of great circles in  $C_{\theta}$  which are Clifford parallel to C.

Each one is a foliation of  $\mathbb{E}^3 - \{Z - axis\}$ .

Then we have got the Villarceau vector field

$$V = d\mathbf{E}_o(H), \qquad H = \text{Hopf},$$

whose integral curves are the Villarceau circles.

The V-helices are the Villarceau helices, among then we find out the quotes Weiner curves  $\gamma_n$ .

Anyway, they are geometrically described by the following result.

## Villarceau helices

The V-helices are loxodromes, regarding the Villarceau flow, in conformal Hopf tubes. So, they are completely determined by:

(1) A function playing the role of the curvature function of the conformal cross section; and

(2) A real number determining the slope with respect to the flow.

The closed V-helices will come determined by a rational link on the slope.

## The variational problem in $\mathbb{S}^3$

We have just seen that V-helices are exactly the geodesics of Hopf tubes. Then, they are determined by a function playing the role of the curvature, in  $S^2$ , of the cross section of the tube and a constant giving the slope of the geodesic in the tube.

### Helices in $\mathbb{S}^3$ (variational version)

These helices are, also, extremals of an energy functional of the following type

$$\mathcal{F}_{np}: \Lambda o \mathbb{R}, \qquad \mathcal{F}_{np}\left(\gamma\right) = \int_{\gamma} \left(p + n \,\kappa + p \,\tau\right) ds.$$

p/n being the slope of the helices.

## **Closing helices**

Let  $\delta$  be a curve in  $S^2(1/2)$  and  $\mathbf{T}_{\delta}$  a torus. To find its isometry group we will consider the covering map

$$X:\mathbb{E}^2 o \mathbf{T}_\delta, \quad X(s,t)=e^{it}\,ar{\delta}(s),$$

and we see that  $\mathbf{T}_{\delta}$  is isometric to  $\mathbb{E}^2/\Gamma$ , where  $\Gamma$  is the planar lattice spanned by (L, 2A) and  $(0, 2\pi)$ , L > 0 is the length of  $\delta$  and  $A \in (-\pi, \pi)$  is the enclosed area by  $\delta$  in  $\mathbb{S}^2(1/2)$ . As a consequence, a helix in  $\mathbb{S}^3$  is closed if, and only if, its slope h satisfies

$$h=rac{1}{L}(2A+q\pi), \quad q\in \mathbb{Q} ext{ rational.}$$

The existence of closed helices in a Hopf torus is guaranteed by means of the isoperimetric inequality in  $\mathbb{S}^2(1/2)$ . Indeed,

$$L^2 + (2A - \pi)^2 \ge \pi^2.$$

## **Closing helices**

Therefore, in the plane (L, 2A) we define the region

$$\Delta = \{ (L, 2A) \ : \ L^2 + (2A - \pi)^2 \ge \pi^2 \text{ and } 0 \le A \le \pi \}.$$

For any point  $a = (L, 2A) \in \Delta$  there exists an embedded and closed curve  $\delta_a$  in  $\mathbb{S}^2(1/2)$  of length L and enclosing an area A. The geodesic of slope h in the Hopf torus  $\mathbf{T}_{\delta_a}$  is closed if, and only if  $h = \frac{1}{L}(2A + q\pi)$ , i. e., the straight line in the plane (L, 2A) of slope h meets the 2A-axis at a height which is a rational multiple of  $\pi$ . Then, the space of closed helices in  $\mathbb{S}^3$  is the planar region

$$\Delta \cap \left( \cup_{q \in \mathbb{Q}} \left( \frac{p}{n} L - 2A = q\pi \right) \right).$$

## **Closing helices**

The space of closed helices has been obtained via a quatization principle and it is sketched in the figure.



Closed extremals for a given slope.

The y-coordinates are rational multiples of  $\pi$ .

## The principles

We have found out three principles to explain helices in Nature

#### Variational

The helicoidal structures in Nature are critical points of a certain energy.

### Topological

Topology becomes essential to explain closed helicoidal structures.

#### Quantization

For a given slope *h*, the helices close provided the *y*-coordinate of the straight line from which they come is a rational multiple of  $\pi$ .