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Epicycloids generating Hamiltonian minimal surfaces in the complex quadric

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ABSTRACT

We show that Hopf tubes on Lancret curves shaped over an epicycloid are Hamiltonian minimal surfaces in the complex quadric. Moreover they are the only Hopf tubes that are Hamiltonian minimal there. This allows one to connect two apparently unrelated topics, such as Hamiltonian minimal surfaces and curves with constant precession, and more generally slant helices. Furthermore, Hamiltonian minimal Hopf tubes encode the phases of particles described according to the gyroscopic force theory.

1. Hamiltonian minimal surfaces

Let M be a surface and let $\mathcal{I}(M, \mathbb{S}^3)$ be the space of immersions of M into the three-dimensional unit sphere $\mathbb{S}^3$. The Gauss map $g_\phi : M \to \mathcal{G}_2(\mathbb{R}^4)$ of an immersion $\phi \in \mathcal{I}(M, \mathbb{S}^3)$, taking values in the Grassmannian of two planes $\mathcal{G}_2(\mathbb{R}^4)$ in $\mathbb{R}^4$, is given by

$$g_\phi(p) = \phi(p) \wedge N_\phi(p),$$

where N_ϕ stands for the unit normal vector field of (M, ϕ) in $\mathbb{S}^3$. Hence, g_ϕ maps any point of M to its normal plane, via ϕ, in $\mathbb{R}^4$. It is well known that this Grassmannian is naturally identified with $\mathbb{S}^2(\frac{1}{\sqrt{2}}) \times \mathbb{S}^2(\frac{1}{\sqrt{2}})$. It can be regarded as a complex hypersurface of $\mathbb{CP}^3(4)$, more precisely, it is the only compact nonlinear complex hypersurface of the complex

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projective three space, with constant holomorphic sectional curvature 4, which is Einstein. In this sense, it is usually known as the complex quadric. The corresponding Kaehler form ω defines a symplectic structure on the complex quadric such that $g_\phi^*(\omega) = 0$ along each Gauss map. Then $\mathcal{H} = \{g_\phi : \phi \in \mathcal{I}(M, \mathbb{S}^3)\}$ provides a class of Lagrangian surfaces in $(\mathcal{G}_2(\mathbb{R}^4), \omega)$. Actually each Lagrangian immersion in the complex quadric locally arises as the Gauss map of an immersion in the unit three sphere (see [1]). Furthermore, $\mathcal{H}$ is a Hamiltonian submanifold of $\mathcal{I}(M, \mathcal{G}_2(\mathbb{R}^4))$, that is, any vector field tangent to $\mathcal{H}$ appears as the ω-gradient of a smooth function. Therefore, one has a correspondence between Hamiltonian variations of g_ϕ and variations of g_ϕ through Gauss maps.

Let $\mathcal{A} : \mathcal{I}(M, \mathbb{S}^3) \to \mathbb{R}$ be the functional measuring the area of Gauss mappings, i.e. $\mathcal{A}(\phi) = \mathrm{Area}(g_\phi(M))$. It is not difficult to see that

$$\mathcal{A}(\phi) = \int_M \sqrt{\det(\mathrm{d}N_\phi^2 + I)}\,\mathrm{d}A,$$

where $\mathrm{d}N_\phi$ stands for the shape operator of (M, ϕ) in $\mathbb{S}^3$ and $\mathrm{d}A$ is the area element of $\phi^*(h)$, h being the metric on $\mathbb{S}^3$. Then, the critical points of $\left(\mathcal{I}(M, \mathbb{S}^3), \mathcal{A}\right)$ correspond to the Gauss maps $\phi \in \mathcal{H}$ with vanishing the first variation of area for Hamiltonian variations. These surfaces were called Hamiltonian minimal surfaces (see [2]). It is known that if ϕ is minimal, then g_ϕ is also minimal. These critical points will be called trivial ones. Hence, it is natural to analyze the existence of nontrivial critical points of $\left(\mathcal{I}(M, \mathbb{S}^3); \mathcal{A}\right)$. This problem was studied in [1] when M is a compact surface. Actually the main result there showed that the only compact critical points are the trivial ones which are regularly homotopic to an embedded surface or whose genus is either zero or an odd number. In particular, embedded minimal tori in $\mathbb{S}^3$ are the only immersed Hamiltonian minimal tori in the three sphere. On the other hand, a popular conjecture by Lawson, [3], states that the only minimally embedded in $\mathbb{S}^3$ is the Clifford torus (see [4] for related problems). Consequently if Lawson's conjecture holds, then the Clifford torus should be the only Hamiltonian minimal torus in the three sphere. It seems natural to study this constrained variational problem for tubes, and more generally for surfaces with boundary.

In [5], the Hamiltonian minimal variational problem of extremizing the area among surfaces in $\mathbb{C}^2$, that are Lagrangian and fill in a given boundary, was considered. This question is, on the other hand, motivated by issues arising in nonlinear elasticity and mirror symmetry. Now, we introduce the corresponding constrained variational problem for tubes in the three sphere. A tube is a cylinder with boundary made up of two regular closed curves. Nevertheless, the extension to other kinds of boundaries can be introduced similarly.

The first order boundary conditions. Let $\Gamma(t) = \{\alpha_1(t), \alpha_2(t)\}$ be a pair of nonintersecting regular closed curves in $\mathbb{S}^3$. For a unit vector field $\eta(t)$ along $\Gamma(t)$ with $\langle\Gamma'(t), \eta(t)\rangle = 0$, we have the field of two planes in $\mathbb{R}^4$ along $\Gamma(t)$ defined by $\mathrm{R}(t) = \Gamma(t) \wedge \eta(t)$.

The boundary value problem. Associated with the first order boundary data (Γ, R), we have the following boundary value problem. Let $M = [a_1, a_2] \times \mathbb{S}^1$ be the surface with boundary $\partial M = \mathbf{C}_1 \cup \mathbf{C}_2$ and $\mathbf{C}_j = \{a_j\} \times \mathbb{S}^1$, $1 \le j \le 2$. Let $\mathcal{I}_{\mathcal{B}}(M, \mathbb{S}^3)$ denote the space of immersions $\phi : M \to \mathbb{S}^3$ satisfying the following boundary conditions

1. $\phi(\partial M) = \Gamma$, or $\phi(\mathbf{C}_j) = \alpha_j$, $1 \le j \le 2$, and
2. $g_\phi(p) = \mathrm{R}(\phi(p))$, $\forall p \in \partial M$.

The dynamics associated with the Hamiltonian minimal variational problem with boundary $\left(\mathcal{I}_{\mathcal{B}}(M, \mathbb{S}^3); \mathcal{A}\right)$ can be roughly stated as follows. By identifying each immersion $\phi \in \mathcal{I}_{\mathcal{B}}(M, \mathbb{S}^3)$ with its graph $\phi(M)$, viewed as a surface with boundary in $\mathbb{S}^3$, we propose the study of the Lagrangian $\mathcal{A}$ in the class of tubes with the same boundary and the same Gauss map along the common boundary. In this sense, it could be considered as a nonlinear $\mathbf{SO}(4)$ sigma model with boundary (see [6]).

2. Hamiltonian minimal Hopf tubes

Let $\Pi : \mathbb{S}^3 \to \mathbb{S}^2(\frac{1}{2})$ be the usual Hopf map, where we have considered the two sphere with radius $1/2$ so that Π becomes a Riemannian submersion. Let $\gamma : I \subset \mathbb{R} \to \mathbb{S}^2(\frac{1}{2})$ be an immersed curve which we can assume, without loss of generality, arc length parametrized. The complete lift $M_\gamma = \Pi^{-1}(\gamma)$ of γ is a surface in $\mathbb{S}^3$ that can be covered by the following map

$$\Phi : I \times \mathbb{R} \to M_\gamma, \qquad \Phi(s, t) = \mathrm{e}^{\mathrm{i}t}\bar{\gamma}(s),$$

where $\bar{\gamma}$ stands for a horizontal lift of γ. This map can be used to parametrize the surface whose coordinate curves are the horizontal lifts of γ (t constant) and the fibres (s constant), respectively. Then $\langle\Phi_s, \Phi_s\rangle = \langle\Phi_t, \Phi_t\rangle = 1$ and $\langle\Phi_s, \Phi_t\rangle = 0$, showing that $M_\gamma = \Pi^{-1}(\gamma)$ is a flat surface in $\mathbb{S}^3$ which we will call the Hopf tube on γ. Its unit normal vector field in $\mathbb{S}^3$ is $N = i\,\Phi_s$.

The shape operator of M_γ, relative to the frame $\{\Phi_s, \Phi_t\}$, is given by

$$\mathrm{d}N_\gamma \equiv \begin{pmatrix} \bar{\kappa} & 1 \\ 1 & 0 \end{pmatrix},$$

κ and $\bar{\kappa} = \kappa \circ \Pi$ being the curvature functions of γ and $\bar{\gamma}$ in $\mathbb{S}^2(1/2)$ and $\mathbb{S}^3$, respectively.

Choose now a pair $\Gamma(t) = \{\alpha_1(t), \alpha_2(t)\}$ of nonintersecting equidistant geodesics in $\mathbb{S}^3$. Certainly, up to motions in $\mathbb{S}^3$, they can be seen as fibres, via the Hopf map, over a couple of different points

$$\alpha_j = \Pi^{-1}(p_j), \quad p_j \in \mathbb{S}^2(1/2), \ 1 \le j \le 2, \quad p_1 \ne p_2.$$

Then, take a unit vector field $\eta(t)$, along $\Gamma(t)$, such that $\langle \eta(t), \alpha_j'(t)\rangle = 0$ and invariant under the action of $\mathbb{S}^1$ on $\mathbb{S}^3$. Note, in particular, that $\eta(t)$ is horizontal along $\Gamma(t)$ and it projects onto the two sphere to give a pair of unit vectors $y_j \in T_{p_j}\mathbb{S}^2(\frac{1}{2})$, $1 \le j \le 2$. Finally, we define the boundary Gauss map associated with the field $\mathrm{R}(t) = \Gamma(t) \wedge \eta(t)$ of two planes along $\Gamma(t)$.

Remark. In [7], K. Enomoto observed that the Gauss map of any flat surface in $\mathbb{S}^3$, which is obviously non-degenerate, is locally the Riemannian product of two curves δ_1 and δ_2, where δ_1 lies in the first factor of $\mathcal{G}_2(\mathbb{R}^4) = \mathbb{S}^2 \times \mathbb{S}^2$ and δ_2 in the second one. Later, J. L. Weiner, [8] (see also [9]), obtained the complete classification of those Riemannian products, $\delta_1 \times \delta_2$, of closed curves in $\mathcal{G}_2(\mathbb{R}^4) = \mathbb{S}^2 \times \mathbb{S}^2$ which are images under the Gauss maps of flat tori in $\mathbb{S}$. This characterization is given in terms of both the total curvature of δ_i in the two sphere, which must be zero, and the total curvature of any sub-arc of δ_i, which must be less than $\pi/2$. In particular, when the torus is a Hopf torus, then one of the curves, say δ_1, is a great circle in the corresponding two-sphere factor. As we are considering Hopf cylinders with boundary, the curve δ_2 is a curve in $\mathbb{S}^2$ connecting two points which are obtained from the boundary conditions. They satisfy, following the same computations as in [8], that the total curvature of any sub-arc is always less than $\pi/2$.

With this choice of $\mathbb{S}^1$-invariant boundary conditions, we consider the constrained variational problem $\left(\mathcal{I}_{\mathcal{B}}(M, \mathbb{S}^3); \mathcal{A}\right)$, where $M = [a_1, a_2] \times \mathbb{S}^1$ and $[a_1, a_2] \subset I$, and state the following

Problem. How can we choose γ so that $M_\gamma = \Pi^{-1}(\gamma)$ will be Hamiltonian minimal, i. e., a critical point of $\left(\mathcal{I}_{\mathcal{B}}(M, \mathbb{S}^3); \mathcal{A}\right)$?

It is clear that the unit circle $\mathbb{S}^1$ acts on $\mathbb{S}^3$ through isometries to get $\mathbb{S}^2(\frac{1}{2})$ as space of orbits. This action can be naturally extended to $\mathcal{I}_{\mathcal{B}}(M, \mathbb{S}^3)$. In fact, for any $\phi \in \mathcal{I}_{\mathcal{B}}(M, \mathbb{S}^3)$ and $t \in \mathbb{R}$, we define $\mathrm{e}^{\mathrm{i}t} \circ \phi \in \mathcal{I}_{\mathcal{B}}(M, \mathbb{S}^3)$ by

$$(\mathrm{e}^{\mathrm{i}t} \circ \phi)(p) = \mathrm{e}^{\mathrm{i}t}\phi(p).$$

The set of immersions invariant under this action, i.e.

$$\mathcal{S}_{\mathcal{B}}(M, \mathbb{S}^3) = \{\phi \in \mathcal{I}_{\mathcal{B}}(M, \mathbb{S}^3) \ : \ \mathrm{e}^{\mathrm{i}t} \circ \phi = \phi, \ \forall t \in \mathbb{R}\},$$

constitutes a submanifold of $\mathcal{I}_{\mathcal{B}}(M, \mathbb{S}^3)$ which we will call the submanifold of symmetric points.

On the other hand, as $\mathcal{A}(\mathrm{e}^{\mathrm{i}t} \circ \phi) = \mathcal{A}(\phi)$, the variational problem $\left(\mathcal{I}_{\mathcal{B}}(M, \mathbb{S}^3); \mathcal{A}\right)$ is also invariant under that action, which provides a setting where the principle of symmetric criticality [10] works out. As a consequence, the critical points of $\left(\mathcal{I}_{\mathcal{B}}(M, \mathbb{S}^3); \mathcal{A}\right)$ which are symmetric, that is, the symmetric Hamiltonian minimal surfaces, in the sense of [2], are nothing but the critical points of $\left(\mathcal{S}_{\mathcal{B}}(M, \mathbb{S}^3); \mathcal{A}\right)$.

The space of symmetric points is just that of Hopf tubes on curves in $\mathbb{S}^2(\frac{1}{2})$ that are defined in $[a_1, a_2] \subset I \subset \mathbb{R}$, and whose tangent vectors at the ending points are orthogonal to y_j, $1 \le j \le 2$, respectively,

$$\mathcal{S}_{\mathcal{B}}(M, \mathbb{S}^3) \equiv \left\{ M_\gamma = \Pi^{-1}(\gamma) \mid \gamma : [a_1, a_2] \to \mathbb{S}^2\left(\frac{1}{2}\right), \ \langle \gamma'(a_j), y_j\rangle = 0, \ 1 \le j \le 2 \right\}.$$

To compute the restriction of $\mathcal{A}$ on $\mathcal{S}_{\mathcal{B}}(M, \mathbb{S}^3)$, it should be observed that the corresponding Lagrangian density is

$$\sqrt{\det(\mathrm{d}N_\gamma^2 + I)} = \sqrt{\kappa^2 + 4} \circ \Pi.$$

Now, we have

$$\mathcal{A}(M_\gamma) = \int_{\gamma \times \mathbb{S}^1} \left(\sqrt{\kappa^2 + 4} \circ \Pi\right) \mathrm{d}s\,\mathrm{d}t = 2\pi \int_\gamma \sqrt{\kappa^2 + 4}\,\mathrm{d}s,$$

so the problem is reduced to that for clamped curves to get

Answer: *A Hopf tube $M_\gamma = \Pi^{-1}(\gamma)$ is Hamiltonian minimal if and only if it comes from a curve γ in $\mathbb{S}^2(\frac{1}{2})$ which is a critical point of the action $\mathcal{C} : \Lambda \to \mathbb{R}$ defined by*

$$\mathcal{C}(\gamma) = \int_\gamma \sqrt{\kappa^2 + 4}\,\mathrm{d}s,$$

where $\Lambda = \{\gamma : [a_1, a_2] \to \mathbb{S}^2(1/2) \mid \langle \gamma'(a_j), y_j\rangle = 0, \ 1 \le j \le 2\}$.

Therefore, the problem of finding out Hamiltonian minimal Hopf tubes with boundary is reduced to that of looking for clamped curves in the two sphere which are critical points of an energy action $\mathcal{C}$ that measures the total curvature of these curves in the Euclidean space. This variational problem was considered in [11], where it was shown that critical points are just plane curves. However, $\mathcal{C}$ only acts on curves in the two sphere (compare with [12]). Then, we can state a general action, with Lagrangian density $P(\kappa)$ satisfying $P'(\kappa) = \frac{dP}{d\kappa} \neq 0$ anywhere, defined by

$$\mathcal{Q}(\gamma) = \int_\gamma P(\kappa(s))\,ds,$$

acting on a space $\mathbf{C}$ of curves defined in some interval $I \subset \mathbb{R}$ and so, in principle, without any boundary conditions (see for example [13,14]). The first variation $\delta\mathcal{Q}_\gamma : \mathbf{TC}_\gamma \to \mathbb{R}$ can be computed as usual to get

$$\delta\mathcal{Q}_\gamma(W) = \int_I \langle \Omega(\gamma), W\rangle ds + \int_I \frac{d}{ds}\mathcal{B}(\gamma, W) ds, \qquad W \in \mathbf{TC}_\gamma,$$

where $\Omega(\gamma)$ and $\mathcal{B}(\gamma, W)$ stand for the Euler–Lagrange and the, a priori, boundary operators, respectively. One can use the Frenet equations for curves in the two sphere to get the following expressions for these operators [13,15]

$$\Omega(\gamma) = \left[\frac{d^2}{ds^2}\left(P'(\kappa)\right) + (\kappa^2+4)P'(\kappa) - \kappa P(\kappa)\right]\xi,$$

and

$$\mathcal{B}(\gamma, W) = P'(\kappa)\langle \xi, \nabla_T W\rangle - \frac{d}{ds}\left(P'(\kappa)\right)\langle \xi, W\rangle + \left(P(\kappa) - \kappa\, P'(\kappa)\right)\langle T, W\rangle,$$

where $\{T = \frac{d\gamma}{ds}, \xi\}$ stands for a Frenet frame along the curve γ and ∇ denotes the Levi-Civita connection of the round two sphere.

The critical points are those curves γ satisfying $\delta\mathcal{Q}(\gamma)[W] = 0$ for any variational field W along γ. Hence, suitable choices of W yield the Euler–Lagrange equation $\Omega(\gamma) = 0$. Conversely, to get a characterization of critical points, we need some boundary conditions. For example, by considering the variational problem associated with a functional $\mathcal{Q}$ acting on the space $\Lambda = \{\gamma : [a_1, a_2] \to \mathbb{S}^2(\frac{1}{2}) \mid \langle \gamma'(a_j), y_j\rangle = 0,\ 1 \le j \le 2\}$ of clamped curves, we have

$$\int_{a_1}^{a_2} \frac{d}{ds}\mathcal{B}(\gamma, W) ds = [\mathcal{B}(\gamma, W)]_{a_1}^{a_2}.$$

On the other hand, a variation of γ within Λ, that is, along clamped curves, is a proper variation, so $W(a_1) = W(a_2) = 0$. Moreover, a direct computation gives

$$\nabla_T W = \nabla_W T + W(\log v)\,T,$$

where v denotes the speed of curves in the variation associated with the variational field W. Then

$$[\mathcal{B}(\gamma, W)]_{a_1}^{a_2} = 0,$$

and so the formula for the first variation of the action $\mathcal{Q}$ on the space of clamped curves Λ is

$$\delta\mathcal{Q}(\gamma)[W] = \int_I \langle \Omega(\gamma), W\rangle ds.$$

Then the critical points of $(\Lambda, \mathcal{Q})$ are just the solutions of the Euler–Lagrange equation $\Omega(\gamma) = 0$.
In particular, when the action is $\mathcal{C}$, that is $P(\kappa) = \sqrt{\kappa^2+4}$, then the field equation turns out to be

$$\frac{d^2}{ds^2}\left(\frac{\kappa}{\sqrt{\kappa^2+4}}\right) = 0.$$

To summarize, the critical points are described as follows.

The critical points of the variational problem $(\Lambda, \mathcal{C})$ *are closed pieces of the class of maximal curves described as follows. For any couple of constant* $a > 0$ *and* b*, the arc length parametrized curve*

$$\gamma_{ab} : \left(\frac{-1-b}{a}, \frac{1-b}{a}\right) \subset \mathbb{R} \to \mathbb{S}^2(1/2),$$

with curvature function

$$\kappa_{ab}(s) = \frac{2(as+b)}{\sqrt{1-(as+b)^2}}$$

is a critical point of the functional $\mathcal{C}$ *acting on suitable curves in* $\mathbb{S}^2(\frac{1}{2})$*. Furthermore, all critical points are obtained in this way.*

These solutions γ_{ab} are completely determined from its curvature function in $\mathbb{S}^2(\frac{1}{2})$. Now, they can be seen in $\mathbb{R}^3$, where such curvature functions are given by

$$\bar{\kappa}_{ab}(s) = \sqrt{\kappa_{ab}^2 + 4} = \frac{2}{\sqrt{1-(as+b)^2}}.$$

So the action $\mathcal{C}$ measures the total curvature of curves in Euclidean space. To completely determine them we also have to know the torsion function τ_{ab}, which is given by

$$\bar{\tau}_{ab} = \frac{2}{\bar{\kappa}_{ab}\sqrt{\bar{\kappa}_{ab}-4}}\left(\frac{\mathrm{d}}{\mathrm{d}s}\bar{\kappa}_{ab}\right),$$

yielding

$$\bar{\tau}_{ab} = \frac{a}{\sqrt{1-(as+b)^2}}.$$

As a consequence, we have

$$\tau_{ab} = \frac{a}{2}\,\bar{\kappa}_{ab},$$

showing that the solutions are Lancret curves in the Euclidean space with slope $\cot\theta = a/4$ (see [16] and references therein). From a classical result [17], we know that any spherical Lancret curve in Euclidean space projects down onto a plane orthogonal to its axis to gives an epicycloid.

Then, we have obtained a 4-step geometric algorithm to construct all Hamiltonian minimal Hopf tubes. For the sake of simplicity, as the parameter b does not play an essential role, we choose it to be zero.

1. In a plane $\mathbf{P} \subset \mathbb{R}^2$, let us choose a one-parameter family $\{\beta_\theta : \theta \in \mathbb{R}\}$ of epicycloids which are determined by a couple of radii $R_1 = \frac{1}{2}\cos\theta$ and $R_2 = \frac{1}{4}(1-\cos\theta)$.
2. Let $\mathbf{C}_\theta$ be the right cylinder shaped on the epicycloid β_θ and choose the geodesic γ_θ with slope θ in $\mathbf{C}_\theta$. This curve is a Lancret one in the Euclidean space with curvature and torsion given, in terms of its arc length s, by
$$\bar{\kappa_\theta}(s) = \frac{2}{\sqrt{1-m^2s^2}}, \qquad \bar{\tau}_\theta(s) = \frac{m}{\sqrt{1-m^2s^2}},$$
where $m = 2\cot\theta$.
3. The curves $\{\gamma_\theta : \theta \in \mathbb{R}\}$ lie in $\mathbb{S}^2(\frac{1}{2})$ and they are determined, up to motions in $\mathbb{S}^2(\frac{1}{2})$, by the following curvature function
$$\kappa_\theta(s) = \frac{2m\,s}{\sqrt{1-m^2s^2}}.$$
This family of curves constitutes the moduli space of solutions of the variational problem $(\mathbb{S}^2(\frac{1}{2}); \mathcal{C})$.
4. The class of Hopf tubes $\{\Pi^{-1}(\gamma_\theta) : \theta \in \mathbb{R}\}$ constitutes the whole space of Hopf tubes which are Hamiltonian minimal.

3. Some applications

The family of Hamiltonian minimal Hopf tubes, that we have just obtained, can also be constructed using curves of constant precession. This method presents a double advantage. On one hand, it allows us to connect Hamiltonian minimal Hopf tubes with the precessional genesis of atomic structure. On the other hand, it will provide Hamiltonian minimal Hopf tori with a finite number of singular circles.

Let $\delta(s)$ be a unit speed curve in $\mathbb{R}^3$ with Frenet apparatus $\{T_\delta, N_\delta, B_\delta, \kappa_\delta, \tau_\delta\}$. When the curve is traversed, an instantaneous rotation appears which is determined by an angular velocity vector, the Darboux vector $\Omega_\delta(s)$. It is usually called the centrode of the curve, which is given by $\Omega_\delta = \tau_\delta\, T_\delta + \kappa_\delta\, B_\delta$.

The curve $\delta(s)$ is said to be of constant precession if its centrode revolves about a fixed line, in $\mathbb{R}^3$, with constant angle and constant speed. A curve is of constant precession if and only if $\kappa_\delta(s) = p\,\sin qs$ and $\tau_\delta(s) = p\,\cos qs$, up to reflection or phase shift or arclength, for constants p and q. The *solving natural equations problem* and the *closed curve problem* for curves of constant precession were solved in [18]. In particular, it was proved that a curve of constant precession lies on the one-sheet hyperboloid of revolution

$$x^2 + y^2 - \frac{q^2}{p^2}z^2 = \frac{4q^2}{p^4}.$$

Furthermore it closes if and only if $\frac{q}{\sqrt{p^2+q^2}}$ is rational.

Now, the main point is that the tangent indicatrix $T_\delta(s)$ of a curve $\delta(s)$ of constant precession is a Lancret curve with axis $A_\delta = \Omega_\delta(s) - q\,N(s)$. In fact, it is clear that if $\delta(s)$ is of constant precession, then $A' = 0$, so A is actually a fixed vector. On the other hand, the angle (i.e. the slope) θ that the tangent indicatrix $T_\delta(s)$ makes with A satisfies that

$$\cos\theta = \frac{q}{\sqrt{p^2+q^2}}.$$

Therefore, the tangent indicatrix of any curve of constant precession is a spherical Lancret helix, sometime called three-dimensional epicycloid.

Observe, however, that for our purposes, we have to select spherical Lancret helices in the sphere with radius $\frac{1}{2}$. Then, in the above argument regarding a curve of constant precession, we will consider the spherical Lancret helix $\frac{1}{2}\,T_\delta(s)$ in $\mathbb{S}^2(\frac{1}{2})$.

For any Pythagorean triple $(p, q, r = \sqrt{p^2+q^2}) = (n^2 - m^2, 2mn, m^2 + n^2)$, $m, n \in \mathbb{N}$, we have a closed curve, $\delta(s) = (x(s), y(s), z(s))$ of constant precession in $\mathbb{R}^3$

$$x(s) = \frac{1}{2r}\left[\frac{r+q}{r-q}\,\sin(r-q)s - \frac{r-q}{r+q}\,\sin(r+q)s\right],$$
$$y(s) = \frac{1}{2r}\left[-\frac{r+q}{r-q}\,\cos(r-q)s + \frac{r-q}{r+q}\,\cos(r+q)s\right],$$
$$z(s) = \frac{p}{qr}\sin qs.$$

The curve $\frac{1}{2}\,T_\delta(s)$ is a closed piecewise smooth curve. Namely, it is a sequence of arcs of spherical Lancret helices in $\mathbb{S}^2(\frac{1}{2})$ joining each other by cusps at their endpoints, just those points where $\kappa_\delta(s) = 0$.

Now, it should be observed that the variational problem describing Hamiltonian minimal Hopf tubes can be easily extended to piecewise smooth curves. We can also consider the free closed version where surfaces are tori. Then $\Pi^{-1}(\frac{1}{2}\,T_\delta(s))$ provides a rational one-parameter class of Hamiltonian minimal tori with a finite number of singular orbits. This result should be compared with that in [1], where any Hamiltonian minimal torus must be minimal and so trivial.

The class of curves of constant precession is a subclass of that of slant helices [19]. They are curves without inflection points and whose principal normal lines makes a constant angle with a fixed direction. It is known that the tangent and binormal spherical indicatrices of a slant helix are both Lancret curves. Then, we can reobtain Hamiltonian minimal Hopf tubes as the Hopf tubes shaped on either the tangent indicatrix or the binormal indicatrix of slant helices.

In the quantum mechanical formalism, three of the four essential quantum numbers involve angular momentum. Of course, this is unacceptable from a classical point of view. The gyroscopic force theory provides a reasonable classical explanation of the angular momentum approach (see [20]). The seminal principle of this framework is that all motion, even rectilinear one, is comprised of some component of angular momentum and therefore all motions and their concomitant forces may be expressed in terms of such. In this context, one has a precessional genesis of atomic structure and according to which the orbital path of an electron in an atom is characterized as the centrode of a curve undergoing constant precession. Therefore, the Hamiltonian minimal Hopf tubes can be regarded as surfaces encoding all possible phases of the wavefunction of an electric charge moving into a vicinity of a magnetic monopole.

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References

[1] B. Palmer, Hamiltonian minimality and Hamiltonian stability of Gauss maps, Differ. Geom. Appl. 7 (1997) 51–58.
[2] Y.G. Oh, Volume minimization of Lagrangian submanifolds under Hamiltonian deformations, Math. Z. 212 (1993) 175–192.
[3] H.B. Lawson, Complete minimal surfaces in $\mathbb{S}^3$, Ann. Math. 92 (1970) 355–374.
[4] M. Barros, A. Ferrández, P. Lucas, Lawson-type problems in non standard 3-spheres, Quart. J. Math. Oxford 50 (1999) 385–388.
[5] R. Schoen, J. Wolfson, Minimizing area among Lagrangian surfaces, J. Differential Geom. 58 (2001) 1–86.
[6] M. Barros, A geometric algorithm to construct new solitons in the $\mathbf{O}(3)$ nonlinear sigma model, Phys. Lett. B 553 (2003) 325–331.
[7] K. Enomoto, The Gauss image of flat surfaces in $\mathbb{R}^4$, Kodai Math. J. 9 (1986) 19–32.
[8] J.L. Weiner, Flat tori in $\mathbb{S}^3$ and their Gauss Maps, Proc. London Math. Soc. 62 (1991) 54–76.
[9] K. Enomoto, Global properties of the Gauss image of flat surfaces in $\mathbb{R}^4$, Kodai Math. J. 10 (1987) 272–284.
[10] R.S. Palais, The principle of symmetric criticality, Commun. Math. Phys. 69 (1979) 19–30.
[11] J. Arroyo, M. Barros, O.J. Garay, Some examples of critical points for the total mean curvature functional, Proc. Edinb. Math. Soc. 43 (2000) 587–603.
[12] L. Santaló, Curvas sobre una superficie extremales de una función de la curvatura y torsión, Abh. Hambrugische Univ. 20 (1956) 216–222.
[13] J. Arroyo, Presión calibrada total, estudio variacional y aplicaciones al problema de Chen-Willmore, Ph.D. Thesis U.P.V., 2001.
[14] A. Feoli, V.V. Nesterenko, G. Scarpetta, Functionals linear in curvature and statistics of helical proteins, Nuclear Phys. B 705 (2005) 577–592.
[15] M.A. Javaloyes, Sumersiones pseudo-Riemannianas y modelos geométricos de partículas relativistas, Ph.D. Thesis, Universidad de Murcia, 2004.
[16] M. Barros, General helices and a theorem of Lancret, Proc. AMS 125 (1997) 1503–1509.
[17] D.J. Struik, Lectures on Classical Differential Geometry, Dover, New-York, 1988.
[18] P.D. Scofield, Curves of constant precession, Amer. Math. Monthly 102 (1995) 531–537.
[19] L. Kula, Y. Yayli, On slant helix and its spherical indicatrix, Appl. Math. Comput. 169 (2005) 600–607.
[20] W. Johnson Jr., The Gyroscopic Force Theory, a classical theory on the unification of electromagnetism and gravity, introducing tovacian chemistry. http://www.rci.rutgers.edu/williebo/.