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Null scrolls as solutions of a sigma model

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Abstract
The two-dimensional $\mathbf{O}(2, 1)$ nonlinear sigma model with boundary is considered. We calibrate the size of its space of field configurations by exhibiting new and wide classes of solutions. We first construct solutions by evolving, under a certain group of transformations, free elastic curves in any surface, either Riemannian or Lorentzian, of constant curvature. Furthermore, we show that any null scroll can provide a solution of this sigma model. This surprising phenomenon, which obviously has no Euclidean counterpart, guarantees the existence of an ample class of solutions which are generated by null (or lightlike) curves evolving through null ruling flows.

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1. Introduction

The existence of a close link between the differential geometry of nondegenerate surfaces in the three-dimensional Lorentz–Minkowski space, $\mathbb{L}^3$, and a wide variety of nonlinear phenomena is well known in physics and mathematics. For instance, problems in geometric analysis, Einstein and Ernst equations, quantum gravity, continuum mechanics, string theories, biological membranes, etc (see [3] and references therein). Many of those problems are related with functionals of the type

$$\mathcal{B}(M) = \int_M F(\mathrm{d}N)\,\mathrm{d}A,$$

where M is a nondegenerate surface in $\mathbb{L}^3$ with Gauss map N, F is a certain function and $\mathrm{d}A$ is the element of area, on M, of the induced metric from the Lorentzian one.

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We are interested in looking for critical points of such a functional in a given class of surfaces satisfying certain constraints, either a topological (free Willmore, free $\mathbf{O}(2, 1)$ nonlinear sigma model, genus one biological membranes, constant mean curvature surfaces, etc) or/and boundary conditions (Plateau, Willmore with boundary, $\mathbf{O}(2, 1)$ nonlinear sigma model with boundary, etc).

We pay attention to the $\mathbf{O}(2, 1)$ nonlinear sigma model because it is the simplest noncompact sigma model. As well as appearing in condensed matter and high energy physics, the two-dimensional nonlinear sigma model is interesting in connection with differential geometry (see [2, 3, 15] and references therein for general knowledge on this topic).

The two-dimensional $\mathbf{O}(2, 1)$ nonlinear sigma model is known to admit important solutions. The complete classification of those admitting a rotational symmetry was obtained in [3]. In particular, Lorentzian solutions with rotational symmetry are obtained by evolving, through a suitable one-dimensional group of transformations, free elastic curves ([12]) in one of the following three surfaces: a hyperbolic plane, a de Sitter plane or an anti-de Sitter plane.

It should be noted that these surfaces are in the class of six models, up to topology, of surfaces, Riemannian and Lorentzian, of constant curvature. Therefore, two questions naturally arise:

Q1. *Can one construct solutions of the sigma model by evolving curves in a round 2-sphere*?

Q2. *Can one construct solutions of this sigma model by evolving curves in either a Euclidean plane or a Lorentzian one*?

We will answer affirmatively to both questions and, once more, the problems will be concerned with searching for free elastic curves in these surfaces which, when evolved suitably, provide solutions of the sigma model. Nevertheless, the arguments that we use to reduce symmetry in both problems are quite different. To answer the former question we will construct the solutions in an anti-de Sitter three space using the so-called sausage decomposition. In this framework, we look for solutions which are invariant under a compact, one-dimensional group of isometries and then we can apply the principle of symmetric criticality in the sense of Palais [16] to reduce variables. Therefore, the problem is dealing with the free clamped elastic curves in a round 2-sphere. These curves evolve through a Killing vector field, in the anti-de Sitter three space, to generate surfaces which can be stereographically projected onto the Lorentz–Minkowski three space to provide a wide class of solutions of the sigma model.

To solve the second problem, we will use a direct approach to reduce symmetry. In fact, we will construct right cylinders over non-null planar curves and then we see that the Euler–Lagrange equation, associated with the sigma model, over these cylinders is concerned with that of free elasticity for their cross sections in the corresponding planes.

Once we have solved these problems, two other natural questions related with null (or lightlike) curves arise. Right cylinders with non-null cross sections are nondegenerate ruled surfaces whose rulings are never null curves. On the other hand, elastic curves, no matter where, are never null curves. Thus, we ask the following questions:

Q3. *Do there exist solutions of the sigma model among those ruled surfaces with null rulings*?

Q4. *Can one construct solutions of the sigma model by evolving null curves*?

In this paper, we give an ample, simple and surprising response to these, *a priori*, unrelated questions by showing that *any null scroll in* $\mathbb{L}^3$ provides a solution for the two-dimensional $\mathbf{O}(2, 1)$ *nonlinear sigma model*. In other words, the presence, in a Lorentzian surface, of a null ruling flow does not impose any restriction on the directrix to generate a ruled surface, with null rulings, which is a solution of the sigma model. In particular, we can construct solutions starting from arbitrary null curves. Furthermore, this remarkable result allows us to obtain the classification of those solutions with constant mean curvature. Besides the family of stationary surfaces, the family of B-scrolls also appears. The moduli spaces of both families of Lorentzian surfaces are made up of infinitely many members. This result has no Euclidean counterpart, where minimal surfaces and pieces of round spheres provide the only constant mean curvature solutions of the two-dimensional $\mathbf{O}(3)$ nonlinear sigma model [2].

2. The nonlinear sigma model with boundary

The two-dimensional $\mathbf{O}(2, 1)$ nonlinear sigma model with boundary actually covers two different but closely related field theories. In both cases, the source space is a surface, $\mathbf{S}$, with boundary, $\partial\mathbf{S}$. Nevertheless, in the first one the target space is the de Sitter plane, $\mathbb{S}^2_1$, while in the second one that space is the hyperbolic plane, $\mathbb{H}^2$. Now, the elementary fields are maps, $\mathbf{N}$, from $\mathbf{S}$ into either $\mathbb{S}^2_1$ or $\mathbb{H}^2$ with the constraint that $\mathbf{N}(\partial\mathbf{S})$ is fixed and the models are governed by the energy:

$$\mathcal{D}(\mathbf{N}) = \int_{\mathbf{S}} \|\mathrm{d}\mathbf{N}\|^2 \,\mathrm{d}A,$$

where $\mathrm{d}A$ is the element of area for the pullback metric on the surface. Now, the solutions of these models are those satisfying the following vectorial field equation:

$$\Delta\mathbf{N} - (\mathbf{N}.\Delta\mathbf{N})\,\mathbf{N} = 0, \tag{1}$$

with the obvious meaning, Δ being the Laplacian of the above-mentioned metric.

The Gauss map of any nondegenerate surface in the Lorentz–Minkowski three space, $\mathbb{L}^3$, can be viewed as an elementary field of this sigma model with target space being $\mathbb{S}^2_1$ or $\mathbb{H}^2$ according to the surface is timelike (Lorentzian) or spacelike (Riemannian), respectively. Consequently, we use the geometrical approach that was initiated in [2, 15] for the sigma model with symmetry $\mathbf{O}(3)$ and then continued in [3] for this noncompact sigma model. Accordingly, the space of dynamical variables of the two-dimensional $\mathbf{O}(2, 1)$ nonlinear sigma model is identified with that of Gauss maps of nondegenerate surfaces in $\mathbb{L}^3$. Since the case of spacelike surfaces is quite similar to the Euclidean one (see [3]), in order to mark differences we will restrict ourselves to timelike surfaces so that the model target space is $\mathbb{S}^2_1$.

Let ϕ be a timelike immersion from $\mathbf{S}$ in $\mathbb{L}^3$, then its Gauss map, $\mathbf{N}_\phi$, is defined as a map from $\phi(\mathbf{S})$ onto $\mathbb{S}^2_1$. Therefore, in order to translate the usual boundary condition to this setting, it is necessary to consider immersions fixing $\partial\mathbf{S}$, which, of course, does not imply any additional constraint. Consequently, the space of elementary fields of this sigma model can be described as follows. Start from a set of non-null regular curves, Γ, in $\mathbb{L}^3$ and a spacelike unit normal vector field, N_o, along Γ. Then we consider the space $\mathbf{I}_\Gamma(\mathbf{S}, \mathbb{L}^3)$ of timelike immersions satisfying the following first-order boundary conditions

$$\phi(\partial\mathbf{S}) = \Gamma, \qquad \mathbf{N}_\phi/_\Gamma = N_o.$$

Roughly speaking, if we identify each immersion $\phi \in \mathbf{I}_\Gamma(\mathbf{S}, \mathbb{L}^3)$ with its graph, $\phi(\mathbf{S})$, viewed as a surface with boundary in $\mathbb{L}^3$, then $\mathbf{I}_\Gamma(\mathbf{S}, \mathbb{L}^3)$ can be viewed as the space of timelike surfaces in $\mathbb{L}^3$ having the same boundary and being tangent along the common boundary. It should be noted that the first-order boundary conditions have been considered in the compact case along the literature (see for example [2, 7] and some references therein). The energy governing the model $\mathcal{D} : \mathbf{I}_\Gamma(\mathbf{S}, \mathbb{L}^3) \to \mathbb{R}$ is now written as

$$\mathcal{D}(\phi) = \int_{\mathbf{S}} \|\mathrm{d}\mathbf{N}_\phi\|^2 \,\mathrm{d}A_\phi,$$

where $\mathrm{d}A_\phi$ stands for the element of area of $(\mathbf{S}, \phi^*(\bar{g}))$ and the solutions of the two-dimensional $\mathbf{O}(2, 1)$ nonlinear sigma model are just the critical points of $\big(\mathbf{I}_\Gamma(\mathbf{S}, \mathbb{L}^3); \mathcal{D}\big)$.

This way to see the model has several advantages, perhaps the more interesting is that it allows us to describe the solutions in terms of their geometrical invariants. This can be done directly, though we will use a more elegant procedure which, in addition, will allow us to show its conformal invariance. It will be convenient to introduce, on the same space of elementary fields, the Willmore problem with boundary (see [2, 3, 7, 17] and references therein), which is associated with the Willmore energy $\mathcal{W} : \mathbf{I}_\Gamma(\mathbf{S}, \mathbb{L}^3) \to \mathbb{R}$ defined by

$$\mathcal{W}(\phi) = \int_{\mathbf{S}} \mathbf{H}_\phi^2 \, \mathrm{d}A_\phi - \int_{\partial\mathbf{S}} \kappa_\phi \, \mathrm{d}s,$$

where $\mathbf{H}_\phi$ stands for the mean curvature of the immersion $\phi(\mathbf{S})$ and κ_ϕ is the geodesic curvature of $\phi(\partial\mathbf{S})$ in $\phi(\mathbf{S})$. The critical points, i.e. the solutions of this model, are known as Willmore surfaces in $\mathbb{L}^3$. It should be pointed out that this variational problem is invariant under conformal changes in $\mathbb{L}^3$, so it is actually stated in the conformal class, $[\bar{g}]$, of the Lorentz–Minkowski metric $\bar{g}$. It will be convenient to remark that, in both cases, a critical point means a critical point for the corresponding induced problems in non-null polygons, that is, those polygons whose boundaries are made of non-null pieces. The amazing fact is that these two field theories are equivalent. The solutions of the two-dimensional $\mathbf{O}(2, 1)$ nonlinear sigma model are just the Willmore surfaces. In particular, the nonlinear sigma model is invariant under conformal changes in $\mathbb{L}^3$.

To show this claim, we first observe that the main parts of both Lagrangian densities are related by

$$\|\mathrm{d}\mathbf{N}_\phi\|^2 = 4\mathbf{H}_\phi^2 - 2\mathbf{K}_\phi, \qquad \forall\phi \in \mathbf{I}_\Gamma(\mathbf{S}, \mathbb{L}^3),$$

where $\mathbf{K}_\phi$ denotes the Gaussian curvature of $(\mathbf{S}, \phi^*(\bar{g}))$. Now, we can apply the Gauss–Bonnet formula for general non-null polygons (see [3] for details):

$$-\int_{\mathbf{P}} \mathbf{K}_\psi \, \mathrm{d}A_\psi + \int_{\partial\mathbf{P}} \kappa(s) \, \mathrm{d}s + \sum_{j=1}^{r} \theta_j = 0,$$

where κ stands for the geodesic curvature of the boundary, while θ_j are the angles in the corners of the boundary. Therefore,

$$\mathcal{D}(\mathbf{P}) = 4\mathcal{W}(\mathbf{P}) + 2\int_{\partial\mathbf{P}} \kappa(s) \, \mathrm{d}s - 2\sum_{j=1}^{r} \theta_j,$$

and we note that, under the boundary conditions that we are considering, the term $\int_{\partial\mathbf{P}} \kappa(s) \, \mathrm{d}s - \sum_{j=1}^{r} \theta_j$ is constant under corresponding variations, which concludes the proof.

Now, the two-dimensional $\mathbf{O}(2, 1)$ nonlinear sigma model with boundary is invariant under conformal changes in the Lorentz–Minkowski. Moreover, the solutions of this model, that is the solutions of equation (1), appear as Willmore surfaces with boundary. Consequently, we can translate this equation to the new setting by computing the field equation providing the Willmore surfaces. It was done in [3] (see also [17] for the Riemannian case), so we will give some brief indications. First, it should be observed that the tangent space to the space $\mathbf{I}_\Gamma(\mathbf{S}, \mathbb{L}^3)$ at ϕ is naturally identified with that of vector fields, $\mathbf{V}$, along the immersion ϕ that vanish on $\partial\mathbf{S}$. Consequently, we use the boundary conditions to obtain $\partial\mathcal{D}(\phi)[\mathbf{V}] = 4\,\partial\mathcal{W}(\phi)[\mathbf{V}]$. Next, one can use standard variational arguments involving several integrations by parts to see that

$$\partial\mathcal{W}(\phi)[\mathbf{V}] = \int_S \left(\Delta_\phi \mathbf{H}_\phi + 2\mathbf{H}_\phi \left(\mathbf{H}_\phi^2 - \mathbf{K}_\phi\right)\right) \bar{g}(\mathbf{N}_\phi, \mathbf{V}) \, \mathrm{d}A_\phi.$$

Therefore, the solutions of (1) are the same as those of the following field equation for Willmore surfaces:

$$\Delta\mathbf{H} + 2\mathbf{H}\left(\mathbf{H}^2 - \mathbf{K}\right) = 0. \tag{2}$$

3. Solutions through elastic curves in the 2-sphere

We will take advantage of two chief points to exhibit solutions of the two-dimensional $\mathbf{O}(2, 1)$ nonlinear sigma model which are obtained from free elastic curves in the round 2-sphere.

(1) We will use the conformal invariance of the model to see $\mathbb{L}^3$, via the stereographic projection, as an anti-de Sitter three space.
(2) Since the evolution of profile curves, to find the solutions, is obtained through a one-dimensional compact group, we can apply the Palais symmetric criticality principle to reduce the symmetry [16].

The anti-de Sitter three space with curvature 1 may be defined as the hyperquadric

$$\mathbb{H}^3_1 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 - |z_2|^2 = -1\},$$

endowed with the induced metric, $\tilde{g}$, from that of $\mathbb{C}^2_1 \equiv \mathbb{R}^4_2$. Our first proposal is to see this space as a warped product 1 (see [5] for a detailed description). To do it, we take a hyperbolic plane endowed with its metric, g, with curvature 1, say

$$\mathbb{H}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = -1, z > 0\}.$$

Next, define the positive function $f : \mathbb{H}^2 \to \mathbb{R}$ by $f(x, y, z) = z$ and use it as a warping function to obtain $\mathbb{H}^2 \times_f -\mathbb{S}^1$, which is nothing but the semi–Riemannian product with the metric $g - f^2\, \mathrm{d}t^2$. Finally, consider the map $F : \mathbb{H}^2 \times \mathbb{S}^1 \to \mathbb{H}^3_1$ defined by $F((x, y, z), \mathrm{e}^{\mathrm{i}t}) = (x + \mathrm{i}y, z\mathrm{e}^{\mathrm{i}t})$ to see that it is a global isometry between $\mathbb{H}^2 \times_f -\mathbb{S}^1$ and $\mathbb{H}^3_1$. Therefore, we can see the anti-de Sitter three space as $\mathbb{H}^2 \times_f -\mathbb{S}^1$ and consequently, we have a natural action of $\mathbb{S}^1$ on this space. This action, in the previous picture, is viewed as follows:

$$\mathbb{S}^1 \times \mathbb{H}^3_1 \to \mathbb{H}^3_1, \qquad \mathrm{e}^{\mathrm{i}s}(z_1, z_2) = (z_1, \mathrm{e}^{\mathrm{i}s} z_2).$$

It is clear that the surfaces invariant under this action are obtained just in the following way: a surface $\mathbf{S}$ is invariant if and only if there exists a curve γ in $\mathbb{H}^2$ such that $\mathbf{S} = \gamma \times \mathbb{S}^1$. Then we will write $\mathbf{S}_\gamma = \gamma \times \mathbb{S}^1$. Of course, all invariant surfaces are Lorentzian. Therefore, we have the following question:

Q5. *How should the curve γ be chosen in order for $\mathbf{S}_\gamma$ to be a solution of the sigma model, or equivalently a Willmore surface in $\mathbb{H}^3_1$?*

In other words, we wish to determine all invariant Willmore surfaces. To be coherent with the symmetry of the problem, we should begin with boundary conditions which are invariant themselves. Therefore, the boundary, Γ, is made up of two orbits or fibres, that is,

$$\Gamma = \{\gamma_1, \gamma_2\}, \qquad \gamma_i = \{p_i\} \times \mathbb{S}^1, \qquad p_i \in \mathbb{H}^2, \qquad 1 \leqslant i \leqslant 2.$$

Since both curves are timelike, the vector field N_o along Γ is spacelike. The topology of $\mathbf{S}$ is chosen to be $\mathbf{S} = [a_1, a_2] \times \mathbb{S}^1$ and the space of elementary fields, $\mathbf{I}_\Gamma(\mathbf{S}, \mathbb{H}^3_1)$, is made up of immersions $\phi : \mathbf{S} \to \mathbb{H}^3_1 \equiv \mathbb{H}^2 \times_f (-\mathbb{S}^1)$ satisfying

(1) $\phi(\{a_i\} \times \mathbb{S}^1) = \{p_i\} \times \mathbb{S}^1$, $1 \leqslant i \leqslant 2$; and
(2) ϕ has spacelike Gauss map with $\mathbf{N}_\phi/_\Gamma = N_o$.

To answer the above question, we use an argument with two main ingredients. First, we make a conformal change in the metric of the anti-de Sitter space, which turns out to be obvious after the previous warped product decomposition. Just note that the conformal anti-de Sitter space, $\left(\mathbb{H}^3_1, \hat{g} = \frac{1}{f^2}\tilde{g}\right)$, is isometric to the semi-Riemannian product $\left(\mathbb{H}^2, \frac{1}{f^2} g\right) \times (\mathbb{S}^1, -\mathrm{d}t^2)$. Furthermore, it is not difficult to see that $\left(\mathbb{H}^2, \frac{1}{f^2} g\right)$ is isometric to the once punctured 2-sphere with constant curvature 1. In particular, both conformal metrics provide the same Willmore energy.

On the other hand, the space of invariant surfaces is $\Sigma = \{\mathbf{S}_\beta\}$, where β is a clamped curve in $\mathbb{H}^2$, that is,

$$\beta : [a_1, a_2] \to \mathbb{H}^2, \qquad \beta(a_i) = p_i, \qquad \beta'(a_i) = \vec{v}_i, \qquad 1 \leqslant \mathrm{i} \leqslant 2,$$

where $\vec{v}_i$ are fixed vectors obtained from the second boundary condition. Furthermore, the $\mathbb{S}^1$ action on $\mathbb{H}^3_1$ is extended, in a natural way, to $\mathbf{I}_\Gamma\big(\mathbf{S}, \mathbb{H}^3_1\big)$ and the Willmore energy remains invariant under that action. Consequently, the principle of Palais [16] applies in the following terms. The surface $\mathbf{S}_\gamma$ is Willmore if and only if it is a critical point of the Willmore energy restricted to the space of symmetric surfaces Σ. To compute this restriction, we use the metric $\hat{g}$. Now the mean curvature of $\mathbf{S}_\gamma$ is given by $\mathbf{H} = \frac{1}{2}\kappa$, κ being the curvature of γ in the once punctured sphere $\big(\mathbb{H}^2, \frac{1}{f^2}\, g\big)$.

Now, the second term appearing in the Willmore energy vanishes identically because the tangent plane of any invariant surface is a mixed section in a semi-Riemannian product. The boundary term in the Willmore energy also vanishes; in fact any fibre is a geodesic in $\big(\mathbb{H}^3_1, \hat{g} = \frac{1}{f^2}\, \tilde{g}\big)$ and consequently it is also a geodesic in $\mathbf{S}_\gamma$. Therefore,

$$\mathcal{W}(\mathbf{S}_\gamma) = \frac{\pi}{2} \int_\gamma \kappa^2 \, \mathrm{d}s.$$

The curves, in any semi-Riemannian space, which are critical points of the total squared curvature are called free elastic curves. So we answer the above-stated question in the following terms.

Theorem 3.1. $\mathbf{S}_\gamma$ *is a solution of the two-dimensional* $\mathbf{O}(2, 1)$ *nonlinear sigma model if and only if its profile curve,* γ*, is a clamped free elastic curve in the unit sphere.*

To calibrate the size of the class of free elastic curves in a round 2-sphere, and so the family of solutions for the sigma model that we have just obtained, we recall that the Euler–Lagrange equation associated with the above elastic energy is given by [12]

$$2\kappa'' + \kappa^3 + 2\kappa = 0.$$

It is clear that the geodesics ($\kappa = 0$) are the only solutions with constant curvature. Let us indicate how to obtain the whole class of solutions (see [12] for details). The above equation can be completely integrated, in fact, we use $u(s) = \kappa^2(s)$ and the equation can be written as $u' = P(u)$, for a certain third degree polynomial $P(u)$. Now, the nonconstant solutions must take on values at which $P(u) > 0$ and consequently the polynomial has three real roots satisfying $-a_1 < 0 = a_2 < a_3$ and $a_1 - a_3 = 4$. Now, the equation can be completely integrated using Jacobi elliptic functions, the general solutions are

$$u(s) = \kappa^2(s) = a_3\left(1 - \mathbf{sn}^2\left(\frac{1}{2}\sqrt{a_1 + a_3}, p\right)\right), \qquad p^2 = \frac{a_3}{a_3 + a_1};$$

therefore, the whole class of free elastic curves in the unit 2-sphere is made up of geodesics and the one parameter class of curves whose curvature functions are

$$\kappa(s) = \frac{2p}{\sqrt{1 - 2p^2}}\mathbf{cn}\left(\frac{1}{\sqrt{1 - 2p^2}}s, p\right), \qquad 0 < p < \frac{\sqrt{2}}{2}.$$

These curves are known as wavelike elasticae. Each wavelike elastica is known to oscillate across a geodesic, its axis, so that one can define its wavelength as the amount of progess it makes along its axis in a complete curvature period. In this setting, one can study the so-called *closed curve problem* for wavelike elasticae to get a rational one parameter class of elastic curves in the unit 2-sphere. Consequently, we obtain a rational one parameter class of Willmore tori in the anti-de Sitter three space and so, via the stereographic projection, in $\mathbb{L}^3$. The final conclusion is that we have a rational one parameter class of tori, in $\mathbb{L}^3$ that provide solutions of the two-dimensional O(2,1) nonlinear sigma model. It should be noted that the case of closed surfaces (compact and boundary free) is a special case of the model with boundary that we are considering.

4. Solutions through null scrolls

We already know the existence of wide classes of Willmore surfaces, in general with boundary, in the conformal Lorentz–Minkowski three space. All these surfaces are obtained by evolving certain non-null curves which we will call profile curves. The evolution is governed by the action of a one-dimensional group of either isometries (see [3]) or conformal transformations, like in the class of surfaces in $\mathbb{H}^3_1$ that we have exhibited in the previous sections. Furthermore, the profile curves are elastic ones in either of the standard surfaces: 2-sphere $\mathbb{S}^2$, hyperbolic plane $\mathbb{H}^2$, de Sitter plane $\mathbb{S}^2_1$ and anti-de Sitter plane $\mathbb{H}^2_1$.

To enlarge that list with both Euclidean and Lorentzian planes, we use the following argument. Let γ be any non-null plane curve in $\mathbb{L}^3$ and $\mathbf{P}$ be the plane containing the curve. It is clear that $\mathbf{P}$ is either Euclidean, $\mathbb{R}^2$, or Lorentzian, $\mathbb{L}^2$. In any case, it is possible to construct the right cylinder on the given curve, $\mathbf{C}_\gamma$, which turns out to be a nondegenerate flat surface in $\mathbb{L}^3$. Furthermore, it is Lorentzian if $\mathbf{P}$ is Euclidean, while when $\mathbf{P}$ is Lorentzian then the cylinder will be Lorentzian or Riemannian according to the causal character of the curve. For suitable boundary conditions, one can use the field equation associated with the sigma model. Then, compute all the ingredients of $\mathbf{C}_\gamma$ appearing in that equation, which becomes the Euler–Lagrange equation for free elastic curves in the plane $\mathbf{P}$. In other words, we have that $\mathbf{C}_\gamma$ is a solution of the sigma model if and only if γ is a free elastic curve in the plane $\mathbf{P}$. This argument allows us to obtain, up to motions in $\mathbb{L}^3$, two other one parameter classes of solutions. The first one is made up of right cylinders whose cross sections are free elastic curves in a Euclidean plane and so the corresponding curvature functions are the solutions of

$$2\kappa'' + \kappa^3 = 0,$$

that is,

$$\kappa(s) = \sqrt{\alpha}\,\mathbf{cn}\left(\sqrt{\frac{\alpha}{2}}s, \sqrt{\frac{1}{2}}\right), \qquad \alpha > 0.$$

The second class of solutions is provided for timelike elastic curves in a Lorentzian plane, which appear associated with curvature functions that are solutions of the following Euler–Lagrange equation:

$$2\kappa'' - \kappa^3 = 0,$$

which can be also integrated in terms of Jacobi elliptic functions.

It should be noted that the above two classes of solutions constitute the whole class of right cylinders with non-null ruling flows that give solutions for the two-dimensional O(2,1) nonlinear sigma model with boundary. Consequently, we can construct solutions of the the two-dimensional O(2,1) nonlinear sigma model with boundary by evolving profile curves that are elastic ones in any surface, Riemannian or Lorentzian, with constant curvature.

Theorem 4.1. *There exist wide classes of solutions of the two-dimensional* $\mathbf{O}(2,1)$ *nonlinear sigma model obtained from elastic curves in any surface, Riemannian or Lorentzian, with constant curvature.*

The last two families of solutions that we have just obtained are right cylinders, and so ruled surfaces, with non-null rulings in $\mathbb{L}^3$. Therefore, we pay attention to those solutions that appear as ruled surfaces with null rulings. In particular, these solutions must be Lorentzian surfaces (timelike surfaces) which are solutions of (2). It is obvious that every surface with vanishing mean curvature (stationary surface) is automatically a solution for this model. In addition, Lorentzian surfaces with nonzero constant mean curvature are solutions of this sigma model if and only if $\mathbf{K} = \mathbf{H}^2$ and so it is also constant. This is the case of the five classes of helicoidal surfaces, orbits of plane curves under Lorentzian screw motions, studied in [11], which turn out to be ruled surfaces with null rulings and so examples of the null scrolls that we are going to consider.

On the other hand, elastic curves anywhere refer to non-null curves and we would like to investigate the existence of solutions being shaped when evolving null curves. This leads us to consider questions **Q3** and **Q4**. The null scrolls will come, once more, to help us (see [1] for a detailed study, [9] and [13]) to answer both questions through a simple and surprising response.

A null scroll is a ruled surface with a null ruling in $\mathbb{L}^3$. That is, we have a suitable curve $\beta(s)$, no matter its causal character, in $\mathbb{L}^3$ and a null vector field $B(s)$ along the curve such that the null scroll with directrix β and ruling flow $B(s)$ is parametrized by

$$\mathbf{X}(s,t) = \beta(s) + tB(s),$$

such that $\langle \beta'(s), B(s)\rangle \neq 0$ and $\langle B(s), B(s)\rangle = 0$. When the directrix β is non-null, then we can reparametrize it and then normalize the ruling flow if necessary to have

$$\langle \beta'(s), \beta'(s)\rangle = \varepsilon = \pm 1, \qquad \langle \beta'(s), B(s)\rangle = -1.$$

Now, we can determine a function $t(s)$ in order to $\gamma(s) = \beta(s) + t(s)B(s)$ be a null curve. Just obtain $t(s)$ as a solution of the differential equation (see [1])

$$\langle \gamma'(s), \gamma'(s)\rangle = \varepsilon + 2t(s)\langle \beta'(s), B'(s)\rangle + t^2(s)\langle B'(s), B'(s)\rangle - 2t'(s) = 0.$$

Consequently, a null scroll $\mathbf{S}(\gamma, B)$ can be reparametrized such that directrix and ruling flow being both null, namely

$$X(s,t) = \gamma(s) + tB(s)$$

and

$$\langle \gamma'(s), \gamma'(s)\rangle = \langle B(s), B(s)\rangle = 0, \qquad \langle \gamma'(s), B(s)\rangle = -1.$$

In this parametrization the induced metric on $\mathbf{S}(\gamma, B)$ is given by the following matrix:

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 2t\langle \gamma'(s), B'(s)\rangle + t^2\langle B'(s), B'(s)\rangle & -1 \\ -1 & 0 \end{pmatrix}.$$

Consequently, it is a Lorentzian surface whose Laplacian, in this reference, is given by

$$\Delta = -2\frac{\partial^2}{\partial s \partial t} - 2\left(\langle \gamma', B'\rangle + t\langle B', B'\rangle\right)\frac{\partial}{\partial t} - \left(2t\langle \gamma', B'\rangle + t^2\langle B', B'\rangle\right)\frac{\partial^2}{\partial t^2}.$$

To compute the Gauss map and the shape operator of $\mathbf{S}(\gamma, B)$, we define $C(s) = \gamma'(s) \times B(s)$. Then, it is a unit spacelike vector field along $\gamma(s)$, which is anywhere orthogonal to $\mathbf{S}(\gamma, B)$ and so it defines the Gauss map of the null scroll along its directrix. The Gauss map is given by

$$N(s,t) = \frac{X_s(s,t) \times X_t(s,t)}{|g_{11}g_{22} - g_{12}^2|} = C(s) + tB'(s) \times B(s).$$

Now, an easy computation allows us to see that

$$B'(s) \times B(s) = -f(s)B(s), \quad \text{where} \quad f(s) = \langle \gamma'(s), B'(s) \times B(s)\rangle = \det[\gamma'(s), B'(s), B(s)],$$

and then the Gauss map of the null scroll is given by

$$N(s,t) = -tf(s)B(s) + C(s).$$

Sometimes the function $-f(s)$ is called the *parameter of the distribution* of the ruled surface $\mathbf{S}(\gamma, B)$.

A straightforward computation allows one to obtain

$$N_s = -fX_s - \left(\langle\gamma', \gamma'' \times B\rangle + tf'\right) X_t$$

and

$$N_t = -fX_t,$$

so the matrix of the shape operator dN is

$$\mathrm{d}N \equiv \begin{pmatrix} f & tf' + \langle\gamma', \gamma'' \times B\rangle \\ 0 & f \end{pmatrix}.$$

Therefore, mean and Gauss curvatures functions of the null scroll $\mathbf{S}(\gamma, B)$ are given by

$$\mathbf{H}(s, t) = f(s) \qquad \text{and} \qquad \mathbf{K}(s, t) = f(s)^2.$$

This gives us an important information, namely

$$\Delta\mathbf{H}(s, t) = 0, \qquad \mathbf{H}^2(s, t) - \mathbf{K}(s, t) = 0,$$

which we bring to the Euler–Lagrange equation (2) to obtain the following.

Theorem 4.2.

(1) Every null scroll is a Willmore surface in $\mathbb{L}^3$.
(2) Null scrolls provide solutions of the two-dimensional $\mathbf{O}(2, 1)$ *nonlinear sigma model.*

It should be noted that the helicoidal surfaces studied in [11] are special examples of null scrolls. They have both constant mean curvature and Gaussian curvature. Usually, null scrolls with constant mean curvature are called B-scrolls. It should also be observed that the equation $\mathbf{H}^2 = \mathbf{K}$, in Euclidean three space $\mathbb{E}^3$, is only satisfied for umbilical surfaces (pieces of either planes or 2-spheres). In this sense, sometimes B-scrolls are called generalized umbilical surfaces [14]. In this paper, we have shown that null scrolls satisfy $\mathbf{H}^2 = \mathbf{K}$ (see also [8]), that is, they are quadratic Weingarten surfaces. In addition, it can be proved that null scrolls are the only Lorentzian surfaces in the Lorentz–Minkowski three space which satisfy that relationship between mean and Gaussian curvatures. To show this statement, we show that, locally, any Lorentzian surface in $\mathbb{L}^3$ can be parametrized through two families of null curves, [10]. Let $\mathbf{Y}(u, v)$ be such a parametrization and write the metric and the shape operator of the surface in this frame as

$$\begin{pmatrix} 0 & g_{12} \\ g_{12} & 0 \end{pmatrix}, \qquad \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix},$$

and then we obtain

$$\mathbf{H}^2 - \mathbf{K} = \frac{h_{11}h_{22}}{g_{12}^2} = 0.$$

Without loss of generality, we may assume that $h_{22} = 0$ and consequently $\mathbf{Y}_{vv} = \lambda(u, v)\,\mathbf{Y}_u + \mu(u, v)\,\mathbf{Y}_v$. However, $\langle\mathbf{Y}_{vv}, \mathbf{Y}_v\rangle = 0$ and so $\lambda = 0$. Now, we compute a first integral of $\mathbf{Y}_{vv} = \mu(u, v)\,\mathbf{Y}_v$ to obtain

$$\mathbf{Y}_v(u, v) = \left(\exp \int_0^v \mu(u, r)\mathrm{d}r\right)\vec{x},$$

where $\vec{x}$ is a fixed null vector. This shows that v-curves of the parametrization are null lines and the surface is a null scroll.

Consequently, we obtain the following characterization of null scrolls among Lorentzian surfaces in $\mathbb{L}^3$.

Theorem 4.3. *A Lorentzian surface in the Lorentz–Minkowski three space is a null scroll if and only if* $\mathbf{H}^2 - \mathbf{K} = 0$*.*

Furthermore, as a consequence of the last two theorems, we can obtain the classification of those Lorentzian surfaces with constant mean curvature providing solutions of the sigma model.

Corollary 4.4. *Let* $\mathbf{S}$ *be a Lorentzian surface, with constant mean curvature, in* $\mathbb{L}^3$*. Then, it is a solution of the two-dimensional* $\mathbf{O}(2, 1)$ *nonlinear sigma model if and only if either*

(1) $\mathbf{S}$ *is stationary or*

(2) $\mathbf{S}$ *is a B-scroll.*

Certainly, one can find surfaces belonging to both families in the last statement. In fact, the existence of infinitely many congruence classes of stationary *B*-scrolls has been shown in [4], where, for instance, the moduli space (or space of congruence classes) of a stationary *B*-scroll with the same directrix has been identified with $\mathbb{S}^1$.

The Lorentzian ruled surfaces with non-null ruling flows which are stationary were classified in [6]. The complete moduli space is made up of five congruence classes associated with a plane, three helicoids and a Enneper surface (see [6] for details on their parametrizations and more). We combine this statement with ours to give the following uniqueness result which provides the whole space of congruence classes of solutions for the two-dimensional O(2,1) nonlinear sigma model with boundary.

Corollary 4.5. *Let* $\phi(\mathbf{S})$ *be a ruled surface, with constant mean curvature in* $\mathbb{L}^3$*. Then, it is a solution of the* $\mathbf{O}(2, 1)$ *nonlinear sigma model if and only if one of the following statements holds:*

(1) $\phi(\mathbf{S})$ *has non-null ruling flow and then it is congruent to a surface in the following list:*

- *(1.1) A Lorentzian plane.*
- *(1.2) A helicoid of the first kind.*
- *(1.3) A helicoid of the second kind.*
- *(1.4) A helicoid of the third kind.*
- *(1.5) The conjugate surface of Enneper of the second kind.*

(2) $\phi(\mathbf{S})$ *has null ruling flow and then it is congruent to a surface in the following list:*

- *(2.1) A Lorentzian plane.*
- *(2.2) A B-scroll.*

Remark 4.6. The following points should be observed. The helicoids and the Enneper surface appearing in the above classification are ruled surfaces with non-null ruling flows ([6]) which have zero mean curvature (they are stationary). However, *B*-scrolls are ruled surfaces with null ruling flows that have constant mean curvature, which can or cannot be zero. Certainly, this family contains infinitely many congruence classes. For example, the complete and explicit description of the moduli space of stationary ($\mathbf{H} = 0$) *B*-scrolls has been obtained in [4], while this problem for nonzero constant mean curvature *B*-scrolls is still an open problem.

5. Conclusions

The space of field configurations of the two-dimensional $\mathbf{O}(2, 1)$ nonlinear sigma model contains the following classes of solutions:

(1) Right cylinders whose section is a free elastic curve in a Euclidean plane.
(2) Right cylinders whose section is a timelike free elastic curve in a Lorentzian plane.
(3) Rotational surfaces with either timelike or null axis and profile curve being a timelike free elastic curve in an anti-de Sitter plane.
(4) Rotational surfaces with a spacelike axis and profile curve being a timelike free elastic curve in the de Sitter plane.
(5) Rotational surfaces with a spacelike axis and profile curve being a free elastic curve in the hyperbolic plane.
(6) Surfaces obtained as images, under the stereographic map, of tubes whose section is a free elastic curve in the 2-sphere.
(7) Null scrolls.

The richness of solutions of the two-dimensional $\mathbf{O}(2, 1)$ nonlinear sigma model should be noted. Its space of field configurations contain wide classes of rotational surfaces (3)–(5), as well as a class of surfaces which admits a one-dimensional conformal symmetry (6). It also contains classes of ruled surfaces with either non-null rulings, (1) and (2), or null rulings (7). However, the following facts, which once more show the importance of the null scrolls, should be pointed out. While solutions in classes (1)–(6) have their counterparts in the corresponding Euclidean sigma model with symmetry $\mathbf{O}(3)$, surfaces in class (7) have no counterpart in this framework. Even more, if we restrict ourselves to ruled solutions, the presence of null rulings does not impose any restriction on the directrices, which can also be chosen to be null. *Whenever a curve propagates in the spacetime through a geodesic flow, it generates a solution of the simplest noncompact model*. This phenomenon obviously does not occur if curves evolve through non-null geodesic flows generating worldsheets that are solutions only under very strong restrictions on the propagating curve.

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