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# Willmore energy estimates in conformal Berger spheres

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ABSTRACT

We obtain isoperimetric inequalities for the Willmore energy of Hopf tori in a wide class of conformal structures on the three sphere. This class includes, on the one hand, the family of conformal Berger spheres and, on the other hand, a one parameter family of Lorentzian conformal structures. This allows us to give the best possible lower bound of Willmore energies concerning isoareal Hopf tori.



## 1. Introduction

The conformal geometry of surfaces $\mathbf{S}$ in semi-Riemannian spaces $(\mathbf{M}, h)$ mainly deals with those properties of $\mathbf{S}$ remaining invariant when the metric $h$ moves within its conformal class $[h]$. In other words, it is concerned with conformal invariants of surfaces.

Up to where we know, the subject was initiated by Darboux in the latter part of the nineteenth century and was taken up again by Blaschke [9] and Thomsen [28] in the 1920's. They proved that the energy density $(H^2 + R)dA$, for surfaces in the Euclidean 3-space $(\mathbb{E}^3, h)$, is conformally invariant in the conformal class of the Euclidean metric, where $H$ is the mean curvature, $R$ is the sectional curvature of the space on the tangent to the surface (sometimes called the extrinsic Gaussian curvature) and $dA$ is the area element, all of them computed with respect to metrics in that conformal class. The strategy of Blaschke was extended to any conformal class of semi-Riemannian metrics.

In the 1960's, Willmore proposed the study of the variational problem associated with that conformally invariant energy density defined on the space of closed (compact and without boundary) surfaces in $\mathbb{E}^3$. In this respect, one has the functional, nowadays known as *Willmore functional* or *Willmore energy*, defined by

$$\mathcal{W}(\mathbf{S}) = \int_{\mathbf{S}} (H^2 + R)dA.$$

Certainly, the Willmore variational problem has been extended in several directions including the following ones: different conformal classes (see [2,17,23] and references therein), surfaces with boundary (see [3,10,11] and references therein), and Lorentzian conformal classes (see [4,6] and references therein). Those extensions have given place to what we will call the Willmore program, which has turned into one of the most popular problems in the geometry of surfaces. This popularity is due, partly, to the Willmore conjecture which states that, in the Euclidean conformal class, the Willmore energy, constrained to closed surfaces of genus one, admits $2\pi^2$ as a lower bound and it is reached precisely on a revolution torus with radii in the ratio $\sqrt{2}$ (the stereographic projection of a Clifford torus). As far as we know, the related conjecture in $\mathbb{S}^3$ has been solved in some special cases, including tori in the three sphere which are invariant under the antipodal map (see [26] and references therein). It should be noted that the existence of Willmore surfaces of arbitrary genus in $\mathbb{E}^3$ was shown by Simon in [27].

The Willmore energy of surfaces is important not only in variational geometry but also in Physics. It constitutes the core of enough apparently unrelated nonlinear phenomena in Physics. We mention some of them:

(1) The elastic theories of membranes. They are governed by energy actions which essentially coincide with the Willmore elastic energy from its origins [12,24] to nowadays [13,18,31].
(2) The bosonic string theories. The Willmore energy also appear as a string action governing the bosonic string theory in the sense of Kleinert [14] and Polyakov [25].

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(3) The two dimensional nonlinear sigma models with either spherical symmetry or Poincaré symmetry. The Willmore action is equivalent, via the Gauss–Bonnet formula, to the boundary free compact, two dimensional, non linear, sigma model with spherical symmetry [19]. However, this equivalence also holds for suitable boundary conditions [3]. Even more, that equivalence also works in Lorentzian sigma models [4].

It is clear that Hopf tori in the three sphere are invariant under the antipodal map. Then $2\pi^2$ is the best possible lower bound for Willmore energy, measured in the round conformal class, of Hopf tori. Nevertheless, by fixing the area of Hopf tori, then the previous estimate is not the best one. So a natural problem is to search for the best possible estimates for the Willmore energy in the class of Hopf tori having the same area. In some sense, we are proposing a kind of isoperimetric problem for the Willmore energy. In this note, we solve this problem in the three sphere, not only in the round conformal structure, but also in a one parameter family of conformal structures which contains the class of conformal Berger spheres, as well as a one parameter family of Lorentzian conformal structures. Each conformal class in that family includes just a metric of constant scalar curvature. Then, up to constant curvature, they represent the high level of rigidity in dimension three from the point of view of the curvature.

The main result states as follows:

**Theorem.** *Let $\alpha$ be an immersed closed curve in $\mathbb{S}^2(1/2) = (\mathbb{S}^2, g)$ with length L, then the Willmore energy of $\mathbf{S}_\alpha$ in $\left(\mathbb{S}^3, [\bar{g}_r^\varepsilon]\right)$ satisfies*

$$\mathcal{W}_r^\varepsilon(\mathbf{S}_\alpha) \geqslant \max\left\{2\pi r^2[\pi + L(\varepsilon r^2 - 1)]; 2\pi r^2\left[\frac{\pi^2}{L} + L(\varepsilon r^2 - 1)\right]\right\}$$

*with equality holding if and only if $\alpha$ is a circle of $\mathbb{S}^2(1/2)$ and so $\mathbf{S}_\alpha$ is a rotational torus with area $2\pi rL$ in $\left(\mathbb{S}^3, \bar{g}_r^\varepsilon\right)$.*

As a consequence, we give some applications. For instance, choosing a point $q_o \in \mathbb{S}^3$, we use the stereographic projection $\mathbf{E}_o : \mathbb{S}^3 - \{q_o\} \to \mathbb{E}^3$. Then take $L_o > 0$ and consider the subclass of tori $\mathbf{E}_o(\mathcal{T}_o) = \{\mathbf{E}_o(\mathbf{S}_\alpha) : \text{Length}(\alpha) = L_o\}$. Then, we get the best possible lower bound

$$\mathcal{W}_o(\mathbf{E}_o(\mathbf{S}_\alpha)) \geqslant \max\left\{2\pi^2, \frac{2\pi^3}{L}\right\}$$

for the Willmore energy in the class $\mathbf{E}_o(\mathcal{T}_o)$, with equality holding if and only if $\mathbf{E}_o(\mathbf{S}_\alpha)$ is an anchor ring with known radii.

## 2. On Berger 3-spheres

The 3-dimensional Berger spheres appeared by the first time in [7] where Berger obtained the classification of simply connected normal homogeneous Riemannian spaces with positive sectional curvature. They can be geometrically realized as geodesic spheres of the complex projective plane $\mathbb{CP}^2$ endowed with the Fubini-Study metric. Therefore, Berger spheres are hypersurfaces of $\mathbb{CP}^2$ which, with the induced metric, have constant scalar curvature.

It should be noted that in 3-dimensional Riemannian geometry, the constancy of scalar curvature provides the second degree of rigidity after the constancy of the sectional curvature, because an Einstein metric has, automatically, constant curvature. Using the usual Hopf map, Berger spheres can be also viewed as 3-spheres endowed with bundle-like or Kaluza–Klein metrics, which appear in the so called canonical variation of the standard round metric. Indeed, it is a special case of the general construction of those kind of metrics.

To proceed, we start with the usual Hopf map $\mathfrak{p} : \mathbb{S}^3 \to \mathbb{S}^2$, defined by

$$\mathfrak{p}(z_1, z_2) = \left(z_1\bar{z}_2, \frac{1}{2}(|z_1|^2 - |z_2|^2)\right),$$

which is a Riemannian submersion when the spheres are endowed with metrics $\bar{g}$ and $g$ of constant curvature 1 and 4, respectively. Setting $\mathbb{S}^3 = \{z = (z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$, then $\mathbb{S}^1$ naturally acts on $\mathbb{S}^3$ by $e^{it} \cdot z = (e^{it} \cdot z_1, e^{it} \cdot z_2)$ and the orbit flow is generated by the unit Killing vector field $V(z) = iz$, usually called Hopf vector field. Now $\mathbb{S}^2$ can be identified with the space of orbits and the Hopf map sends each point to its orbit.

The Hopf map can also be regarded as a circle bundle endowed with a natural connection, or gauge potential, $\omega = V^\sharp$. With these ingredients, the metric $\bar{g}$ on $\mathbb{S}^3$ can be written as $\bar{g} = \mathfrak{p}^*(g) + \omega^*(dt^2)$. Now, we can use scaling factors, or squashing parameters, on the fibres to obtain a one parameter class of metrics on $\mathbb{S}^3$ that constitute the canonical variation of $\bar{g}$. To be precise, we define the family

$$\{(\mathbb{S}^3, \bar{g}_r), \text{ with } \bar{g}_r = \mathfrak{p}^*(g) + r^2\omega^*(dt^2) \text{ and } r > 0\},$$

of Berger spheres.

It will be convenient, for later use, to summarize the properties of these metrics:

(1) $\left(\mathbb{S}^3, \bar{g}_r\right)$ has constant scalar curvature.
(2) $\mathfrak{p} : \left(\mathbb{S}^3, \bar{g}_r\right) \to (\mathbb{S}^2, g)$ is still a Riemannian submersion with fibres being geodesics in the Berger sphere. Thus the first O'Neill invariant $\mathbf{T}_r$ vanishes identically (see [8,20,21]).
(3) The action of $\mathbb{S}^1$ on $\mathbb{S}^3$ is carried out through isometries of $\left(\mathbb{S}^3, \bar{g}_r\right)$.
(4) The second O'Neill invariant $\mathbf{A}_r$, which measures the obstruction to integrability of the horizontal distribution, can be computed to be

$$(\mathbf{A}_r)_{\overline{X}}V = r^2 i\overline{X}, \quad \overline{X} \text{ being horizontal}.$$

It is known (see for example [30]) that any odd dimensional sphere, in particular $\mathbb{S}^3$, admits a standard Lorentz metric which may be written in the above context as follows

$$\bar{g}^{\mathrm{L}} = \mathfrak{p}^*(g) - \omega^*(dt^2),$$

so that we have just changed the causal character of the fibres and now they appear as timelike geodesics in $(\mathbb{S}^3, \bar{g}^{\mathrm{L}})$.

Certainly $\mathfrak{p} : (\mathbb{S}^3, \bar{g}^L) \to (\mathbb{S}^2, g)$ is a semi-Riemannian submersion with geodesic fibres and so we can proceed, as above, to define the corresponding canonical variation. Then, we obtain a one parameter class of Lorentzian metrics on the family of Lorentzian 3-spheres

$$\left\{\left(\mathbb{S}^3, \bar{g}^L_r\right), \text{ with } \bar{g}^L_r = \mathfrak{p}^*(g) - r^2\ \omega^*(dt^2) \text{ and } r > 0\right\},$$

providing a family of semi-Riemannian submersions with timelike geodesics fibres.

We can unify both families of metrics under the same treatment by writing

$$\left\{\left(\mathbb{S}^3, \bar{g}^\varepsilon_r\right), \text{ with } \bar{g}^\varepsilon_r = \mathfrak{p}^*(g) + \varepsilon r^2 \omega^*(dt^2)\ r > 0 \text{ and } \varepsilon \pm 1\right\}.$$

Therefore, we obtain a class of constant scalar curvature metrics on the 3-sphere which are either Riemannian or Lorentzian according to $\varepsilon = 1$ or $\varepsilon = -1$, respectively. Furthermore, the Hopf flow $V$ associated with the Hopf map is carried out through isometries of any of the metrics in this class and the O'Neill invariants of the semi-Riemannian submersion $\mathfrak{p} : (\mathbb{S}^3, \bar{g}^\varepsilon_r) \to (\mathbb{S}^2, g)$ satisfy

$$\mathbf{T}^\varepsilon_r = 0, \quad (\mathbf{A}^\varepsilon_r)_{\overline{X}} V = \varepsilon r^2 i\overline{X}, \quad \overline{X} \text{ being horizontal.} \tag{1}$$

## 3. Willmore Hopf tori

Let $\alpha$ be a closed curve immersed in $(\mathbb{S}^2, g)$ with length $L > 0$ and enclosing an oriented area $A \in (-\pi/2, \pi/2)$. Then the surface $\mathbf{S}_\alpha = \mathfrak{p}^{-1}(\alpha)$ is a Hopf torus, which can be described from its universal covering

$$\Phi : \mathbb{R}^2 \to \mathbf{S}_\alpha = \mathfrak{p}^{-1}(\alpha), \quad \Phi(s,t) = e^{it}\bar{\alpha}(s),$$

$\bar{\alpha}$ being a lifting of $\alpha$. This map can also be used to parametrize this torus with coordinate curves being fibres ($s$ = constant) and horizontal liftings of $\alpha$ ($t$ = constant).

Hopf tori can be viewed in $\left(\mathbb{S}^3, \bar{g}^\varepsilon_r\right)$ and then, with the induced metric, they are flat tori. We assume that $\alpha$ is arc length parametrized and so the metric in the above frame is

$$\bar{g}^\varepsilon_r(\Phi_s, \Phi_s) = 1, \quad \bar{g}^\varepsilon_r(\Phi_s, \Phi_t) = 0, \quad \text{and} \quad \bar{g}^\varepsilon_r(\Phi_t, \Phi_t) = \varepsilon r^2.$$

The isometry type of these flat tori is determined to be that associated with the lattice $\Gamma_r(\alpha) = \mathrm{Span}\{(L, 2A); (0, 2\pi r)\}$ in either $\mathbb{R}^2$ if $\varepsilon = 1$ or $\mathbb{L}^2$ if $\varepsilon = -1$. Note that Hopf tori are Lorentzian (or time-like) surfaces in the latter case.

The extrinsic geometry of Hopf tori in $(\mathbb{S}^3, \bar{g}^\varepsilon_r)$ is mainly encoded in the shape operator. In the above framework, it is given by the following matrix [5]

$$\begin{pmatrix} \kappa_\alpha \circ \mathfrak{p} & r \\ r & 0 \end{pmatrix},$$

where $\kappa_\alpha$ is the curvature function of $\alpha$ in $(\mathbb{S}^2, g)$. In particular, the mean curvature function, $H_\alpha$, of a Hopf torus, $\mathbf{S}_\alpha$, is related with the curvature function of its cross section $\alpha$ by

$$H_\alpha = \frac{1}{2}\kappa_\alpha \circ \mathfrak{p}. \tag{2}$$

As a consequence, a Hopf torus in $\left(\mathbb{S}^3, \bar{g}^\varepsilon_r\right)$ has constant mean curvature if and only if its cross section $\alpha$ is a circle in $(\mathbb{S}^2, g)$. In particular, minimal Hopf tori correspond with geodesics in $(\mathbb{S}^2, g)$. From now on, the constant mean curvature Hopf tori, even in $\left(\mathbb{S}^3, \bar{g}^\varepsilon_r\right)$, will be called **rotational** tori. Also, minimal Hopf tori, even in $\left(\mathbb{S}^3, \bar{g}^\varepsilon_r\right)$, will be referred to as **Clifford** tori.

The critical points of the Willmore energy are called Willmore surfaces (see [3,10,11] for boundary surfaces satisfying Dirichlet boundary conditions). The class of Willmore Hopf tori was obtained in [23] for the canonical conformal structure on the 3-sphere $(\mathbb{S}^3, [\bar{g}])$ and in [2] for conformal Berger spheres $(\mathbb{S}^3, [\bar{g}_r])$ (see [17] for other conformal classes close to the standard one). It should be noted that all these metrics define different conformal structures, so we can not find two of them in the same conformal class. The method used in [2,23] can be adapted to the Lorentzian partners to determine the class of Willmore tori in $\left\{\left(\mathbb{S}^3, [\bar{g}^\varepsilon_r]\right) : r > 0,\ \varepsilon = \pm 1\right\}$. Let us briefly recall how this approach works.

(1) First, consider the $\mathbb{S}^1$-action on $\mathbb{S}^3$ and note that a torus in the three sphere is invariant under this action if and only if it is a Hopf torus. Consequently the space of $\mathbb{S}^1$ symmetric tori can be identified with that of immersed closed curves in $(\mathbb{S}^2, g)$.

(2) Since the group $\mathbb{S}^1$ is compact, we can apply the principle of symmetric criticality, [22]. Therefore, a Hopf torus is Willmore if it is a critical point of the Willmore energy restricted to the space of Hopf tori.

(3) To compute the Willmore energy of a Hopf torus in $\left(\mathbb{S}^3, \bar{g}^\varepsilon_r\right)$, we observe that the tangent plane along a Hopf torus is a mixed section which admit the following orthonormal frame $\{\Phi_s = e^{it}\bar{\alpha}'(s); \frac{1}{r}\Phi_t = \frac{1}{r}V(\Phi(s,t))\}$, where we have assumed that $s$ is the arclength of the cross section. Thus, we can compute the sectional curvature appearing in the Willmore density as in [8] and then use (1) to obtain

$$R = \frac{\varepsilon}{r^2}\left|(A^\varepsilon_r)_{\Phi_s}\Phi_t\right|^2 = \varepsilon r^2 \mathfrak{p}.$$

Now, we use (2) and $dA_\alpha = r\,ds\,dt$, to get

$$\begin{aligned} \mathcal{W}^\varepsilon_r(\mathbf{S}_\alpha) &= \int_\alpha \left(\int_0^{2\pi r} r\left(\frac{1}{4}\kappa_\alpha^2 + \varepsilon r^2\right)\mathfrak{p}\,dt\right)ds \\ &= \frac{\pi r^2}{2}\int_\alpha (\kappa_\alpha^2 + 4\varepsilon r^2)\,ds. \end{aligned}$$

(4) Consequently, the Hopf torus $\mathbf{S}_\alpha$ is Willmore in $(\mathbb{S}^3, [\bar{g}^\varepsilon_r])$ if and only if its cross section $\alpha$ is a critical point of the elastic energy $\int_\alpha (\kappa_\alpha^2 + 4\varepsilon r^2)ds$ acting on the space of closed curves in $(\mathbb{S}^2, g)$. These curves are known to be elasticae (see [15,16] for details about closed elastic curves).

## 4. Obtaining Hopf tori

Let $\mathbb{S}^3(1) = (\mathbb{S}^3, \bar{g})$ be the 3-sphere with radius one in $\mathbb{C}^2$ and $\mathbb{S}^2(1/2) = (\mathbb{S}^2, g)$ the 2-sphere with radius 1/2 in $\mathbb{C} \times \mathbb{R}$. Let $\mathfrak{p} : \mathbb{S}^3(1) \to \mathbb{S}^2(1/2)$ the Hopf map defined as above. The fibre over a point $p = (a + ib, c) \in \mathbb{S}^2(1/2)$ is easily computed as

$$\mathfrak{p}^{-1}(p) = \left\{ \left( \frac{\sqrt{1+2c}}{\sqrt{2}} e^{it}, \frac{\sqrt{2}}{\sqrt{1+2c}} (ae^{it} + be^{i(t-\pi/2)}) \right) : t \in \mathbb{R} \right\}.$$

Likewise, given a curve $\alpha(s) = (a(s), b(s), c(s)), s \in \mathbb{R}$, in $\mathbb{S}^2(1/2)$, its Hopf tube is nothing but

$$\begin{aligned} \mathbf{S}_\alpha &= \mathfrak{p}^{-1}(\alpha) \\ &= \left\{ \left( \frac{\sqrt{1+2c(s)}}{\sqrt{2}} e^{it}, \frac{\sqrt{2}}{\sqrt{1+2c(s)}} (a(s)e^{it} + b(s)e^{i(t-\pi/2)}) \right), \right. \\ &\quad \left. \times (s,t) \in \mathbb{R}^2 \right\}. \end{aligned}$$

To parametrize $\mathbf{S}_\alpha = \mathfrak{p}^{-1}(\alpha)$ by means of fibres and horizontal lifts of the cross section $\alpha$, we need to determine a horizontal lift $\bar{\alpha}$ of $\alpha$. To do that, write $t = h(s)$ in the above formula to get $\bar{\alpha}$ from the horizontality condition $\langle i\bar{\alpha}(s), \bar{\alpha}'(s) \rangle = 0$. Then, we have

$$h'(s) = B'(s)A(s) - A'(s)B(s), \quad \text{where}$$
$$A(s) = \frac{\sqrt{2}a(s)}{\sqrt{1+2c(s)}}, \quad B(s) = \frac{\sqrt{2}b(s)}{\sqrt{1+2c(s)}},$$

so that

$$\bar{\alpha}(s) = \left( \frac{\sqrt{1+2c(s)}}{\sqrt{2}} e^{ih(s)}, A(s)e^{ih(s)} + B(s)e^{i(h(s)-\pi/2)} \right),$$

where

$$h(s) = \int_{s_o}^{s} (A(s)B'(s) - A'(s)B(s))ds.$$

Therefore

$$\begin{aligned} \mathbf{S}_\alpha &= \mathfrak{p}^{-1}(\alpha) \equiv X(s,t) \\ &= e^{it} \left( \frac{\sqrt{1+2c(s)}}{\sqrt{2}} e^{ih(s)}, A(s)e^{ih(s)} + B(s)e^{i(h(s)-\pi/2)} \right). \end{aligned}$$

Hopf tori are, of course, Hopf tubes whose cross section $\alpha$ is closed. Let us exhibit some explicit examples of Hopf tori.

**(1) Clifford tori as Hopf tori**. Perhaps, the more popular compact surface, at least of genus one, in the three sphere is the so called Clifford torus. It appears in a lot of problems and there are some interesting and old conjectures regarding that surface. The simplest way to define a Clifford torus is to start with the map

$$Y : \mathbb{R}^2 \to \mathbb{C}^2, \quad Y(u,v) = \frac{\sqrt{2}}{2} \left( e^{i\sqrt{2}u}, e^{i\sqrt{2}v} \right). \tag{3}$$

It is clear that $Y(\mathbb{R}^2) \subset \mathbb{S}^3$. This map is bi-periodic with period $\sqrt{2}\pi$ and consequently, it defines an embedding, also denoted by $Y$, of the squared torus or the Riemannian product of two circles with radii $\sqrt{2}/2$ into the unit three sphere, which is the well known Clifford torus. Now, we wish to see the Clifford torus as a Hopf torus with cross section being a geodesic of $\mathbb{S}^2(1/2)$.

Let us choose

$$\alpha : \mathbb{R} \to \mathbb{S}^2(1/2), \quad \alpha(s) = \left( \frac{1}{2}\cos 2s, \frac{1}{2}\sin 2s, 0 \right).$$

We use the above argument to see that $h(s) = s$, so that the horizontal lift is given by

$$\bar{\alpha} : \mathbb{R} \to \mathbb{S}^3, \quad \bar{\alpha}(s) = \frac{\sqrt{2}}{2} (e^{is}, e^{-is}).$$

Now, the Clifford torus can be parametrized by horizontal lifts ($t$ = constant) and fibres ($s$ = constant) by

$$X : \mathbb{R}^2/\Gamma \to \mathbb{S}^3 \subset \mathbb{C}^2, \quad X(s,t) = \frac{\sqrt{2}}{2} (e^{i(s+t)}, e^{i(-s+t)}),$$

where $\Gamma$ is the lattice in $\mathbb{R}^2$ spanned by $(\pi,\pi)$ and $(0,2\pi)$.

**(2) Rotational tori as Hopf tori with cross sections geodesic circles**. The Clifford torus is the only minimal flat tori in the three sphere. Now the above construction can be extended to flat tori with constant mean curvature in the three sphere. This class of tori constitutes, up to congruences, a one parameter family which can be viewed as follows

$$\left\{ Y_\theta : \mathbf{T}_\theta = \mathbb{S}^1(r_1) \times \mathbb{S}^1(r_2) \to \mathbb{S}^3 : r_1 = \cos\theta, \right.$$
$$\left. r_2 = \sin\theta \text{ and } \theta \in \left(0, \frac{\pi}{2}\right) \right\},$$

where

$$Y_\theta(u,v) = \left( r_1 e^{iu/r_1}, r_2 e^{iv/r_2} \right).$$

This class, obviously contains the minimal Clifford torus, which is obtained for $\theta = \pi/4$. Now, any torus in this family can be viewed as Hopf torus over a geodesic circle in $\mathbb{S}^2(1/2)$. So we start with a geodesic circle in $\mathbb{S}^2(1/2)$, say

$$\begin{aligned} \alpha_m &: \mathbb{R} \to \mathbb{S}^2(1/2), \quad \alpha_m(s) \\ &= \left( a\cos\frac{s}{a}, a\sin\frac{s}{a}, m \right), \quad a^2 + m^2 = \frac{1}{4}. \end{aligned}$$

Working as above, we find that $h(s) = \lambda s$, where $\lambda = \sqrt{\frac{1-2m}{1+2m}}$, and the following curve defines a horizontal lift of $\alpha$

$$\bar{\alpha}_m(s) = \left( \sqrt{\frac{1+2m}{2}} e^{i\lambda s}, \sqrt{\frac{1-2m}{2}} e^{i(\lambda - 1/a)s} \right).$$

The corresponding Hopf torus is given by

$$\begin{aligned} X_m &: \mathbb{R}^2/\Gamma_m \to \mathbb{S}^3 \subset \mathbb{C}^2, \quad X_m(s,t) \\ &= \left( \sqrt{\frac{1+2m}{2}} e^{i(\lambda s + t)}, \sqrt{\frac{1-2m}{2}} e^{i((\lambda - 1/a)s + t)} \right), \end{aligned}$$

where $\Gamma_m$ is the lattice in $\mathbb{R}^2$ spanned by either

(i) $(L = \pi\sqrt{1-4m^2}, 2A = \pi(1-2m))$ and $(0,2\pi)$ if $m > 0$; or
(ii) $(L = \pi\sqrt{1-4m^2}, 2A = \pi(1+2m))$ and $(0,2\pi)$ if $m < 0$.

Now choose $\theta \in (0,\pi/2)$ with $\cos\theta = r_1 = \sqrt{\frac{1+2m}{2}}$ and $\sin\theta = r_2 = \sqrt{\frac{1-2m}{2}}$. Then, it is not difficult to check that

$X_m(s,t) = Y_\theta(r_1(\lambda s + t), r_2((\lambda - 1/a)s + t))$.

Moreover, the length $L = \pi\sqrt{1-4m^2}$ of $\alpha_m$ and the angle $\theta$ are related by $L = \pi \sin 2\theta$.

The following facts should be noted:

(1) Even though the curve $\alpha$ was closed, the horizontal lifts $\bar{\alpha}$, in general, are not. This is because the holonomy of the potential gauge associated with the Hopf map is not trivial. However, one can check that $\bar{\alpha}$ *closes, may be after a number of liftings, if and only if the curve* $\alpha$ *in* $\mathbb{S}^2(1/2)$ *encloses an oriented area which is a rational multiple of* $\pi$. According to the latter computations, the horizontal lifts of a circle are closed if and only if $m$ is a rational number.

(2) It can be proved (see [1]) that an anchor ring in $\mathbb{E}^3$ is the stereographic projection of a rotational torus in the unit three sphere if an only it is associated with radii given by $R = \frac{1}{\cos\theta}$ and $\rho = \frac{\sin\theta}{\cos\theta}$.

**(3) The Hopf torus with cross section the Viviani's curve**. In 1692, Vincenzo Viviani (1622–1703), a student of Galileo, proposed the following problem: How is it possible that a hemisphere has four windows of such a size that the remaining surface can be exactly squared?

The answer to this question involves the so called Viviani's curve. This curve, regarded in $\mathbb{S}^2(1/2)$, is obtained when intersecting this sphere with the right cylinder $(x - 1/4)^2 + y^2 = 1/16$. Therefore, we have the curve

$$\alpha(s) = \left(\frac{1+\cos s}{4}, \frac{\sin s}{4}, \frac{\sin\frac{s}{2}}{2}\right),$$

which closes with period $4\pi$, for example in $-2\pi \leqslant s \leqslant 2\pi$. It should be noted that this curve is not parametrized by its arc-length, however this is no great matter. Consequently, we can follow step by step the stated argument to obtain

$$h'(s) = \frac{1}{4}\left(1 - \sin\frac{s}{2}\right), \quad h(s) = \frac{1}{4}\left(s + 2\cos\frac{s}{2}\right).$$

Now a horizontal lifting for Viviani's curve turns out to be

$$\bar{\alpha}(s) = \left(\frac{\sqrt{1+\sin\frac{s}{2}}}{\sqrt{2}} e^{ih(s)}; \frac{\sqrt{2}}{4\sqrt{1+\sin\frac{s}{2}}}\left(e^{ih(s)} + e^{i(h(s)-s)}\right)\right),$$

and $h(s) = \frac{1}{4}\left(s + 2\cos\frac{s}{2}\right)$.

This allows us to compute the following nice parametrization of the Hopf torus over the Viviani's curve

$$X : [-2\pi, 2\pi] \times \mathbb{R} \to \mathbf{S}_\alpha \subset \mathbb{S}^3 \subset \mathbb{C}^2,$$

given by

$$X(s,t) = \left(\frac{\sqrt{1+\sin\frac{s}{2}}}{\sqrt{2}} e^{i(h(s)+t)}; \frac{\sqrt{2}}{4\sqrt{1+\sin\frac{s}{2}}}\left(e^{i(h(s)+t)} + e^{i(h(s)-s+t)}\right)\right),$$

where $h(s) = \frac{1}{4}\left(s + 2\cos\frac{s}{2}\right)$.

## 5. Willmore energy estimates

Given a positive real number $A_o > 0$, consider the class $\mathcal{T}_o$ of Hopf tori with area $A_o$. Then we state the following isoperimetric problem: **Find the best possible lower bound of the Willmore energy** $\{\mathcal{W}_r^\varepsilon, r > 0, \varepsilon = \pm 1\}$ **in the class** $\mathcal{T}_o$.

Certainly this problem can be also stated in terms of the length of the cross sections, because, in $(\mathbb{S}^3, \bar{g}_r^\varepsilon)$, we have Area $(\mathbf{S}_\alpha) = 2\pi r$ Length $(\alpha)$. So we are proposing the problem of minimizing Willmore energies in the class of Hopf tori whose cross sections have the same length. The theorem we have stated in the Introduction completely solves this problem.

**Proof of the Theorem**. We already know that the Willmore energy of a Hopf torus in $\left(\mathbb{S}^3, [\bar{g}_r^\varepsilon]\right)$ is given by

$$\mathcal{W}_r^\varepsilon(\mathbf{S}_\alpha) = \frac{\pi r^2}{2}\int_\alpha (\kappa_\alpha^2 + 4\varepsilon r^2)ds, \quad r > 0, \quad \varepsilon \pm 1.$$

In particular, we see that

$$\mathcal{W}_r^\varepsilon(\mathbf{S}_\alpha) = r^2\mathcal{W}(\mathbf{S}_\alpha) + 2\pi r^2 L(\varepsilon r^2 - 1). \tag{4}$$

Therefore, we only need to control the Willmore energy in the conformal structure $(\mathbb{S}^3, [\bar{g}])$ which satisfies

$$\mathcal{W}(\mathbf{S}_\alpha) = \frac{\pi}{2}\int_\alpha (\kappa_\alpha^2 + 4)ds = \frac{\pi}{2}\int_\alpha \tilde{\kappa}_\alpha^2 ds,$$

where $\tilde{\kappa}_\alpha$ is the curvature function of $\alpha$ in the Euclidean space $\mathbb{E}^3$. The Schwartz and Fenchel inequalities then imply that

$$\mathcal{W}(\mathbf{S}_\alpha) \geqslant \frac{2\pi^3}{L}.$$

On the other hand, every Hopf torus in the unit three sphere is invariant under the antipodal map. In fact

$$A(\Phi(s,t)) = -\Phi(s,t) = -e^{it}\bar{\alpha}(s) = e^{it+\pi}\bar{\alpha}(s) \in \mathbf{S}_\alpha.$$

Now the Willmore conjecture for tori in the real projective three space, equivalently for tori in the unit three sphere which are invariant under the antipodal map, has been proved in [26], so we have

$$\mathcal{W}(\mathbf{S}_\alpha) \geqslant 2\pi^2.$$

This proves that $\mathcal{W}(\mathbf{S}_\alpha) \geqslant \max\left\{2\pi^2, \frac{2\pi^3}{L}\right\}$ getting the inequality. As for equality, we distinguish two cases. If $L \geqslant \pi$, equality yields $\mathcal{W}(\mathbf{S}_\alpha) = 2\pi^2$ and then $\mathbf{S}_\alpha$ is a Clifford torus in $(\mathbb{S}^3, \bar{g})$, which implies that $\alpha$ is a geodesic in $\mathbb{S}^2(1/2)$ and so $\mathbf{S}_\alpha$ is a minimal rotational torus in $\left(\mathbb{S}^3, \bar{g}_r^\varepsilon\right)$. Otherwise, if $L \leqslant \pi$, equality says that $\mathcal{W}(\mathbf{S}_\alpha) = \frac{2\pi^3}{L}$, which we bring to Schwartz and Fenchel inequalities to ensure that $\alpha$ is a circle in $\mathbb{S}^2(1/2)$ and so $\mathbf{S}_\alpha$ is a rotational torus in $\left(\mathbb{S}^3, \bar{g}_r^\varepsilon\right)$.

The above result, and its proof, has important consequences and applications, some of them are listed below.

**(1) Willmore energy estimates**. The above stated result allows one to obtain the best possible estimates for Willmore energy on the class of Hopf tori with the same area. In some sense it can be considered as a kind of isoperimetric inequality for Willmore energy. Let $L_o$ be a positive real number and consider the class of Hopf tori in $\left(\mathbb{S}^3, \bar{g}_r^\varepsilon\right)$ with area $2\pi r\, L_o$, that is

$\mathcal{T}_o = \{\mathbf{S}_\alpha : \alpha$ is a closed curve in $\mathbb{S}^2(1/2)$ with length $(\alpha) = L_o\}$.

Then we have

(i) *If* $L_o \geqslant \pi$ *then* $\mathcal{W}_r^\varepsilon(\mathbf{S}_\alpha) \geqslant 2\pi^2 r + 2r\pi L_o(\varepsilon r^2 - 1)$ *and the equality holds if and only if* $\mathbf{S}_\alpha$ *is the minimal rotational torus (the Clifford torus) in* $\left(\mathbb{S}^3, \bar{g}_r^\varepsilon\right)$.

(ii) *If* $L_o \leqslant \pi$ *then* $\mathcal{W}_r^\varepsilon(\mathbf{S}_\alpha) \geqslant \frac{2\pi^3 r}{L_o} + 2r\pi L_o(\varepsilon r^2 - 1)$ *and the equality holds if and only if* $\mathbf{S}_\alpha$ *is a rotational torus with area* $2\pi r L_o$ *in* $\left(\mathbb{S}^3, \bar{g}_r^\varepsilon\right)$.

**(2) A conformal picture in** $\mathbb{E}^3$. We can use a stereographic projection to see a conformal picture, in $\mathbb{E}^3$. Pick a point $q_o \in \mathbb{S}^3$ and denote by $\mathbf{E}_o : \mathbb{S}^3 - \{q_o\} \to \mathbb{E}^3$ the stereographic projection with pole $q_o$. Choose $\mathbf{C}$ to be a great circle through $q_o$ and $\mathbf{E}_o(\mathbf{C})$ as the $\{z\}$-axis in $\mathbb{E}^3$. Now, the Hopf vector field, $V$, projets down to the vector field $Z = d\mathbf{E}_o(V)$ which is a conformal infinitesimal translation in the Euclidean metric, $\bar{g}_o$. In particular, it defines a flow of circles in $\mathbf{M} = \mathbb{E}^3 - \mathbf{E}_o(\mathbf{C})$ which are called circles of Villarceau (see [7,29]). This Villarceau flow can be explicitly obtained as follows. First, note that any Villarceau circle intersects in exactly one point the half plane $P = \{(x,0,z) : x > 0\}$. Then, for any $p = (x,0,z) \in P$, the Villarceau circle, $\gamma_p : [-\pi,\pi] \to \mathbb{E}^3$, passing through $p$ is given by

$$\gamma_p(t) = \frac{1}{1 - x_2\sin t - y_2\cos t}(x_1\cos t, x_1\sin t, x_2\cos t - y_2 \times \sin t),$$

where

$$x_1 = \frac{2x}{1+x^2+z^2}, \quad x_2 = \frac{2z}{1+x^2+z^2}, \quad y_2 = \frac{-1+x^2+z^2}{1+x^2+z^2}$$

Now, the class of Hopf tori projects down in the class of tori that are foliated by Villarceau circles, namely $\{\mathbf{E}_o(\mathbf{S}_\alpha) : \alpha$ is a closed curve in $\mathbb{S}^2(1/2)\}$. So these tori are invariant under the one parameter group of conformal transformations $\{\psi_t = \mathbf{E}_o \cdot \varphi_t \cdot \mathbf{E}_o^{-1} : t \in \mathbb{R}\}$, where $\psi_t : \mathbb{S}^3 \to \mathbb{S}^3$ is given by $\varphi_t(z) = e^{it}z$.

For $L_o > 0$ consider the subclass of tori $\mathbf{E}_o(\mathcal{T}_o) = \{\mathbf{E}_o(\mathbf{S}_\alpha) : \text{Length}(\alpha) = L_o\}$. Then, we have the following best possible lower bound for Willmore energy, of $\mathbb{E}^3$, in the class $\mathbf{E}_o(\mathcal{T}_o)$

$$\mathcal{W}_o(\mathbf{E}_o(\mathbf{S}_\alpha)) \geqslant \max\left\{2\pi^2, \frac{2\pi^3}{L}\right\}$$

and the equality holds if and only if $\mathbf{E}_o(\mathbf{S}_\alpha)$ is an anchor ring with radii $R > \rho$ given by

$$R = \frac{\sqrt{2}\pi}{\sqrt{\pi + \sqrt{\pi^2 - L_o^2}}} \quad \text{and} \quad \rho = \frac{\sqrt{\pi - \sqrt{\pi^2 - L_o^2}}}{\sqrt{\pi + \sqrt{\pi^2 - L_o^2}}}.$$

More precisely, we have

(i) If $L_o \geqslant \pi$, then $\mathcal{W}_o(\mathbf{E}_o(\mathbf{S}_\alpha)) \geqslant 2\pi^2$ and the equality holds if and only if $\mathbf{E}_o(\mathbf{S}_\alpha)$ is an anchor ring with radii in the ratio $\frac{R}{\rho} = \sqrt{2}$.

(ii) If $L_o \leqslant \pi$, then $\mathcal{W}_o(\mathbf{E}_o(\mathbf{S}_\alpha)) \geqslant 2\pi^3/L_o$, with equality holding if and only if $\mathbf{E}_o(\mathbf{S}_\alpha)$ is an anchor ring with radii in the ratio $\frac{R}{\rho} = \frac{\sqrt{2}\pi}{\sqrt{\pi - \sqrt{\pi^2 - L_o^2}}}$.

**(3) Estimates in Anti de Sitter worlds**. Let $\alpha$ be an immersed closed curve in $\mathbb{S}^2(1/2) = (\mathbb{S}^2, g)$ with curvature function $\kappa_\alpha$ and length $L$. Then its total squared curvature can be written as

$$\int_\alpha \kappa_\alpha^2 ds = \frac{2}{\pi}\mathcal{W}(\mathbf{S}_\alpha) - 4L,$$

so that

$$\int_\alpha \kappa_\alpha^2 ds \geqslant \max\left\{0; \frac{4(\pi^2 - L^2)}{L}\right\} \tag{5}$$

and the equality holds if and only if $\alpha$ is a circle of $\mathbb{S}^2(1/2)$.

On the other hand, the an anti de Sitter metric $\tilde{g}$, with constant curvature $-4$, can be materialized by choosing the model

$$\mathbb{H}_1^3 = \left\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 - |z_2|^2 = -\frac{1}{4}\right\},$$

endowed with the induced metric from that of $\mathbb{C}_1^2 \equiv \mathbb{R}_2^4$. However, we wish to see the anti de Sitter metric as a warped product and so as a bundle-like one associated with a semi-Riemannian submersion where the horizontal distribution is integrable and consequently the second O'Neill invariant $A$ vanishes identically. To do it, consider the hyperbolic plane, endowed with the metric $g$ with constant curvature $-\frac{1}{4}$, viewed in $\mathbb{L}^3$ as

$$\mathbb{H}^2 = \left\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = -\frac{1}{4}, z > 0\right\}.$$

Now, define the positive function $f : \mathbb{H}^2 \to \mathbb{R}$ by $f(x,y,z) = z$ and use it as a warping function to see $\mathbb{H}^2 \times \mathbb{S}^1$ equipped with the metric $g - f^2 dt^2$, also written as the warped product $\mathbb{H}^2 \times_f (-\mathbb{S}^1)$.

Next, define the map $F : \mathbb{H}^2 \times \mathbb{S}^1 \to \mathbb{H}_1^3$ by $F((x,y,z), e^{it}) = (x + iy, ze^{it})$. It is not difficult to see that $F$ is a global isometry between $\mathbb{H}^2 \times_f (-\mathbb{S}^1)$ and $(\mathbb{H}_1^3, \tilde{g})$. As a consequence, an obvious conformal change allows us to see the conformal anti de Sitter space $\left(\mathbb{H}_1^3, \frac{1}{f^2}\tilde{g}\right)$ isometric to the semi-Riemannian product $\left(\mathbb{H}^2, \frac{1}{f^2}g\right) \times (\mathbb{S}^1, -dt^2)$. It is not difficult to see that $\Sigma = (\mathbb{H}^2, \frac{1}{f^2}g)$ is isometric to the once punctured round two sphere with radius 1/2. Now, for any closed curve $\alpha$ in this once punctured two sphere, one can consider the Lorentzian tube $\mathrm{T}_\alpha = F(\alpha \times (-\mathbb{S}^1))$. As the Willmore energy is invariant under conformal changes in the ambient metric, we can compute the Willmore energy of those tubes in the anti de Sitter space using the Riemannian product conformal metric. Then, we have

$$\widetilde{\mathcal{W}}(T_\alpha) = \frac{\pi}{2}\int_\alpha \kappa_\alpha^2 ds,$$

where $\kappa_\alpha$ denotes the curvature function of $\alpha$ in the once punctured two sphere $\Sigma = \mathbb{S}^2(1/2) - \{p\}$. Then, we can use (5) to obtain

$$\widetilde{\mathcal{W}}(T_\alpha) \geqslant \max\left\{0; \frac{2\pi(\pi^2 - L^2)}{L}\right\}$$

and the equality holds if and only if $\alpha$ is a circle in the unit two sphere.

**Final remark**. A natural question is to study how the Willmore energy of a given Hopf torus $\mathbf{S}_\alpha$ varies when the scaling factor moves. That variation can be measured from (4) which can be written as

$$\mathcal{W}_r^\varepsilon(\mathbf{S}_\alpha) = r^2(2\pi L\varepsilon r^2 + \widetilde{\mathcal{W}}(T_\alpha)).$$

Therefore, the behavior of the Willmore energy is different enough according to whether the metric is either Riemannian or Lorentzian. In Berger spheres, that is, when $\varepsilon = 1$, the Willmore energy of Hopf tori increases with the scaling factor. However, as for Lorentzian partners we have

$$\mathcal{W}_r^{\varepsilon=-1}(\mathbf{S}_\alpha) = r^2(-2\pi L r^2 + \widetilde{\mathcal{W}}(T_\alpha))$$

and so the Willmore energy reaches its maximum when the scaling factor is

$$r_o^2 = \frac{1}{4\pi L}\widetilde{\mathcal{W}}(T_\alpha).$$

Then the maximum Willmore energy of a Lorentzian Hopf torus is

$$\mathcal{W}_{r_o}^{\varepsilon=-1}(\mathbf{S}_\alpha) = \frac{1}{8\pi L}(\widetilde{\mathcal{W}}(T_\alpha))^2.$$

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