A conformal variational approach for helices in nature

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We propose a two step variational principle to describe helical structures in nature. The first one is governed by an energy action which is a linear function in both curvature and torsion allowing to describe nonclosed structures including elliptical, spherical, and conical helices. These appear as rhumb lines in right cylinders constructed over plane curves. The model is completed with a conformal alternative which, in particular, gives a description of closed structures. The energy action is linear in the curvatures when computed in a conformal spherical metric. Now, helices appear as making a constant angle with a Villarceau flow and so they are loxodromes in surfaces which are stereographic projections of Hopf tubes, in particular, anchor rings, revolution tori, and Dupin cyclides. The model satisfies the requirements of simplicity and beauty as reflected in the three main principles that head its construction: least action, topological, and quantization. According to the latter, the main entities and quantities associated with the model should not be multiplied unnecessarily but they are quantized. In this sense, a quantization principle, *a la* Dirac, is obtained for closed structures and also for the critical levels of energy. © *2009 American Institute of Physics.* [doi:10.1063/1.3236683]

I. INTRODUCTION: HELICAL CONFIGURATIONS IN NATURE

Helical configurations are structures commonly found in nature. They appear in microscopic systems (biomolecules, bacterial fibers, nanosprings, protein chains in particular DNA, etc.) as well as in macroscopic phenomena (strings, ropes, climbing plants, coiled springs, horns of mountain goats, vortices, etc.) (see, for example, Refs. 2, 5, 10–12, 21, and 25 and references therein). In particular, they are very important and ubiquitous in biology as a consequence of the following known, in the biological community since the work of Pauling, theorem: *Identical objects, regularly assembled, form a helix* (see Ref. 6 and references therein). These structures are so basic ingredients of the spectacle of the universe, which becomes so much grander and so much beautiful when one gets a small number of laws, most wisely established, which will suffice to obtain mathematical models to describe the experimental phenomena. Several mathematical models have been proposed to describe helices and protein fold, including lattice models, statistical mechanical models, random energy models, and molecular dynamics simulations (see references in Refs. 19 and 20). In general, perhaps for simplicity, helical structures are usually identified, in the literature, with the simplest idea of circular helix (see, for example, Ref. 8). However, that does not fit the real world. Nobody can believe that squirrels chasing one another up and around tree trunks follow a path of circular helix. First because the cross section of a tree trunk is not circular, but also because the axis of a three trunk is not exactly a straight line. As another example we find many types of bacteria, such as certain strains of *Escherichia coli* or *Salmonella typhimorium*, swim by rotating flagellar filaments. These are polymers which are flexible enough to switch between

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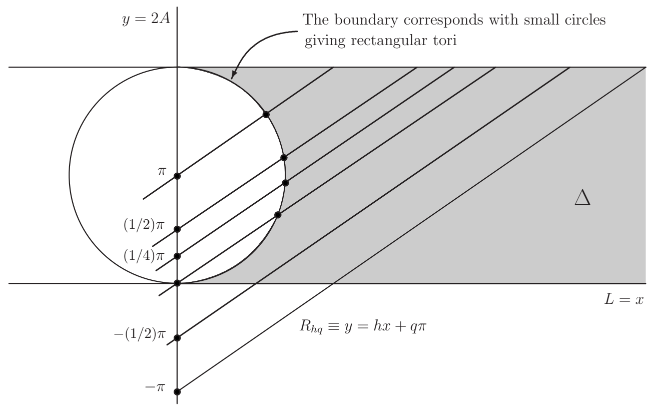

FIG. 1. Closed extremals for a given slope. The y-intersect points are rational multiples of π.

different helical forms, which are really far from circular helices. Therefore the question is: *What kind of helices are there in nature*? In this paper we try to answer this question by proposing a two steps variational model to describe helices in nature.

(1) The first step describes a wide class of helices. That made up of Lancret helices or curves making a constant angle with a flow of Euclidean parallel straight lines. They can be geometrically seen as geodesics of right cylinders whose cross section lies in a plane orthogonal to the flow. Furthermore, they appear, variationally, as extremals of an energy action whose density is a linear function in both curvature and torsion. A quantization principle works for critical values of that functional: "the energy of a helix is not arbitrary, but it comes as a natural multiple of some basic quantity of energy." In particular, the energy is constant if the homotopy class of the cross section is preserved.

(2) The submodel we have just described is very rich in solutions. However, it does not allow to get closed helical structures due, in part, to noncompactness of their flow lines. Then, we have to consider a second step which, in particular, should allow us to describe closed structures. Let us draw up the main ingredients to create it.

(1) Note first that the helical concept is related to a vector field or flow lines. Therefore, under obvious considerations, it is preserved under conformal mappings.

(2) Then change the topology of the space in order to close Euclidean parallel straight lines. The simplest way to do that is reached by adding a point at infinity to get a round three-sphere.

(3) The new space is endowed with a kind of parallelism of great circles, which is known as Clifford parallelism. Then we use the flow of Clifford parallel great circles to solve the problems associated with helices and, in particular, the so-called closed curve problem. We will see, for instance, that helices will appear as geodesics of certain flat surfaces known as Hopf tubes and they can be closed provided Hopf tubes are compact genus one surfaces, i.e., Hopf tori. Furthermore, helices will also be extremals of an energy action whose density is a linear function in both curvature and torsion.

(4) Now, use the stereographic projection, which is known to be conformal, to view the Clifford parallelism as a flow of Villarceau circles. Then, Villarceau helices will be curves making a constant angle with a Villarceau flow and they will appear as loxodromes in surfaces which are stereographic projection images of Hopf tubes.

(5) We will finally exhibit a new quantization principle working out for closed Villarceau helices. The corresponding moduli space can be identified with a certain domain in the plane (see Fig. 1).

II. THE FIRST STEP: A LEAST ACTION PRINCIPLE FOR HELICES

The intuitive idea of helical structure, as a one-dimensional configuration, is a curve that makes a constant angle with an axis. The simplest idea of axis is a unit vector, say $\vec{x}$. These curves are known in the literature as generalized helices, Lancret helices or curves with constant slope, *Böschungslinien* (see, for instance, Ref. 3 and references therein). They are analytically characterized by the constancy of the ratio between torsion and curvature. Geometrically, they are characterized as geodesics of right cylinders over plane curves. As those surfaces are flat, a Lancret helix is completely determined from the following data: first the curvature function of a plane curve which makes the role of cross section in the cylinder and then the ratio between torsion and curvature which works as the slope of the helix regarded as a geodesic in this cylinder. Next, we give a simple variational characterization of Lancret helices.

Least action. Lorsqu'il arrive quelque changement dans la Nature la quantité d'action, nécessaire pour ce changement, est la plus petite possible (Pierre Louis Moreau de Maupertuis, Lyon 1756, Vol IV, p. 36). Admissible helical structures in Nature should be, as possible, extremals of a reasonable elastic energy action. Obviously the choice of such an energy action involves some requirements. Therefore, it must be invariant not only by reparametrizations but also by motions of the Euclidean space.[22] Consequently, it yields to choose a Lagrangian density that is a function of the geometrical invariants: arc length s, curvature κ, and torsion τ,

$$\mathcal{E}(\gamma) = \int_\gamma \mathbf{F}(\kappa,\tau)ds.$$

The Euler–Lagrange equations, also called field equations, for these kind of functionals, acting on suitable spaces of curves, can be obtained using standard arguments that involve several integrations by parts. Actually, they were obtained in Ref. 19 for more general actions where density also involves the first derivatives with respect to s, i.e., $\mathbf{F}(\kappa,\tau,\kappa',\tau')$. These equations have been manipulated in Refs. 19 and 20, having no outstanding progress, even in special cases. For example, the case where the energy is a linear combination of both length, total bending, and total twisting,

$$\mathbf{F}(\kappa,\tau) = m + n\kappa + p\tau \quad m,n,p \in \mathbb{R},$$

is not sufficiently exploited there. The authors affirm that the only solutions, of the Euler–Lagrange equations in this model, are circular helices. However, this is quite false. Precisely the case where $m=0$ provides a simple model, with a wide space of field configurations, which is able to describe a lot of helices in nature.

Let Λ be the space of curves connecting two points $x,y \in \mathbb{R}^3$ in the Euclidean space and having the same Frenet frame at those points (*clamped* curves). For any three real numbers, $m,n,p \in \mathbb{R}$, we consider the action

$$\mathcal{F}_{mnp}:\Lambda \to \mathbb{R}, \quad \mathcal{F}_{mnp}(\gamma) = \int_\gamma (m + n\kappa + p\tau)ds.$$

The field equations for these functionals have been computed in several places, including Ref. 19, getting

$$m\kappa + (n\tau - p\kappa)\tau = 0, \quad n\tau' - p\kappa' = 0.$$

These equations can be easily solved. If $m \neq 0$, the solutions are circular helices (both curvature and torsion are constant) as asserted in Ref. 19. However, when $m=0$, the space of field configurations consists, up to motions in $\mathbb{R}^3$, of those curves such that the ratio between torsion and curvature is a constant, namely,

$$\frac{\tau}{\kappa} = \frac{p}{n}.$$

Hence, given a pair of real numbers, $n,p \in \mathbb{R}$, the space of field configurations of the energy action,

$$\mathcal{F}_{np}:\Lambda \to \mathbb{R}, \quad \mathcal{F}_{np}(\gamma) = \int_{\gamma} (n\kappa + p\tau)ds,$$

is, up to congruences in the Euclidean space, that of Lancret helices with slope p/n. Consequently, it can be identified with the space of plane curves. In other words, the corresponding moduli space is just the space of real valued functions of one variable, each function working as the curvature function of a cross section. The following algorithm allows one to construct all helical configurations of the model $\mathcal{F}_{np}$.

(1) Choose any plane curve, say $\alpha(s)$, s being the arc length parameter, and let $\vec{\xi}$ be a unitary vector normal to the plane that contains the curve.

(2) The right cylinder $\mathbf{C}_\alpha$, with cross section $\alpha(s)$, is defined by the map

$$\phi(s,v) = \alpha(s) + v\vec{\xi}.$$

(3) In $\mathbf{C}_\alpha$ we choose the geodesic with slope $h=p/n$, that is,

$$\gamma_h(t) = \phi(nt,pt) = \alpha(nt) + pt\vec{\xi}.$$

Then γ_h is a Lancret helix which is an extremal of $\mathcal{F}_{np}$. Furthermore, each extremal of this energy action is constructed in this way.

It should be noted that if the weight of the twisting effect, in the action, increases, then the slope of helical configurations of the model also increases. However, if the weight of bending effect increases then, the slope of helical configurations decreases. As it was pointed out in Ref. 19 (see also Ref. 28) besides circular helices there are many different shapes of helical configurations that might also be of considerable interest for protein folding, including elliptical, spherical, and conical. Then, as an illustration, they exhibited conical helices and tried to get them as extremal of energy actions, however, the result is confused and unnecessary complicated. In our model, not only conical but also elliptical and spherical helices appear naturally as extremals. Next, we exhibit these helical structures as an illustration. Namely, we get elliptical, spherical, and conical helices in the space of field configurations associated with the energy action $\mathcal{F}_{np}$ for any pair of real numbers n,p.

Elliptical helices. Besides protein folding, they apply to different contexts, from construction of antennas (see, for instance, Ref. 31) to nanotechnology (see, for instance, Ref. 18). These helical structures appear as geodesics, with slope $h=p/n$, of right cylinders with elliptic cross section. Therefore, we start with an ellipse, say in the plane $z=0$,

$$\alpha(u) = (r_1 \cos u, r_2 \sin u, 0).$$

The arc length function is given by $s(u)=r_2\int_0^u\sqrt{1-\lambda \sin^2 \theta}d\theta$, which is the elliptic integral of second kind, where $\lambda=1-r_1^2/r_2^2$ and $r_2>r_1$. This function, as well as its inverse, can be numerically handled because they are standard in mathematical software. For instance, one can use MATHEMATICA as in Ref. 14 to get numerical solutions of elliptical helices.

Spherical helices. Energy functionals such as $\mathcal{F}_{np}$ have extremals lying in spheres. They are essentially geodesics of right cylinders with cross sections being epicycloids. An epicycloid is a planar curve traced out by a point on a circle (of radius b) rolling outside another circle (of radius a).[9] In fact, pick out a Lancret helix γ_h, with slope $h=p/n$, which is an extremal of $\mathcal{F}_{np}$. Then $\tau=h\kappa$. Moreover, γ_h is contained in a sphere of radius, say r, if and only if $R^2+(TR')^2=r^2$, where $R=1/\kappa$ and $T=1/\tau$. We can solve both equations to obtain

$$\kappa = \frac{1}{\sqrt{r^2 - h^2 s^2}}, \quad \tau = \frac{h}{\sqrt{r^2 - h^2 s^2}}.$$

An easy computation allows one to see that the corresponding cross section is the epicycloid,

$$\alpha(u) = \left((a+b)\cos u - b\cos\frac{(a+b)u}{b}; (a+b)\sin u - b\sin\frac{(a+b)u}{b} \right),$$

where radii are $a = rh/\sqrt{1+h^2}$ and $b = (r/2)(1 - rh/\sqrt{1+h^2})$. The arc length function is given by $s(u) = -(4(a+b)/ab)\cos(au/2b)$. Therefore,

$$\gamma_h(t) = \left((a+b)\cos t(ns) - b\cos\frac{(a+b)t(ns)}{b}; (a+b)\sin t(ns) - b\sin\frac{(a+b)t(ns)}{b}; t(ps) \right),$$

t being the function defined by $t(r) = (2b/a)\arccos(-(ab/4(a+b))r)$.

Conical helices. They appear as geodesics of right cylinders whose cross section is either a logarithmic spiral (like that given in Ref. 19) or an Archimedean spiral. Therefore, consider the former

$$\alpha(u) = (au\cos(b\ln u); au\sin(b\ln u)), \quad u > 0,$$

where u works, up to a scaling constant, as the arc length parameter. Now, given the energy action $\mathcal{F}_{np}$, we choose in the right cylinder $\phi(u,v) = \alpha(u) + v\vec{\partial}_z$, the geodesic with slope $h = p/n$ to get

$$\gamma_h(t) = (ant\cos(b\ln(nt)); ant\sin(b\ln(nt)); pt),$$

which is a conical helix, lying in the cone $x^2 + y^2 = (a^2n^2/p^2)z^2$, which is an extremal of $\mathcal{F}_{np}$. Other conical helices being extremals of this energy action can be obtained starting from an Archimedean spiral. The simplest one is

$$\alpha(u) = (au\cos u; au\sin u),$$

then choose in the right cylinder, with cross section α, the geodesic

$$\gamma_h(t) = (ant\cos(nt); ant\sin(nt); pt)$$

with slope $h = p/n$.

Remark: The set of solutions of the Euler–Lagrange equations associated with $\mathcal{F}_{mnp}$ is summarized in the following table. For simplicity of interpretation, we have represented different cases according to the values of the three coupling parameters specifying the free energy proposed model.

m	n	p	Moduli space of trajectories
$\neq 0$	$=0$	$=0$	Geodesics $\kappa = 0$
$=0$	$=0$	$\neq 0$	Circles κ constant and $\tau = 0$
$=0$	$\neq 0$	$=0$	Plane curves $\tau = 0$
$\neq 0$	$\neq 0$	$=0$	Helices with $\kappa = \frac{-n\tau^2}{m}$
$\neq 0$	$=0$	$\neq 0$	Helices with arbitrary κ and $\tau = \frac{m}{p}$
$=0$	$\neq 0$	$\neq 0$	Lancret curves with $\tau = \frac{p}{n}\kappa$
$\neq 0$	$\neq 0$	$\neq 0$	Helices with $\kappa = \frac{-na^2}{m+ap}$, $\tau = \frac{ma}{m+ap}$ and $a \in \mathbb{R} - \{-\frac{m}{p}\}$

A quantization principle for energy critical values. Pluralitas non est ponenda sine neccesitate (William of Ockham logician and franciscan friar of the 14th cetury). Let $\gamma_h(t) = \phi(nt, pt) = \alpha(nt) + pt\vec{\xi}$, with $h = p/n$, be a critical point of the action $\mathcal{F}_{np}$. Denote by $\{T = \alpha', N\}$ and $\kappa_\alpha(s)$ a Frenet frame and the curvature function of the cross section α, which we have assumed to be arc length parametrized, respectively. As for the Frenet apparatus of γ_h in $\mathbb{R}^3$ we have

$$T_h = \frac{n}{\sqrt{n^2+p^2}}T + \frac{p}{\sqrt{n^2+p^2}}\vec{\xi}.$$

To compute the unit normal and the curvature, one proceeds as usual,

$$\nabla_{T_h} T_h = \kappa N_h = \frac{n^2}{n^2+p^2}\nabla_T T = \frac{n^2}{n^2+p^2}\kappa_\alpha N,$$

so that

$$N_h = \pm N, \quad \kappa = \frac{n^2}{n^2+p^2}|\kappa_\alpha|.$$

Finally, the unit binormal and the torsion are given by

$$B_h = T_h \wedge N_h = \frac{n}{\sqrt{n^2+p^2}}\vec{\xi} - \frac{p}{\sqrt{n^2+p^2}}T,$$

$$\nabla_{T_h} B_h = -\tau N_h = -\frac{np}{n^2+p^2}\nabla_T T = -\frac{np}{n^2+p^2}\kappa_\alpha N,$$

and

$$\tau = \frac{np}{n^2+p^2}|\kappa_\alpha|.$$

Assume now that $\alpha:[0,L]\to\mathbb{R}^2$ is parametrized by the arc length, then the Lancret curve $\gamma_h(t)=\alpha(nt)+pt\vec{\xi}$, with $h=(p/n)$, is parametrized by $\gamma_h:[0,L/n]\to\mathbb{R}^3$ with arc length function $\bar{s}$ satisfying $d\bar{s}=\sqrt{n^2+p^2}/n ds=\sqrt{n^2+p^2}dt$. Then

$$\mathcal{F}_{np}(\gamma_h) = \int_{\gamma_h}(n\kappa+p\tau)d\bar{s} = \int_0^{L/n}\left(n+\frac{p^2}{n}\right)\kappa(t)\|\gamma_h'(t)\|dt.$$

Now $\|\gamma_h'(t)\|=\sqrt{n^2+p^2}$ and $\kappa=[n^2/(n^2+p^2)]|\kappa_\alpha|$, so that

$$\mathcal{F}_{np}(\gamma_h) = n\sqrt{1+h^2}\int_\alpha|\kappa_\alpha|ds.$$

Therefore, up to a constant, the energy critical values are provided by the total absolute curvature of the cross section.

To describe an explicit situation, let P and Q be any two points in $\mathbb{R}^3$ and $\vec{\xi}=(Q-P)/|Q-P|$. Choose a coordinate system in $\mathbb{R}^3$ with P in the plane $z=0$ and $\vec{\xi}$ parallel to the z-axis. Now, pick up a unit vector, say $\vec{v}$, and define a space Λ of clamped curves $\beta:[a,b]\to\mathbb{R}^3$ satisfying

$$\beta(a)=P, \quad \beta(b)=Q, \quad T(a)=T(b)=\vec{v} \quad \text{and} \quad \langle N(a),\vec{\xi}\rangle = \langle N(b),\vec{\xi}\rangle = 0.$$

Let h be the slope of $\vec{v}$ measured with respect to the plane $z=0$ and consider the energy action $\mathcal{F}_{np}:\Lambda\to\mathbb{R}$ with $h=p/n$. The critical points of $(\Lambda,\mathcal{F}_{np})$ are Lancret helices with slope h and cross section being a closed curve in the plane $z=0$. Therefore, start with a closed curve $\alpha:[0,L]\to\mathbb{R}^2$, in the plane $z=0$, with length $L>0$, such that the corresponding Lancret helix $\gamma_h(t)=\alpha(nt)+pt\vec{\xi}$, starting at P $(\gamma_h(0)=P)$ with slope h, reaches the point Q after d consecutive liftings, i.e., $\gamma_h(Ld/n)=Q$. Now, to compute the critical value $\mathcal{F}_{np}(\gamma_h)$ we must evaluate the total absolute curvature of the d-fold of α. To do it, we consider a convex curve $\tilde{\alpha}$ in the plane $z=0$. Geometrically this curve is obtained from α by a process of symmetrization, namely, reflecting concave parts by using straight lines at the inflection points of α. In other words, $\tilde{\alpha}$ is the arc length parametrized curve with curvature function $\kappa_{\tilde{\alpha}}=|\kappa_\alpha|$. After these remarks we obtain

$$\mathcal{F}_{np}(\gamma_h) = 2\pi nd\sqrt{1+h^2}i(\tilde{\alpha}),$$

where $i(\tilde{\alpha})$ is the rotation number of $\tilde{\alpha}$.

Thus, we get the following Dirac quantization principle for extremals: *The energy of a helical configuration is not arbitrary but it comes only in natural multiples of some basic quantity of energy,* $2\pi n\sqrt{1+h^2}$. *In particular, it only depends on the homotopy class of the corresponding cross section.*

An open problem and a conjecture. In Ref. 8 the authors proposed a model to study protein chains which is governed by a Lagrangian whose density is a linear function in κ, the curvature of the centerline of the protein molecule, namely, $\mathbf{F}(\kappa,\tau)=m+n\kappa$. This is supported in the spiral stationary form of the protein chains. The helical structure of proteins implies the choice of a Lagrangian whose extremals would be only helices. However, the extremals of the corresponding energy are circular helices (see the table above).

On the other hand, several arguments could be given in order to include the torsion in the Lagrangian governing the protein model, even in a linear way as a variable of the energy density. Perhaps, it is enough to mention that, in this way, it is an essential ingredient in the equations of Calugareanu[7] and White,[30] which have became very important in connection with the theory of DNA supercoiling.

Consequently, an interesting, and still open, problem is to determine those energy densities $\mathbf{F}(\kappa,\tau)$, such that the extremals of $\mathcal{E}$ are Lancret curves. We already know that $\mathbf{F}(\kappa,\tau)=m+n\kappa+p\tau$, with $m,n,p\in\mathbb{R}$, provide nice solutions to the above stated problem. Although it is open, we have a conjecture in the sense that linear densities are the only solutions. We know partial answers to this conjecture. In fact, it is true when energy density is linear in either the curvature $\mathbf{F}(\kappa,\tau)=m\kappa+\mathbf{G}(\tau)$ or the torsion $\mathbf{F}(\kappa,\tau)=\mathbf{H}(\kappa)+p\tau$. We will give the details in a forthcoming paper.

III. THE SECOND STEP: A CONFORMAL EXTENSION IS NEEDED

The above model certainly covers a wide variety of helical structures. However, there exist helical configurations which are not yet modeled. To clarify that we start making the following considerations.

(1) The first new ingredient to be considered is that related with the axis. To be precise, the notion of Lancret helix can be nicely expressed in mathematical form as follows. First, observe that the axis $\vec{x}$, via parallel transport, defines in the space a Killing vector field of constant length, i.e., an infinitesimal translation, say X, which generates a flow of curves, namely, straight lines parallel to $\vec{x}$. Now, X-helices are those curves crossing this flow at the same angle. Certainly, this idea may be extended to a more complicated axis. Any vector field X in the Euclidean space $\mathbb{R}^3$ generates, at least locally and via integration, a flow of integral curves. Then we call X-helices those curves crossing this flow under a constant angle. In this case, X works as an axis. Therefore, it seems natural to look at the helices in nature as curves making a constant angle with some geometrical flow. It should be noted that when the helix lies in a certain surface $\mathbf{S}$ of $\mathbb{R}^3$ and the axis is tangent to $\mathbf{S}$, then we get the notion of loxodrome (also called rhumb line) relative to $X\in\mathfrak{X}(\mathbf{S})$. In this way, circular helices (Lancret helices with both curvature and torsion constant) admit a second axis, a rotational Killing vector field Y, perpendicular to $\vec{x}$, which is tangent to the cross sections of the right cylinders containing circular helices. However, the role played by both axes is quite different. In fact, it is clear that Lancret helices are loxodromes in cylinders over plane curves. Now, this property is obtained from the existence of the first axis. The existence of an axis being a rotational Killing vector field does not imply that the helix was contained in a cylinder over a plane curve.

(2) Moreover, it is clear that Lancret helices can never be closed. This happens because the tangent vector field of a Lancret helix always has a positive component, just that in the direction of the axis. On the other hand, cyclic peptides, such as the antibiotic gramicidin S and the immunodepressive agent cyclosporin, have been known for some time in contrast with circular proteins that were little known a decade ago. In recent years, a great number of circular proteins have been discovered in bacteria, plants, and animals. This class includes the so-called rounded proteins or cyclotides (see, for example, Ref. 5). *A priori*, it could

seem natural to admit that circular protein chains are modeled by some kind of helices with a circular axis.

For a better understanding of these difficulties, consider the following test due to Weiner.[29] For any $n \in \mathbb{N}$, the curve $\alpha_n : \mathbb{R} \to \mathbb{R}^3$ defined by

$$\alpha_n(t) = \left(\left(\frac{n - \cos t}{n} \right) \cos\left(\frac{t}{n} \right); \left(\frac{n - \cos t}{n} \right) \sin\left(\frac{t}{n} \right); \frac{\sin t}{n} \right),$$

which is closed with period $2\pi n$, winds n times around the anchor ring obtained when revolving the circle with center (1,0,0) and radius $1/n$ about the z-axis. Thus, it admits an axis being a rotational Killing vector field which is orthogonal to the revolution axis. In addition, these curves have the following curious property (see Ref. 29). For a curve $\alpha : [0,L] \to \mathbb{R}^3$ define the function $\mu : [0,L] \to \mathbb{R}$ by $\mu(t) = \tau(t)/\kappa(t)$ and then the breadth of the graph of μ,

$$B[\mu] = \max_{[0,L]} \mu - \min_{[0,L]} \mu .$$

If μ_n denotes the breadth of α_n in a period, say $[0, 2\pi n]$, then

$$\lim_{n\to\infty} \{B[\mu_n]\} = 0,$$

so that, in this sense, these curves can be regarded as a sort of Lancret at the infinity.

In this section, we propose a conformal variational model to solve those difficulties. It will allow us to describe circular proteins as well as other more complicated helical configurations of protein chains. To this end, we draw up the main ideas and ingredients to create it.

(1) The nonexistence of closed helices in the submodel that we have just exhibited is due, in part, to the noncloseness of the straight lines that constitute its axis flow lines. To close parallel straight lines, we need to change the topology of $\mathbb{R}^3$. In this sense, it seems natural to choose its once-point compactification, that is, the three-sphere $\mathbb{S}^3 = \mathbb{R}^3 \cup \{\zeta_o\}$. Roughly speaking, we add the infinity point.
(2) Then, we equip $\mathbb{S}^3$ with a metric as similar as possible to that Euclidean in $\mathbb{R}^3$. Certainly, the answer is obvious. Choose one in the same conformal class in order to preserve the angles. In addition, we dispose of metrics with constant curvature which allow to see $\mathbb{S}^3$ as a round sphere.
(3) In this background, the straight lines are geodesics, i.e., great circles. Furthermore, we have a classical structure which allows one to talk about parallel straight lines, the so-called Clifford parallelism. Now, flows of Clifford parallel great circles are well described through Hopf vector fields, i.e., infinitesimal translations which can be regarded as the vertical flows of Hopf maps. In this way, we have a kind of cylinders or tubes in the round three-sphere, playing the role of right cylinders, that are obtained by lifting, via a Hopf map, curves in the two-sphere which work as cross sections for tubes. Then we can characterize Lancret helices in the three-sphere, i.e., curves which admit an axis being an infinitesimal translation, or a Hopf vector field, as those curves that are geodesics in a Hopf tube.
(4) However, the surprise continues. We can exhibit Lancret helices in $\mathbf{S}^3$ as extremals of an energy action involving the corresponding geometrical invariants. Moreover, we obtain the whole space of closed Lancret configurations. Said otherwise, solve the so-called *closed curve problem* for Lancret helices in $\mathbb{S}^3$.
(5) Once the problem is solved in the three-sphere, project conformally over the Euclidean space. To do it, we use a stereographic projection, for example, from the added point ζ_o. The Hopf flow goes to a Villarceau flow, so conformal helices make a constant angle with this flow, the axis being a conformal Killing vector field. They can be viewed as loxodromes in conformal Hopf tubes including anchor rings. However, we can stereographically project from an arbitrary point to nicely deform surfaces and helices which can appear as rhumb lines in Dupin cyclides.

(6) Roughly speaking, the underlying idea in the new geometrical picture we are proposing is to see Villarceau flow of circles, which is the conformal image of the Clifford parallelism in the three-sphere, replacing Euclidean flow of parallel straight lines. In this sense, the whole model provides solutions which are helical structures with axis being either a flow of Euclidean parallel straight lines or a flow of Clifford parallel Villarceau circles.

Therefore, once we have chosen the three-sphere with a metric of constant curvature, i.e., a round sphere, without loss of generality we may assume that it has radius one and all this framework will be denoted simply by $\mathbb{S}^3$. From now on, we give some details on the above stated program.

IV. THE CLIFFORD PARALLELISM IN THE THREE-SPHERE

The Hopf map $\mathbb{S}^3 \to \mathbb{S}^2$ is a very important object not only in mathematics but also in physics (see Ref. 27, for a nice survey). In this section, we use this map to describe a classical structure in the three-sphere, the Clifford parallelism, which is reminiscent of the classical parallelism of lines in the Euclidean space.

In the three-sphere, $\mathbb{S}^3 = \{\zeta = (z_1, z_2) \in \mathbb{C}^2 : |\zeta|^2 = |z_1|^2 + |z_2|^2 = 1\}$, we consider the usual action $\mathbb{S}^1 \times \mathbb{S}^3 \to \mathbb{S}^3$ defined by

$$(e^{it}, \zeta) \mapsto e^{it}\zeta = (e^{it}z_1, e^{it}z_2).$$

The orbits under this action are great circles (geodesics) of $\mathbb{S}^3$. If $\mathbf{C}$ and $\mathbf{C}'$ denote any two orbits and d stands for the distance in $\mathbb{S}^3$, then we have

$$d(\zeta, \mathbf{C}') = d(\eta, \mathbf{C}') \quad \text{for any} \quad \zeta, \eta \in \mathbf{C}.$$

Moreover, if $\zeta \in \mathbf{C}$ and $\zeta' \in \mathbf{C}'$ satisfy $d(\zeta, \zeta') = d(\mathbf{C}, \mathbf{C}')$, then any great circle containing ζ and ζ' intersects orthogonally both $\mathbf{C}$ and $\mathbf{C}'$. This leads to the following definition. Two great circles, $\mathbf{C}$ and $\mathbf{C}'$, in $\mathbb{S}^3$ are Clifford parallel if $d(\zeta, \mathbf{C}')$ does not depend on $\zeta \in \mathbf{C}$. If this is the case, then we write $\mathbf{C} \| \mathbf{C}'$. Given a great circle $\mathbf{C}$ and $\theta \in [0, \pi]$, define

$$\mathbf{C}_\theta = \{\zeta \in \mathbb{S}^3 : d(\zeta, \mathbf{C}) = \theta\}.$$

If $\mathbf{C}^\perp$ denotes the great circle associated with the plane, through the origin, that is, orthogonal to that corresponding to $\mathbf{C}$, then $\mathbf{C}_0 = \mathbf{C}$, $\mathbf{C}_{\pi/2} = \mathbf{C}^\perp$, $\mathbf{C}_{\pi-\theta} = -\mathbf{C}_\theta$, and $\mathbf{C}_{\pi/2-\theta} = \mathbf{C}_\theta^\perp$. Therefore, it is enough to consider $\theta \in (0, \pi/2)$ to describe the geometry of these subsets as follows.

(1) For any $\theta \in (0, \pi/2)$, the set $\mathbf{C}_\theta$ is the intersection of $\mathbb{S}^3$ with a cone in $\mathbb{R}^4 = \mathbb{C}^2$. More precisely, in a suitable coordinate system (z_1, z_2) in $\mathbb{C}^2$, we can check that $\mathbf{C}_\theta = \mathbb{S}^3 \cap \boldsymbol{\Omega}_\theta$, where

$$\boldsymbol{\Omega}_\theta = \{\zeta = (z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 \sin^2 \theta - |z_2|^2 \cos^2 \theta = 0\}.$$

(2) Even more, $\mathbf{C}_\theta$ can be identified with the following torus:

$$\mathbf{C}_\theta = \{\zeta = (z_1, z_2) \in \mathbb{C}^2 : |z_1| = \cos \theta, |z_2| = \sin \theta\}.$$

(3) For any $\theta \in (0, \pi/2)$, any great circle $\mathbf{C}$ and $\zeta \in \mathbf{C}_\theta$, there exist exactly two great circles through ζ, $\mathbf{C}'$, and $\mathbf{C}''$, that are Clifford parallel to $\mathbf{C}$. This shows that the Clifford parallelism is not an equivalence relation.

Despite that, the Clifford parallelism can be decomposed in two equivalence relations. Let us sketch how to do it. Given a great circle $\mathbf{C}$, we have $\mathbf{C}^\perp$, so that the planes through the origin $\mathbf{P}$ and $\mathbf{P}^\perp$, containing these two great circles, satisfy $\mathbf{P} \oplus \mathbf{P}^\perp = \mathbb{R}^4$. We also fix an orientation on $\mathbf{P}$ and $\mathbf{P}^\perp$ to get the canonical orientation in $\mathbb{R}^4$ according to the above decomposition. Next, define subgroups of $\mathbf{O}^+(\mathbf{P}) \times \mathbf{O}^+(\mathbf{P}^\perp) \subset \mathbf{O}^+(\mathbb{R}^4)$ by

$$\mathbf{G}_{\mathbf{C}}^+ = \{(\omega; f_+ \circ \omega \circ f_+^{-1}) : \omega \in \mathbf{O}^+(\mathbf{P})\},$$

$$\mathbf{G}_{\mathbf{C}}^{-} = \{(\omega; f_- \circ \omega \circ f_-^{-1}) : \omega \in \mathbf{O}^{+}(\mathbf{P})\},$$

where $f_+ \in \mathbf{Iso}^{+}(\mathbf{P},\mathbf{P}^{\perp})$ ($f_- \in \mathbf{Iso}^{-}(\mathbf{P},\mathbf{P}^{\perp})$) is an orientation preserving (nonpreserving) isometry. It should be noticed that this construction does not depend on f_+ (or f_-). Now, an orbit under the $\mathbf{G}_C^{+}$-action is a great circle, say $\mathbf{C}'$, that is called a Clifford parallel to $\mathbf{C}$ of the first kind, while second kind of Clifford parallels, $\mathbf{C}''$, is obtained via the second subgroup. These are two equivalence relations which are denoted by $\mathbf{C}\|^{+}\mathbf{C}'$ and $\mathbf{C}\|^{-}\mathbf{C}''$. Furthermore, we have the following facts.

(1) The condition $\mathbf{C}\|\widetilde{\mathbf{C}}$ is equivalent to either $\mathbf{C}\|^{+}\widetilde{\mathbf{C}}$ or $\mathbf{C}\|^{-}\widetilde{\mathbf{C}}$.
(2) For each $\zeta \in \mathbb{S}^3$, there exist two great circles through ζ that are Clifford parallel to $\mathbf{C}$, one of the first kind, $\mathbf{C}'$, and one of the second, $\mathbf{C}''$. Furthermore, $\mathbf{C}' \neq \mathbf{C}''$ if $\zeta \in \mathbb{S}^3 \backslash (\mathbf{C} \cup \mathbf{C}^{\perp})$.

The great circle Clifford parallel to $\mathbf{C}$ of first kind (second kind) can be viewed as orbits of a standard action of the group $\mathbf{G}_{\mathbf{C}}^{+}$ ($\mathbf{G}_{\mathbf{C}}^{-}$). In fact, in the appropriate coordinate system, the action of $\mathbf{G}_{\mathbf{C}}^{+}$ on $\mathbb{S}^3$ is the usual one described by the action

$$\mathbf{G}_{\mathbf{C}}^{+} \times \mathbb{S}^3 \to \mathbb{S}^3, \quad (\varphi_t, \zeta) \mapsto \varphi_t(\zeta) = e^{it} \cdot \zeta.$$

Hence, the orbits under this action, i.e., the first kind great circles Clifford parallel to $\mathbf{C}$, are nothing but the fibers of the usual Hopf map $\Pi : \mathbb{S}^3 \to \mathbb{S}^2(1/2)$, $\Pi(z_1,z_2) = \left(z_1\bar{z}_2, \frac{1}{2}(|z_1|^2 - |z_2|^2)\right)$, where $\bar{z}_2$ is the complex conjugate of z_2. To simplify, we write $\mathbf{G}_{\mathbf{C}}^{+} = \{\varphi_t : t \in \mathbb{R}\}$.

Next, we deal with second kind Clifford parallel circles. As before, in a suitable coordinate system, write $\mathbf{G}_{\mathbf{C}}^{-} = \{\chi_t : t \in \mathbb{R}\}$ and the action of $\mathbf{G}_{\mathbf{C}}^{-}$ on $\mathbb{S}^3$ is described by

$$\mathbf{G}_{\mathbf{C}}^{-} \times \mathbb{S}^3 \to \mathbb{S}^3, \quad (\chi_t, (z_1,z_2)) \mapsto \chi_t(z_1,z_2) = (e^{it}z_1, e^{-it}z_2).$$

Similarly to the usual Hopf map, the projection map to the quotient space is

$$\Pi_- : \mathbb{S}^3 \to \mathbb{S}^2(1/2), \quad \Pi_-(z_1,z_2) = \left(z_1 z_2, \tfrac{1}{2}(|z_1|^2 - |z_2|^2)\right).$$

As before, the fibers of Π_- are nothing but the second kind circles Clifford parallel to $\mathbf{C}$.

Now, we just need to see that the isometry $J : \mathbb{S}^3 \to \mathbb{S}^3$, $J(z_1,z_2) = (z_1, \bar{z}_2)$ yields to the following commutative diagram:

$$\begin{array}{ccc} \mathbb{S}^3 & \xrightarrow{J} & \mathbb{S}^3 \\ \Pi\downarrow & & \downarrow\Pi_- \\ \mathbb{S}^2(1/2) & \xrightarrow{Id} & \mathbb{S}^2(1/2). \end{array}$$

Then, up to small changes, we can reduce ourselves to the case of first kind Clifford parallel great circles.

V. LANCRET HELICES IN THE SPHERE

Once we have established the general ideas governing the second part of this model and the main technical ingredient provided by the Clifford parallelism, it should be considered the following program.

- Find the meaning of a helical structure or a Lancret helix in $\mathbb{S}^3$.
- Obtain Lancret helices in $\mathbb{S}^3$ geometrically, i.e., use geometry to solve the natural equations for Lancret helices in $\mathbb{S}^3$.
- Characterize Lancret helices in $\mathbb{S}^3$ as solutions of a variational principle associated with an action that involves only geometrical invariants of trajectories.
- Find an explicit algorithm to characterize the closed Lancret curves in $\mathbb{S}^3$, i.e., solve the so-called closed curve problem for Lancret helices in $\mathbb{S}^3$.

To answer this program, we first observe that the natural way to define the notion of Lancret helix, not only in $\mathbb{S}^3$ but also in any Riemannian manifold, say M, was given in Ref. 3. A curve γ in a Riemannian manifold M is a Lancret one if it makes constant slope with respect to a Killing vector field in M which has constant length along γ. In other words, there exists a Killing vector field V in M satisfying

$$\|V(\gamma(s))\| = \text{constant} \quad \text{and} \quad \frac{\langle V(s), \gamma'(s)\rangle}{\|\gamma'(s)\|} = \text{constant}.$$

It should be noted that this is the natural extension of the classical Lancret curve notion in Euclidean three-space. In this sense we will say that V is an axis of the Lancret helix γ anywhere.

Analytical approach. Lancret helices in $\mathbb{S}^3$, like those in Euclidan three-space, can be nicely characterized in terms of the geometrical invariants, curvature, and torsion (see Ref. 3). Therefore, a curve, γ in $\mathbb{S}^3$ is a Lancret one if and only if either (1) $\tau=0$ and so γ lies in a totally geodesic round two-sphere $\mathbb{S}^2$ or (2) the curvature and the torsion of γ are constrained as follows:

$$\tau = h\kappa \pm 1 \quad \text{for some constant } h.$$

In general, when one works in a round three-sphere with radius $R>0$, then the above equation should be changed by $\tau=h\kappa\pm 1/R$.

Geometrical approach. Lancret helices in $\mathbb{S}^3$ can be also nicely characterized as geodesics in some flat tubes. To better understand this geometrical meaning, note first that the Hopf map becomes a Riemannian submersion by choosing in the two-sphere the metric with constant curvature 4 (see Refs. 4 and 23, for details on Riemannian submersions). Now, the one-parameter group $\mathbf{G}^+_{\mathbf{C}}=\{\varphi_t : t\in\mathbb{R}\}$ generates the following vector field on $\mathbb{S}^3$,

$$V(\zeta) = \left\{\frac{d}{dt}(\varphi_t(\zeta))\right\}_{t=0} = i\zeta, \quad \forall\, \zeta \in \mathbb{S}^3,$$

which is a Killing vector field with constant length that defines the vertical flow of that Riemannian submersion. This kind of fields in $\mathbb{S}^3$ is usually called Hopf fields.

Now, for a curve $\delta : I\subset\mathbb{R}\to\mathbb{S}^2(1/2)$ in the round two-sphere with radius of 1/2, its complete lifting $\mathbf{T}_\delta=\Pi^{-1}(\delta)$ is a flat surface in $\mathbb{S}^3$, the Hopf tube over δ or the Hopf tube with cross section δ. This surface can be nicely parametrized by

$$X: I\times\mathbb{R}\to\mathbf{T}_\delta\subset\mathbb{S}^3, \quad X(s,t) = e^{it}\bar{\delta}(s),$$

where $\bar{\delta}$ stands for a horizontal lifting of δ. Thus, the coordinate curves are, respectively, δ-horizontal liftings (t=constant) and orbits or fibers (s=constant). Now, the following result (see Ref. 3) shows, in particular, that the natural equations for Lancret helices in $\mathbb{S}^3$ can be integrated by quadratures.

Theorem 1: *A curve γ in $\mathbb{S}^3$ is a Lancret helix if and only if, up to motions in $\mathbb{S}^3$, it is a geodesic of a Hopf tube.*

Proof: Note first that a geodesic γ in a Hopf tube $\mathbf{T}_\delta$ has curvature and torsion given by

$$\kappa = \frac{\kappa_\delta + 2h}{1+h^2}, \quad \tau = \frac{-1 + h\kappa_\delta + h^2}{1+h^2},$$

where κ_δ is the curvature function of δ in $\mathbb{S}^2(1/2)$ and h denotes the slope, measured with respect to fibers, of γ as a geodesic in the flat surface $\mathbf{T}_\delta$. This automatically implies that γ is a Lancret helix in $\mathbb{S}^3$.

Conversely, if γ is a Lancret helix in $\mathbb{S}^3$, we have $\tau=h\kappa\pm 1$ for some constant h. Now, consider in $\mathbb{S}^2(1/2)$ a curve δ, with curvature function $\kappa_\delta=(1+h^2)\kappa\pm 2h$ (this curve is unique up to motions in the two-sphere). Now, in the Hopf tube $\mathbf{T}_\delta$, choose the geodesic γ_h, making an angle

φ with fibers, where $\cot\varphi = \mp h$ (the geodesic with slope h). It is not difficult to see that γ and γ_h have the same curvature and the same torsion and so, up to parametrization, they are congruent in $\mathbb{S}^3$.

Corollary 2: Up to motions in $\mathbb{S}^3$, Lancret helices can be described by one of the following equivalent two moduli.

(1) *(κ,τ): The curvature and torsion functions which must satisfy a well known constraint.*
(2) *(κ_δ,h): The curvature function in $\mathbb{S}^2(1/2)$ of the cross section δ and the slope as a geodesic in $\mathbf{T}_\delta$.*

In any case, the moduli space of complete Lancret helices can be identified with the space $C^\infty(\mathbb{R})\times\mathbb{R}$.

Variational approach. To complete a round, Lancret helices in $\mathbb{S}^3$ can be also characterized as solutions of a variational principle. In fact, like in Sec. III for Euclidean space, let $q_1,q_2\in\mathbb{S}^3(R)$ (the three-sphere with radius R) and $\{\vec{x}_1,\vec{y}_1\}$, $\{\vec{x}_2,\vec{y}_2\}$ orthonormal vectors in $T_{q_1}\mathbb{S}^3(R)$ and $T_{q_2}\mathbb{S}^3(R)$, respectively. We consider the space of clamped curves,

$$\Lambda = \{\gamma:[t_1,t_2]\to\mathbb{S}^3(R):\gamma(t_i)=q_i, T(t_i)=\vec{x}_i, N(t_i)=\vec{y}_i, 1\le i\le 2\}.$$

In this space of curves, one has the following three-parameter family of functionals, $\{\mathcal{F}_{mnp}:\Lambda\to\mathbb{R}:m,n,p\in\mathbb{R}\}$, defined by

$$\mathcal{F}_{mnp}(\alpha)=\int_\alpha (m+n\kappa+p\tau)ds. \tag{1}$$

The field equations associated with these Lagrangians, for curves in the three-sphere using a standard method (see Ref. 1), are given by

$$-m\kappa+p\kappa\tau+n\left(\frac{1-R^2\tau^2}{R^2}\right)=0,$$

$$(p\kappa-n\tau)'=0,$$

which can be nicely integrated. Moduli spaces of solutions are represented in the following table where, for simplicity, we have distinguished different cases according to the values of parameters.

m	n	p	Moduli space of configurations in $\mathbb{S}^3(R)$
$\neq 0$	$=0$	$=0$	Geodesics $\kappa=0$
$=0$	$=0$	$\neq 0$	Circles κ constant and $\tau=0$
$=0$	$\neq 0$	$=0$	Horizontal lifts, via the Hopf map, of curves in $\mathbb{S}^2$
$\neq 0$	$\neq 0$	$=0$	Helices with $\kappa=\dfrac{n(1-R^2\tau^2)}{mR^2}$
$\neq 0$	$=0$	$\neq 0$	Helices with arbitrary κ and $\tau=\dfrac{m}{p}$
$=0$	$\neq 0$	$\neq 0$	Helices with $\kappa=\dfrac{n(1-a^2R^2)}{apR^2}$ and $\tau=\dfrac{1}{aR^2}$ and $a\in\mathbb{R}-\{0\}$
$\neq 0$	$\neq 0$	$\neq 0$	Helices with $\kappa=\dfrac{n(1-a^2R^2)}{(m+ap)R^2}$, $\tau=\dfrac{p+maR^2}{(m+ap)R^2}$, and $a\in\mathbb{R}-\left\{-\dfrac{m}{p}\right\}$
$\neq 0$	$\neq 0$	$\neq 0$	Lancret helices with $\tau=\dfrac{p}{n}\kappa-\dfrac{m}{p}$ and $R=\pm\dfrac{p}{m}$

As a consequence, we have the following variational approach for Lancret helices in the three-sphere.

Theorem 3: *A curve in the round three-sphere is a Lancret helix if and only if it is an extremal for some action $\mathcal{F}_{mnp}$ when acting on a suitable space of curves.*

Quantization principle for closing Lancret helices. To finish the Lancret program in $\mathbb{S}^3$, we need to describe closed Lancret helices. This is done through the following argument which culminates in a quantization principle.

- Since Lancret helices are geodesic of Hopf tubes which are flat surfaces, one needs to start from a closed curve δ in $\mathbb{S}^2(1/2)$. Now, if the cross section closes then it is not difficult to see that $\mathbf{T}_\delta$ is a torus. To determine the isometry type of this flat torus, we consider the covering map,

$$X:\mathbb{R}^2 \to \mathbf{T}_\delta, \quad X(s,t) = e^{it}\bar{\delta}(s),$$

  and use a well known machinery (see Refs. 15 and 24) to obtain that $\mathbf{T}_\delta$ is isometric to $\mathbb{R}^2/\Gamma$, where Γ is the lattice in the Euclidean plane spanned by $(L,2A)$ and $(0,2\pi)$. Here $L>0$ denotes the length of δ and $A \in (-\pi,\pi)$ is the oriented area enclosed by δ in the two-sphere.
- Consequently, a Lancret helix in $\mathbb{S}^3$ closes if and only if its slope h satisfies the following quantization constraint:

$$h = \frac{1}{L}(2A + q\pi), \quad q \in \mathbb{Q} \text{ rational}.$$

- The existence of closed Lancret helices in any Hopf torus is guaranteed by the isoperimetric inequality in $\mathbb{S}^2(1/2)$. In fact, the length and the enclosed area for an embedded closed curve are related by

$$L^2 + 4A^2 - 4\pi A \geq 0,$$

  which can be written as

$$L^2 + (2A - \pi)^2 \geq \pi^2.$$

  Therefore, in the plane $(L,2A)$, define the region

$$\Delta = \{(L,2A): L^2 + (2A-\pi)^2 \geq \pi^2 \text{ and } 0 \leq A \leq \pi\}.$$

  Then for any point $a=(L,2A) \in \Delta$ there exists an embedded closed curve δ^a in $\mathbb{S}^2(1/2)$ with length L and enclosed area A. We compare with the above slope quantization principle to see that the geodesic with slope h in the Hopf torus $\mathbf{T}_{\delta^a}$ closes if and only if the straight line in the $(L,2A)$-plane with slope h cuts the $2A$-axis at a height which is a rational multiple of π. Therefore, the moduli space of closed Lancret helices in $\mathbb{S}^3$ is identified with the following region of the plane:

$$\Delta \cap \left(\cup_{q\in\mathbb{Q}} \left(\frac{p}{n} L - 2A = q\pi \right) \right).$$

VI. SOME EXAMPLES

In this section, we give a general method to construct Hopf tubes in $\mathbb{S}^3 \subset \mathbb{C}^2$. First of all, we note that the Hopf mapping can be written as

$$\Pi:\mathbb{S}^3 \subset \mathbb{C}^2 \to \mathbb{S}^2(1/2) \subset \mathbb{C}\times\mathbb{R}, \quad \Pi(z_1,z_2) = \left(z_1\bar{z}_2, \tfrac{1}{2}(|z_1|^2 - |z_2|^2) \right).$$

Now, given a point $p=(a+ib,c) \in \mathbb{S}^2(1/2)$, its fiber, in $\mathbb{S}^3$, is given by

$$\Pi^{-1}(p)=\left\{\left(\frac{\sqrt{1+2c}}{\sqrt{2}}e^{it},\frac{\sqrt{2}}{\sqrt{1+2c}}(ae^{it}+be^{i(t-\pi/2)})\right):t\in\mathbb{R}\right\}.$$

Let $\delta:I\subset\mathbb{R}\to\mathbb{S}^2(1/2)$ be a curve $\delta(s)=(a(s),b(s),c(s))$. Then its Hopf tube is the following set:

$$\mathbf{T}_\delta=\Pi^{-1}(\delta)=\left\{\left(\frac{\sqrt{1+2c(s)}}{\sqrt{2}}e^{it},\frac{\sqrt{2}}{\sqrt{1+2c(s)}}(a(s)e^{it}+b(s)e^{i(t-\pi/2)})\right):s\in I,t\in\mathbb{R}\right\}.$$

To parametrize $\mathbf{T}_\delta=\Pi^{-1}(\delta)$ as above, i.e., by means of fibers and horizontal lifts of the cross section δ, we need to determine a horizontal lift $\bar{\delta}$ of δ. To do that, we put $t=\alpha(s)$ in the above formula and then determine $\alpha(s)$ from the horizontality condition $\langle i\bar{\delta}(s),\bar{\delta}'(s)\rangle=0$. We proceed in this way to obtain

$$\alpha'(s)=B'(s)A(s)-A'(s)B(s),\quad \text{where } A(s)=\frac{\sqrt{2}a(s)}{\sqrt{1+2c(s)}},\quad B(s)=\frac{\sqrt{2}b(s)}{\sqrt{1+2c(s)}}.$$

Then

$$\bar{\delta}(s)=\left(\frac{\sqrt{1+2c(s)}}{\sqrt{2}}e^{i\alpha(s)},A(s)e^{i\alpha(s)}+B(s)e^{i(\alpha(s)-\pi/2)}\right),$$

where

$$\alpha(s)=\int_{s_o}^{s}(A(s)B'(s)-A'(s)B(s))ds,$$

so that

$$\mathbf{T}_\delta=\Pi^{-1}(\delta)\equiv X(s,t)=e^{it}\left(\frac{\sqrt{1+2c(s)}}{\sqrt{2}}e^{i\alpha(s)},A(s)e^{i\alpha(s)}+B(s)e^{i(\alpha(s)-\pi/2)}\right).$$

Let us exhibit some explicit examples of Hopf tubes.

The Clifford torus as a Hopf tube. Perhaps, the more popular compact surface, at least of genus one, in the three-sphere is the so-called Clifford torus. It appears in a lot of problems and there are some interesting and old conjectures in relation with that torus. The simplest way to define a Clifford torus is given by the map

$$Y:\mathbb{R}^2\to\mathbb{C}^2,\quad Y(u,v)=\frac{\sqrt{2}}{2}(e^{i\sqrt{2}u},e^{i\sqrt{2}v}). \tag{2}$$

It is clear that $Y(\mathbb{R}^2)\subset\mathbb{S}^3$. Moreover, the map is biperiodic with period $\sqrt{2}\pi$, and consequently, it defines an embedding, which is also denoted by Y, of the squared torus or the Riemannian product of two circles with radii $\sqrt{2}/2$ into the unit three-sphere, that is,

$$Y:\mathbb{S}^1(\sqrt{2}/2)\times\mathbb{S}^1(\sqrt{2}/2)\to\mathbb{S}^3,\quad Y(u,v)=\frac{\sqrt{2}}{2}(e^{i\sqrt{2}u},e^{i\sqrt{2}v}),$$

this is the well known Clifford torus.

Now, we wish to see the Clifford torus as a Hopf tube with cross section being a geodesic of $\mathbb{S}^2(1/2)$. Therefore, we choose

$$\delta:\mathbb{R}\to\mathbb{S}^2(1/2),\quad \delta(s)=\left(\tfrac{1}{2}\cos 2s,\tfrac{1}{2}\sin 2s,0\right).$$

We use the above stated argument to see that $\alpha(s)=s$ and so

$$\bar{\delta}:\mathbb{R} \to \mathbb{S}^3, \quad \bar{\delta}(s) = \frac{\sqrt{2}}{2}(e^{is}, e^{-is}),$$

is a horizontal lift. Now, the Clifford torus can be parametrized by horizontal lifts (t=constant) and fibers (s=constant) as follows:

$$X:\mathbb{R}^2/\Gamma \to \mathbb{S}^3 \subset \mathbb{C}^2, \quad X(s,t) = \frac{\sqrt{2}}{2}(e^{i(s+t)}, e^{i(-s+t)}),$$

where Γ is the lattice, in $\mathbb{R}^2$, spanned by (π,π) and $(0,2\pi)$.

It is obvious that $X(s,t)=Y((1/\sqrt{2})(s+t),(1/\sqrt{2})(-s+t))$, and so we have

$$X_s = \frac{1}{\sqrt{2}}Y_u - \frac{1}{\sqrt{2}}Y_v, \quad X_t = \frac{1}{\sqrt{2}}Y_u + \frac{1}{\sqrt{2}}Y_v.$$

Since the curves u=constant and v=constant are geodesic in the Clifford torus, they are Lancret helices in $\mathbb{S}^3$ with slope $h=1$ and $h=-1$, respectively (in this case they are circular helices because δ has constant curvature).

Rectangular tori as Hopf tubes with cross sections geodesic circles. The above example can be extended to Rectangular tori in the three-sphere which can be regarded as Hopf tubes over geodesic circles in $\mathbb{S}^2(1/2)$. Those tori can be defined by

$$Y:\mathbb{S}^1(r_1) \times \mathbb{S}^1(r_2) \to \mathbb{S}^3, \quad Y(u,v) = (r_1 e^{iu/r_1}, r_2 e^{iv/r_2}),$$

with $r_1^2+r_2^2=1$. As above, the coordinate curves u=constant and v=constant are geodesics in these tori $Y(\mathbb{S}^1(r_1)\times\mathbb{S}^1(r_2))\subset\mathbb{S}^3$.

On the other hand, consider a geodesic circle in $\mathbb{S}^2(1/2)$, i.e., a small circle, say

$$\delta:\mathbb{R} \to \mathbb{S}^2(1/2), \quad \delta(s) = \left(r\cos\frac{s}{r}, r\sin\frac{s}{r}, m\right), \quad r^2+m^2 = \frac{1}{4}.$$

In this case, we see that $\alpha(s)=\lambda s$ with $\lambda=\sqrt{(1-2m)/(1+2m)}$, and so the curve,

$$\bar{\delta}(s) = \left(\sqrt{\frac{1+2m}{2}}e^{i\lambda s}, \sqrt{\frac{1-2m}{2}}e^{i(\lambda-1/r)s}\right),$$

constitutes a horizontal lift of δ.

Therefore, the rectangular torus parametrized as a Hopf tube is given by

$$X:\mathbb{R}^2/\Gamma \to \mathbb{S}^3 \subset \mathbb{C}^2, \quad X(s,t) = \left(\sqrt{\frac{1+2m}{2}}e^{i(\lambda s+t)}, \sqrt{\frac{1-2m}{2}}e^{i((\lambda-1/r)s+t)}\right),$$

where Γ is either (i) the lattice in $\mathbb{R}^2$ spanned by $(\pi\sqrt{1-4m^2},\pi(1-2m))$ and $(0,2\pi)$ if $m>0$ or (ii) the lattice in $\mathbb{R}^2$ spanned by $(\pi\sqrt{1-4m^2},\pi(1+2m))$ and $(0,2\pi)$ if $m<0$.

It is clear that

$$r_1 = \sqrt{\frac{1+2m}{2}}, \quad r_2 = \sqrt{\frac{1-2m}{2}}, \quad \text{and} \quad X(s,t) = Y(r_1(\lambda s+t), r_2((\lambda-1/r)s+t)),$$

so that

$$X_s = \lambda r_1 Y_u + r_2(\lambda - 1/r)Y_v, \quad X_t = r_1 Y_u + r_2 Y_v.$$

As the curves u=constant and v=constant are geodesics in the rectangular torus, they are Lancret helices in $\mathbb{S}^3$ (in this case they are circular helices because δ has constant curvature).

The Hopf tube with cross section the Viviani curve. In 1692, Vincenzo Viviani (1622–1703), a student of Galileo, proposed the following problem: How is it possible that a hemisphere has four windows of such a size that the remaining surface can be exactly squared?

The answer to this question involves the so-called Viviani curve. This curve, regarded in $\mathbb{S}^2(1/2)$, is obtained when intersecting this sphere with the right cylinder $(x-1/4)^2+y^2=1/16$.[26] Therefore, we get the curve,

$$\delta(s) = \left(\frac{1+\cos s}{4}, \frac{\sin s}{4}, \frac{\sin \frac{s}{2}}{2} \right),$$

which closes with period 4π, for example, in $-2\pi \leq s \leq 2\pi$. It should be noted that this curve is not parametrized by the arc length, however, in the above argument to obtain the Hopf tube with a given cross section the parametrization does not matter. Consequently, we can follow step by step the stated argument to obtain

$$\alpha(s) = \frac{1}{4}\left(s + 2\cos\frac{s}{2} \right).$$

Now a horizontal lifting of the Viviani curve can be computed to be

$$\bar{\delta}(s) = \left(\frac{\sqrt{1+\sin\frac{s}{2}}}{\sqrt{2}} e^{i\alpha(s)}; \frac{\sqrt{2}}{4\sqrt{1+\sin\frac{s}{2}}} \left(e^{i\alpha(s)} + e^{i(\alpha(s)-s)} \right) \right).$$

This allows one to compute a nice parametrization of the Hopf torus over the Viviani curve,

$$X:[-2\pi,2\pi] \times \mathbb{R} \to \mathbf{T}_\delta \subset \mathbb{S}^3 \subset \mathbb{C}^2,$$

given by

$$X(s,t) = \left(\frac{\sqrt{1+\sin\frac{s}{2}}}{\sqrt{2}} e^{i(\alpha(s)+t)}; \frac{\sqrt{2}}{4\sqrt{1+\sin\frac{s}{2}}} \left(e^{i(\alpha(s)+t)} + e^{i(\alpha(s)-s+t)} \right) \right).$$

VII. VILLARCEAU FLOWS: A CONFORMAL FIELD THEORY TO DESCRIBE PROTEIN CHAINS

Let $\mathbf{T}$ be a revolution torus (or anchor ring) in $\mathbb{R}^3$. It is well known that $\mathbf{T}$ contains two families of circles, the parallels of latitude, and the meridians. However, it is less known that $\mathbf{T}$ contains other kind of circles, called *Villarceau circles*, as they were first discovered by A. J. Yvon Villarceau (1813–1883) in 1848. Villarceau circles in $\mathbf{T}$ can be found by intersecting $\mathbf{T}$ with a bitangent plane. In this way, one can find two families, $\mathcal{F}_1=\{\Upsilon(t)\}$ and $\mathcal{F}_2=\{\Xi(t)\}$, of these exotic circles. Two circles from different families intersect in exactly two points, while two circles in the same family not only do not intersect but they are also always linked.[13]

Clifford parallel great circles are nicely related with Villarceau circles through a suitable stereographic projection. We take $\zeta_o \in \mathbb{S}^3 \subset \mathbb{R}^4$ and consider the stereographic projection $E_o:\mathbb{S}^3 \setminus \{\zeta_o\} \to \mathbb{R}^3$ which, as it is well known, is the restriction of an inversion in $\mathbb{R}^4$ with pole ζ_o. Now, fix a great circle, say $\mathbf{C}$, going through ζ_o. We choose in $\mathbb{R}^3$ a coordinate system $\{x,y,z\}$, such that the z-axis will be $E_o(\mathbf{C}\setminus\{\zeta_o\})$ and then $E_o(\mathbf{C}^\perp)$ will be the unit circle in the $\{x,y\}$-plane.

In this setting, it is not difficult to see that $\mathbf{T}_\theta = E_o(\mathbf{C}_\theta)$, $\theta \in (0,\pi/2)$, is a revolution torus around $E_o(\mathbf{C}\setminus\{\zeta_o\})$ in $\mathbb{R}^3$. Furthermore, up to similarities, every revolution torus in $\mathbb{R}^3$ is of the form $E_o(\mathbf{C}_\theta)$ for a suitable value $\theta \in (0,\pi/2)$. Now, both families of Villarceau circles in $\mathbf{T}_\theta$

$=E_o(\mathbf{C}_\theta)$ are obtained as images under the stereographic projection E_o of the two kinds of great circles in $\mathbf{C}_\theta$ that are Clifford parallel to $\mathbf{C}$. Then all Villarceau circles in $\mathbb{R}^3 \setminus (z-\text{axis})$ can be described as follows:

$$\mathcal{F}_1 = \{\Upsilon(t)\} = \{E_o(\mathbf{C}'):\mathbf{C}' \text{ is first kind Clifford parallel to } \mathbf{C}\}$$

and

$$\mathcal{F}_2 = \{\Xi(t)\} = \{E_o(\mathbf{C}''):\mathbf{C}'' \text{ is second kind Clifford parallel to } \mathbf{C}\}.$$

From now on, we will refer to these circles as first and second kind Villarceau circles according to they lie in $\mathcal{F}_1$ or $\mathcal{F}_2$, respectively. It should be observed that $\mathcal{F}_1$ ($\mathcal{F}_2$) defines a foliation on $\mathbb{R}^3 \setminus (z-\text{axis})$.

Note that in $\mathcal{F}_1$ and in $\mathcal{F}_2$ we have included a circle which is not of Villarceau type in a revolution torus around the z-axis. That circle is $E_o(\mathbf{C}^\perp)$. However, we will treat it as a Villarceau circle.

The Clifford parallelism is projected down to $\mathbb{R}^3$, and so it can be described in terms of Villarceau circles. Indeed, as $\mathbf{C}$ is the orbit through $\zeta_o \in \mathbf{C}$ and we have chosen $E_o(\mathbf{C})$ to be the z-axis in $\mathbb{R}^3$, we have a group of orientation preserving conformal maps in $\mathbb{R}^3 \setminus (z-\text{axis})$ associated with $\mathbf{C}$, and defined by

$$\mathbf{H}^+_{\mathbf{C}} = E_o \circ \mathbf{G}^+_{\mathbf{C}} \circ E_o^{-1} = \{\psi_t = E_o \circ \varphi_t \circ E_o^{-1} : t \in \mathbb{R}\}.$$

In this setting, the orbits in $\mathbb{R}^3 \setminus (z-\text{axis})$ associated with $\mathbf{H}^+_{\mathbf{C}}$ are just the first kind Villarceau circles over a family of revolution tori around the z-axis.

Note that, given a pair of first kind Villarceau circles, say γ_1 and γ_2, then $\gamma_1 = E_o(\mathbf{C}_1)$ and $\gamma_2 = E_o(\mathbf{C}_2)$ for certain great circles which satisfy $\mathbf{C} \|^+ \mathbf{C}_1$, $\mathbf{C} \|^+ \mathbf{C}_2$, and so $\mathbf{C}_1 \|^+ \mathbf{C}_2$. In other words, those Villarceau circles are images, via a stereographic projection, of two Hopf fibers. However, they can lie on either the same revolution torus or two different revolution tori. The former occurs when $d(\mathbf{C},\mathbf{C}_1) = d(\mathbf{C},\mathbf{C}_2)$, while the latter happens when $d(\mathbf{C},\mathbf{C}_1) \neq d(\mathbf{C},\mathbf{C}_2)$.

Similarly, let

$$\mathbf{H}^-_{\mathbf{C}} = E_o \circ \mathbf{G}^-_{\mathbf{C}} \circ E_o^{-1} = \{E_o \circ \chi_t \circ E_o^{-1} : t \in \mathbb{R}\}$$

be the group of conformal maps that leave invariant the second kind Villarceau circles over a family of revolution tori around the z-axis.

From now on, we will restrict ourselves to first kind Villarceau flows even though a similar theory works for second kind Villarceau flows. Therefore, once we have solved the variational problem whose solutions are Lancret helices in $\mathbb{S}^3$, we project down, via a stereographic map, to obtain helices in Euclidean space whose axis is a conformal vector field (it generates a one-parameter group of conformal transformations in Euclidean space). To be explicit, and without loss of generality, consider $\zeta_o = (0,i) \in \mathbb{S}^3 \subset \mathbb{C}^2$. Then, in the above setting, the stereographic projection with pole ζ_o is given by

$$E_o : \mathbb{S}^3 \setminus \{\zeta_o\} \to \mathbb{R}^3 \equiv \mathbb{C} \times \mathbb{R}, \quad E_o(z_1, z_1) = \left(\frac{z_1}{1 - \mathrm{Im}(z_2)} ; \frac{\mathrm{Re}(z_2)}{1 - \mathrm{Im}(z_2)} \right).$$

Now, choose, for instance, a Clifford torus $\mathbf{T}_{\text{Clifford}} \equiv Y(u,v)$, parametrized as a Riemannian product of circles (2). It is not difficult to see that its stereographic image, $E_o(\mathbf{T}_{\text{Clifford}})$, is an anchor ring in $\mathbb{R}^3$ whose meridians are the curves $E_o(u = \text{constant})$ and latitude parallels are $E_o(v = \text{constant})$.

On the other hand, that torus viewed as a Hopf tube $\mathbf{T}_{\text{Clifford}} \equiv X(s,t)$ is projected in the conformally flat surface $E_o(\mathbf{T}_{\text{Clifford}})$, proving that this is foliated by first kind Villarceau circles, the images under E_o of fibers $s = \text{constant}$. Now, the Lancret helices in $\mathbf{T}_{\text{Clifford}}$ make a constant angle with fibers and so with the curves $v = \text{constant}$. As E_o is a conformal map, the stereographic projection of a Lancret helix makes a constant angle with the latitude parallels in the anchor ring

$E_o(\mathbf{T}_{\text{Clifford}})$. Choosing, in particular, closed Lancret helices in the Clifford torus and projecting them by E_o, we get nice closed configurations which are good candidates to model circular protein chains.

The story, however, does not finish here. It continues, since what we made with the Clifford torus also works for rectangular tori and more generally for any Hopf tube (in particular, any Hopf tori). In fact, the chief point in this discussion is a conformal vector field in the Euclidean space. It is the corresponding stereographic image of the Hopf vector field $V(\zeta)=i\zeta$ that governs the theory of Lancret helices in $\mathbb{S}^3$. Therefore, we define in $\mathbb{R}^3\setminus(z-\text{axis})$ the following vector field:

$$W \in \chi(\mathbb{R}^3 \setminus (z-\text{axis})), \quad W = dE_o(V).$$

This is a vector field providing the Villarceau flow $\mathcal{F}_1$. Alternatively, it is a conformal one, generating the one-parameter group of conformal transformations $\mathbf{H}^+_{\mathbf{C}}$.

First kind Villarceau flows, i.e., W-flows in $\mathbb{R}^3\setminus(z-\text{axis})$, can be explicitly obtained as follows. Observe that any Villarceau circle intersects in exactly one point the half plane $P=\{(x,0,z):x>0\}$ and recall that two Villarceau circles of the same kind do not intersect. Then, for any $p=(x,0,z)\in P$, the first kind Villarceau circle, $\gamma_p:[-\pi,\pi]\to\mathbb{R}^3$, going through p, is given by

$$\gamma_p(t) = E_o(e^{it}x_1, e^{it}((x_2 + iy_2)),$$

where $E_o^{-1}(x,0,z)=(x_1,0,x_2,y_2)$ and consequently

$$\gamma_p(t) = \frac{1}{1 - x_2 \sin t - y_2 \cos t}(x_1 \cos t, x_1 \sin t, x_2 \cos t - y_2 \sin t).$$

The length and the radius of this circle are, respectively,

$$L = \frac{1 + \|p\|^2}{x}\pi, \quad r = \frac{1 + \|p\|^2}{2x}.$$

Now, given any regular curve δ in $\mathbb{S}^2(1/2)$, consider its Hopf tube $\mathbf{T}_\delta$, which is a flat surface in $\mathbb{S}^3$. Then project it, via the stereographic map, to obtain a surface $\mathbf{M}_\delta=E_o(\mathbf{T}_\delta)\subset\mathbb{R}^3$ which we call the *conformal Hopf tube with conformal cross section* δ. These surfaces are tangent to $W=dE_o(V)$ and so they are foliated by Villarceau circles, that is,

$$\mathbf{M}_\delta = E_o(\mathbf{T}_\delta) = \cup_{p\in\mathbf{M}_\delta} E_o(\mathbf{C}_{E_o^{-1}(p)}),$$

where $\mathbf{C}_{E_o^{-1}(p)}$ stands for the fiber in $\mathbb{S}^3$ through $E_o^{-1}(p)$.

Given a Lancret helix γ in $\mathbb{S}^3$, it is a geodesic in some Hopf tube $\mathbf{T}_\delta$, and it makes a constant angle with the Hopf vector field V, which is a Killing vector field on the whole three-sphere. Consequently, its stereographic image $\beta=E_o(\gamma)$ (we call these curves *conformal Lancret helices*) lies in the conformal Hopf tube $\mathbf{M}_\delta=E_o(\mathbf{T}_\delta)$ and it makes a constant angle with the conformal vector field $W=dE_o(V)$. Sais otherwise, the conformal Lancret helices lie in conformal Hopf tubes and they make a constant angle with the Villarceau circles.

In the model that we are proposing to describe helical configurations in nature the solutions are conformal Lancret helices. They are conformal images in Euclidean space of curves in the three-sphere that are extremals of an action involving linearly in its Lagrangian density both bending and twisting effects.[16,17] As a culmination of this discussion, we obtain the following result which from a geometrical point of view can be considered as a conformal Lancret integration theorem.

Theorem 4: *A curve β in Euclidean space is a conformal Lancret helix if and only if it makes a constant angle with the conformal Hopf vector field $W=dE_o(V)$. Moreover, conformal Lancret helices lie in conformal Hopf tubes and they make a constant angle with the Villarceau circles.*

Remark 5:

(i) It should be noted that when the Hopf vector field is provided by the second Hopf map Π_{-}, we obtain helices associated with the second kind Villarceau flow.

(ii) Also, we can project down from a different point in the three-sphere, say ζ. Now helical structures appear as loxodromes, with respect to a conformal vector field, in deformed conformal Hopf tubes such as Dupin cyclides.

VIII. CONCLUSIONS

In the variational model that we are proposing, helices in Nature appear as either (A) Lancret helices, that is, critical points of an action which is linear in both curvature and torsion. In this case helices are loxodromes, with respect to an infinitesimal translation, in right cylinders over plane curves; or (B) conformal Lancret helices, that is, conformal images of critical points in the three-sphere of an action which is linear in both curvature and torsion.

Furthermore, they are either helices with respect to a kind of Villarceau flow and so loxodromes in conformal Hopf tubes (anchor rings, revolution tori) or helices with respect to a deformed Villarceau flow and so loxodromes in deformed conformal Hopf tubes (Dupin cyclides).

In both cases, the space of helical structures is completely determined, up to either rigid motions in the submodel (A) or conformal motions in the submodel (B), by two moduli.

- The cross section of either the right cylinder or the Hopf tube, which can be determined from a real valued function playing the role of its curvature function in the plane or in the two-sphere, respectively.
- A real number playing the role of slope. The geometrical meaning of this moduli is obvious. However, it also has an important variational meaning. The slope of a helix measures the ratio between twisting and bending weights in the energy actions admitting that helix as an extremal.

In both submodels, helices appear as solutions of a simple variational problem. In the former, helices appear directly in the Euclidean space, while in the later they appear in the three-sphere and then we have to project down, conformally, in the Euclidean space. However, the main difference between both submodels comes from the topology. The second submodel admits closed structures while this cannot hold in the first one. Besides these two principles, least action and topological, which are two requirements of our model, a third one must be remarked. A quantization principle works for the main entities of the model. Therefore, the energy of an extremal, i.e., a helical configuration, is not arbitrary but it comes only in natural multiples of some basic quantity of energy. So energy critical values only depend on the homotopy class of cross sections.

The moduli space of closed helical structures in this model is also obtained from a rational constraint between both moduli, the cross section, and the slope. Assume, for instance, we wish to determine the space of closed W-helices with slope h, where $W=dE_o(V)$ is a certain Villarceau flow. These helices are images, under E_o, of closed Lancret helices with slope h in the three-sphere. To construct the corresponding cross sections we need essentially two ingredients: (1) the isoperimetric inequality in the two-sphere $\mathbb{S}^2(1/2)$. This allows us to determine the region $\Delta=\{(x,y):x^2+(y-\pi)^2\geq\pi^2, x>0 \text{ and } 0\leq y\leq 2\pi\}$ in the $\{x,y\}$-plane with the following property: the coordinates of any point $a=(x,y)\in\Delta$ provide the length and the enclosed oriented area of a simple closed curve in the two-sphere according to $L=x$ and $2A=y$. (2) The isometry type of the associated Hopf tori. It allows one to obtain, in terms of the slope, the constraint to close a Lancret helix. Therefore, for any rational number $q\in\mathbb{Q}$, consider the straight line R_{hq} given by $y=hx+q\pi$. Now, for any point $a=(x,y)\in\Delta\cap R_{hq}$, there exists a closed curve $\delta^a_{hq}\subset\mathbb{S}^2(1/2)$ with length $L=x$ and enclosed area $A=y/2$. The Lancret helix $\gamma^a_{hq}(t)=e^{ipt}\bar{\delta}^a(nt)$, with slope $h=p/n$, is closed. Moreover, the moduli space of closed W-helices having W-slope h is obtained as $\{\gamma^a_{hq}:q\in\mathbb{Q} \text{ and } a\in\Delta\cap R_{hq}\}$.

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