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RIEMANNIAN VERSUS LORENTZIAN SUBMANIFOLDS. SOME OPEN PROBLEMS

A. Ferrández

All results contained in this paper have been made in collaboration with my colleagues Manuel Barros, Pascual Lucas and Miguel A. Meroño.

Contents

This is a survey of the following three subjects: B-scrolls, r-elastic curves and Willmore-Chen submanifolds. B-scrolls arose as the first important example of indefinite submanifolds having no Riemannian counterpart. They have played an essential role in a series of classification results of indefinite submanifolds which point out substantial differences between indefinite and Riemannian submanifolds (see [1], [2], [3], [4], [5], [15] and [16]). Then, following Pinkall, [24], we define indefinite Hopf cylinders and find a nice characterization of B-scrolls with constant mean curvature in $\mathbb{H}^3_1(-1)$ in terms of them (see [12] and [9]). Now two remarkable facts should be noticed. On one hand, looking at parametrizations of indefinite Hopf cylinders, we bring to mind the Betchov-Da Rios soliton equation (see [18], [25], [26], [27], [28] and [29]). Then we find solutions of this equation lying on B-scrolls: they are helices. Furthermore, we give a rational one-parameter family of closed solutions and show that the only soliton solutions are the null geodesics of the corresponding B-scroll (see [9]). On the other hand,

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we see that Hopf surfaces in $\mathbb{H}_1^3(-1)$ shaped on closed curves in the hyperbolic plane $\mathbb{H}^2(-1/4)$ are Lorentzian Hopf tori. Then we first determine the isometry group of Lorentzian Hopf tori and, secondly, we try to get solutions of the Willmore problem in $\mathbb{H}_1^3(-1)$. The latter will be solved, following again Pinkall, by means of the Langer and Singer viewpoint on elastic curves ([19], [20] and [21]) and the symmetric criticality principle of Palais [23].

As far as helices is concerned, we start by recalling that a general helix in a Euclidean 3-space is defined by the property that its tangent indicatrix is a planar curve. The straight line perpendicular to this plane is called the axis of the general helix. First of all we need a good definition appropriate for the new ambient spaces. Moreover, we have to consider both degenerate and non-degenerate general helices in $\mathbb{L}^3$, according to the causal character of its axis. Therefore, to define general helices in 3-dimensional De Sitter $\mathbb{S}_1^3$ and anti De Sitter $\mathbb{H}_1^3$ spaces we follow the idea of Langer and Singer, [21], and use Killing vector fields along curves. Namely, let M be a non flat 3-dimensional Lorentzian space form. A curve γ in M is said to be a general helix if there exists a Killing vector field V along γ with constant length and orthogonal to the acceleration vector field of γ. V will be the axis of γ. The helix is said to be degenerate or non-degenerate according to V is, respectively. In [6] Barros has shown that general helices are geodesics either of right general cylinders or of Hopf cylinders, according to the curve lies in $\mathbb{R}^3$ or $\mathbb{S}^3$, respectively. Now, general helices in $\mathbb{L}^3$ are geodesics in right general cylinders or in flat B-scrolls, according to the helix is non-degenerate or degenerate, respectively. In non flat 3-dimensional Lorentzian space forms the Lancret thorem underlines deep differences between pseudo-spherical and pseudo-hyperbolic spaces. The former has no non trivial general helices, the latter being nicely similar to $\mathbb{L}^3$. Whence roles played by $\mathbb{H}_1^3$ and $\mathbb{S}_1^3$ correspond with those played by $\mathbb{S}^3$ and $\mathbb{H}^3$, respectively (see [10]).

The Willmore-Chen variational problem is the natural extension of the Willmore one. The first non trivial examples of Willmore-Chen submanifolds were given by Barros and Garay in [13]. We aim to find Willmore-Chen submanifolds in a pseudo-hyperbolic space $\mathbb{H}_r^n$. That will be done in several steps. After writing $\mathbb{H}_r^n$ as a warped product, we characterize $SO(r+1)$-invariant submanifolds of $\mathbb{H}_r^n$. Then we extend the concept of elastic curves to r-elastic curves and apply the symmetric criticality principle. As a consequence Willmore-Chen submanifolds in $\mathbb{H}_r^n$ are characterized in terms of r-generalized free elaticae in the once punctured unit sphere Σ^{n-r} (see [11]). Furthermore, following the classification of closed free elasticae in the standard 2-sphere obtained by Langer and Singer, [21], we show that there exist infinitely many Lorentzian Willmore tori in the 3-dimensional anti De Sitter space. Examples of Willmore tori in non-standard 3-spheres have been recently given by Barros in [7]. The same author has also found wide families of Willmore tori in warped product manifolds (see [8]).

1 Indefinite Hopf cylinders ([9,12])

Following Pinkall [24], we look for pseudo-Riemannian submersions

$$\pi_s : \mathbb{H}^3_1(-1) \to \mathbb{H}^2_s(-1/4),\ s = 0, 1.$$

Idea:

identify $\mathbb{H}^3_1(-1)$ with an appropriate subset of maps $\mathbb{R}^4_2 \to \mathbb{R}^4_2$.

How to do that:

P be a 2-dimensional subspace in $\mathbb{R}^4_2$ and $\{x, y\}$ an orthonormal basis of P.

Define maps

$$\begin{aligned} &f: P \to P, && f(x) = y, && f(y) = -x,\\ &g: P \to P, && g(x) = y, && g(y) = x,\\ &h: P \to P, && h(x) = -y, && h(y) = -x, \end{aligned}$$

which will be called **rotation, first reflection** and **second reflection** on P, respectively.

Let $\{e_1, e_2, e_3, e_4\}$ be the usual basis of $\mathbb{R}^4_2$ equipped with $(g_{ij}) = \mathrm{diag}[-1, -1, 1, 1]$.

Set $P_i = \mathrm{span}\{e_1, e_i\}$, $i = 2, 3, 4$ such that $\mathbb{R}^4_2 = P_i \oplus P_i^\perp$.

Consider the following maps

$$\begin{aligned} \rho &= f \times f: P_2 \oplus P_2^\perp \to P_2 \oplus P_2^\perp,\\ \sigma &= g \times h: P_3 \oplus P_3^\perp \to P_3 \oplus P_3^\perp,\\ \iota &= g \times g: P_4 \oplus P_4^\perp \to P_4 \oplus P_4^\perp. \end{aligned}$$

Then $\mathcal{F} = \mathrm{span}\{1, \rho, \sigma, \iota\}$ is a 4-dimensional vector space over $\mathbb{R}$ and the following identities hold

$$\begin{aligned} &\rho^2 = -1, && \sigma\rho = -\iota, && \iota\rho = \sigma,\\ &\rho\sigma = \iota, && \sigma^2 = 1, && \iota\sigma = \rho,\\ &\rho\iota = -\sigma, && \sigma\iota = -\rho, && \iota^2 = 1. \end{aligned}$$

Let $\varphi : \mathcal{F} \to \mathbb{R}^4_2$ be the isomorphism given by

$$\varphi(1) = e_1,\ \varphi(\rho) = e_2,\ \varphi(\sigma) = e_3,\ \varphi(\iota) = e_4.$$

Then φ becomes an isometry when $\mathcal{F}$ is endowed with the metric $\varphi^*(g_0)$, g_0 being the standard scalar product on $\mathbb{R}^4_2$.

Both metrics will be denoted by $\langle\, ,\rangle$.

Write $\omega = a + b\rho + c\sigma + d\iota \in \mathcal{F}$, a standing for $a \cdot 1$, a, b, c and d being real numbers.

Define

$$\overline{\omega} = -a + b\rho + c\sigma + d\iota.$$

Then

$$\langle \omega, \omega \rangle = \omega\overline{\omega} = \overline{\omega}\omega.$$

In general

$$\langle \omega_1, \omega_2 \rangle = p_1(\omega_1\overline{\omega}_2),$$

p_1 denoting the projection over the subspace spanned by the identity map.

Hence

$$\overline{\omega_1\omega_2} = -\overline{\omega_2}\,\overline{\omega_1}$$

and

$$\langle \omega_1\omega_2, \omega_1\omega_2 \rangle = -\langle \omega_1, \omega_1 \rangle \langle \omega_2, \omega_2 \rangle .$$

Now set

$$\begin{aligned} \mathbb{H}^3_1(-r^2) &\equiv \{\omega \in \mathcal{F}: \omega\overline{\omega} = -r^2\}, \\ \mathbb{H}^2(-r^2) &\equiv \operatorname{span}\{1, \sigma, \iota\} \subset \mathbb{H}^3_1(-r^2), \\ \mathbb{H}^2_1(-r^2) &\equiv \operatorname{span}\{1, \rho, \sigma\} \subset \mathbb{H}^3_1(-r^2). \end{aligned}$$

Define $\pi_s : \mathbb{H}^3_1(-1) \to \mathbb{H}^2_s(-1/4)$ by

$$\pi_s(\omega) = \frac{1}{2}\tilde{\omega}\omega,$$

where $\omega \to \tilde{\omega}$ denote the antiautomorphism of $\mathcal{F}$ given by

$$\tilde{\omega} = a - b\rho + c\sigma + d\iota, \qquad \text{or} \qquad \tilde{\omega} = a + b\rho + c\sigma - d\iota,$$

according to the base manifold is $\mathbb{H}^2(-1/4)$ or $\mathbb{H}^2_1(-1/4)$, respectively.

As usual, we define $e^{\theta x}$, $\theta \in \mathcal{F}$, by

$$\begin{array}{ll} \cos(x) + \sin(x)\theta, & \text{if } \theta^2 = -1, \\ \cosh(x) + \sinh(x)\theta, & \text{if } \theta^2 = 1. \end{array}$$

That means that the fibers are topologically $\mathbb{S}^1$ and $\mathbb{H}^1$, respectively.

Remark Writing $\sigma = f \times f$ and $\iota = f \times f$, then we obtain in the Euclidean space $\mathbb{R}^4$ the standard quaternionic structure, which was already used by U. Pinkall to describe the usual Hopf fibration $\mathbb{S}^3(1) \to \mathbb{S}^2(1)$.

Let $\overline{\nabla}$ and ∇ be the semi-Riemannian connections of $\mathbb{H}^3_1(-1)$ and $\mathbb{H}^2_s(-1/4)$, respectively, and denote by overbars the lifts of corresponding objects on the base $\mathbb{H}^2_s(-1/4)$.

Then

$$\begin{aligned} \overline{\nabla}_{\overline{X}}\overline{Y} &= \overline{\nabla_X Y} + (-1)^s(\langle JX, Y \rangle \circ \pi_s)V, \\ \overline{\nabla}_{\overline{X}}V &= \overline{\nabla}_V\overline{X} = \theta\overline{X}, \\ \overline{\nabla}_V V &= 0, \end{aligned}$$

where J denotes the standard complex structure of $\mathbb{H}^2_s(-1/4)$ and $\theta = \rho$ when $s = 0$ or $\theta = \iota$ when $s = 1$.

Let $\beta\colon I \to \mathbb{H}^2_s(-1/4)$ be a unit speed curve with Frenet frame $\{T, \xi_2\}$ and curvature κ.

Consider a horizontal lift $\overline{\beta} : I \to \mathbb{H}^3_1(-1)$ of β with Frenet frame $\left\{\overline{T}, \xi_2^*, \xi_3^*\right\}$ and curvatures κ^* and τ^*.

Now, from the Frenet equations, we can deduce that $\xi_2^* = \overline{\xi_2}$ and $\overline{\kappa} = \kappa \circ \pi_s$. In particular ξ_2^* lies in the horizontal distribution along $\overline{\beta}$ and it has the same causal character as ξ_2. Also it is not difficult to see that $\tau^* = \pm 1$ and $\xi_3^* = \pm V$, that is, the binormal ξ_3^* of $\overline{\beta}$ coincides with the unit tangent to the fibers through each point of $\overline{\beta}$.

Proposition

(1) *The horizontal lifts of unit speed curves in* $\mathbb{H}^2(-1/4)$ *are spacelike curves in* $\mathbb{H}^3_1(-1)$ *with torsion* ± 1.

(2) *The horizontal lifts of unit speed timelike curves in* $\mathbb{H}^2_1(-1/4)$ *are timelike curves in* $\mathbb{H}^3_1(-1)$ *with torsion* ± 1.

By pulling back via π_s a non-null curve β in $\mathbb{H}^2_s(-1/4)$ we get the total horizontal lift of β, which is a flat immersed surface M_β in $\mathbb{H}^3_1(-1)$, that will be called the **indefinite Hopf cylinder associated to** β

Notice that if $s = 0$, M_β is a Lorentzian surface, whereas if $s = 1$, M_β is Riemannian or Lorentzian, according to β be spacelike or timelike, respectively.

Theorem

Let M be a Lorentzian surface immersed into $\mathbb{H}^3_1(-1)$. Then M is the semi-Riemannian Hopf cylinder in $\mathbb{H}^3_1(-1)$ associated to a unit speed curve β in $\mathbb{H}^2_s(-1/4)$ if and only if M is the B-scroll over any horizontal lift $\overline{\beta}$ of β.

Let $\beta\colon I \to \mathbb{H}^2_s(-1/4)$ be a unit speed curve with Frenet frame $\{T, \xi_2\}$ and curvature function κ.

Let $\overline{\beta}$ be a horizontal lift of β to $\mathbb{H}^3_1(-1)$ with Frenet frame $\{\overline{T}, \overline{\xi}_2, \xi_3^*\}$ and curvature $\overline{\kappa} = \kappa \circ \pi_s$ and $\tau = 1$. Recall that ξ_3^* is nothing but the unit tangent vector field to the fibers along $\overline{\beta}$.

Then the Hopf cylinder M_β can be orthogonally parametrized as

$$X(t,z) = \begin{cases} \cos(z)\overline{\beta}(t) + \sin(z)\xi_3^*(t), & \text{if } s = 0 \\ \cosh(z)\overline{\beta}(t) + \sinh(z)\xi_3^*(t), & \text{if } s = 1 \end{cases}$$

Setting, as usual, $X_t = \frac{\partial X}{\partial t}$ and $X_z = \frac{\partial X}{\partial z}$, then $\{X_t, X_z\}$ is an orthonormal frame of $T_{X(t,z)}M_\beta$ along X and a direct computation shows that the shape operator S of M_β in this frame can be written as

$$\begin{aligned} S(X_t) &= \overline{\kappa} X_t + \varepsilon X_z, \\ S(X_z) &= X_t, \end{aligned}$$

where $\varepsilon = +1$ if M_β is Riemannian and $\varepsilon = -1$ if M_β is Lorentzian.

Notice that a unit normal vector field to M_β into $\mathbb{H}_1^3(-1)$ is obtained from the complete horizontal lift of ξ_2 and it is, of course, $\overline{\xi}_2$ along each horizontal lift of β.

As a consequence we have that M_β is a flat surface, as we said before, and its mean curvature function α is given by $\alpha = \overline{\kappa}/2$.

According to the description of curves with constant curvature in $\mathbb{H}_s^2(-1/4)$ we can give the following description of Hopf cylinders of constant mean curvature.

Proposition

Let β be a unit speed curve in $\mathbb{H}_s^2(-1/4)$ with constant curvature κ. Then one of the following statements holds:

(1) *M_β is a minimal complex circle ($\kappa = 0$).*

(2) *M_β is a non-minimal complex circle ($0 < \kappa^2 < 4$).*

(3) *M_β is the Hopf cylinder over the horocycle ($s = 0$, $\kappa^2 = 4$) or over the pseudo-horocycle ($s = 1$, $\kappa^2 = 4$).*

(4) *M_β is one of the following semi-Riemannian products*

$$\begin{array}{lll} (4.1) & \mathbb{H}_1^1(-r^2) \times \mathbb{S}^1(r^2 - 1) & \text{if } s = 0 \text{ and } \kappa^2 > 4, \\ (4.2) & \mathbb{H}^1(-r^2) \times \mathbb{S}_1^1(r^2 - 1) & \text{if } s = 1 \text{ and } \kappa^2 > 4. \end{array}$$

(5) *M_β is the Riemannian product $\mathbb{H}^1(-r^2) \times \mathbb{H}^1(-1 + r^2)$ with r satisfying*

$$\frac{1 - 2r^2}{r\sqrt{1 - r^2}} = \kappa.$$

It should be noticed that the above cases (1) through (4) correspond to the Lorentzian character of M_β and so, according to the above theorem, it can be considered as the classification of B-scrolls with constant mean curvature in $\mathbb{H}_1^3(-1)$. The remainder case corresponds with the Riemannian character of M_β.

2 Lorentzian Hopf tori ([9])

Hopf surfaces in $\mathbb{H}_1^3(-1)$ shaped on closed curves in $\mathbb{H}^2(-1/4)$ are Lorentzian flat tori. Now we want to determine the isometry group of these surfaces.

We use standard computations involving the structure equations of the induced connection and [17] to get a similar result to that of Pinkall:

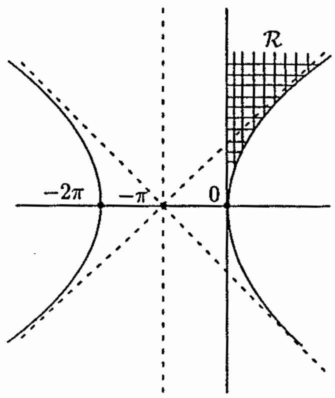
Theorem

Let β be a closed embedded curve in $\mathbb{H}^2(-1/4)$ of length L enclosing an area A. Then M_β is isometric to $\mathbb{L}^2/\Lambda$, Λ being the lattice in the Lorentzian plane $\mathbb{L}^2$ generated by the vectors $(2\pi, 0)$ and $(2A, L)$.

Remark It is worth noting that $(2A, L)$ is only constrained by the isoperimetric inequality in $\mathbb{H}^2(-1/4)$

$$L^2 \geq 4\pi A + 4A^2.$$

Hence the vector $(2A, L)$ must be spacelike. Therefore $(2A, L)$ lies in the shaded region $\mathcal{R}$



3 Willmore tori in $\mathbb{H}^3_1(-1)$ ([9])

Inspired again by Pinkall's paper, we look for Willmore tori in $\mathbb{H}^3_1(-1)$ associated to elastic curves in $\mathbb{H}^2(-1/4)$.

A unit-speed curve γ in M^n_ν is said to be an **elastica** (or **elastic curve**) if it is an extremal point of the functional

$$\mathfrak{G}_\lambda(\gamma) = \int_0^L (\langle \nabla_T T, \nabla_T T\rangle + \lambda) ds = \int_0^1 (\langle \nabla_T T, \nabla_T T\rangle + \lambda) v dt,$$

for some λ, where ds and L stand for the arclength on γ and the length of γ, respectively. It is called a **free elastica** if $\lambda = 0$ (see [21]).

The Euler-Lagrange equation associated to this variational problem is

$$2\nabla^3_T T + \varepsilon_1 \nabla_T((3\varepsilon_2\kappa^2 - \lambda)T) - 2R(\nabla_T T, T)T = 0.$$

Frenet equations for γ:

$$\begin{aligned}
\nabla_T T &= \varepsilon_2 \kappa \xi_2, \\
\nabla_T \xi_2 &= -\varepsilon_1 \kappa T - \varepsilon_3 \tau \xi_3, \\
\nabla_T \xi_3 &= \varepsilon_2 \tau \xi_2 + \delta,
\end{aligned}$$

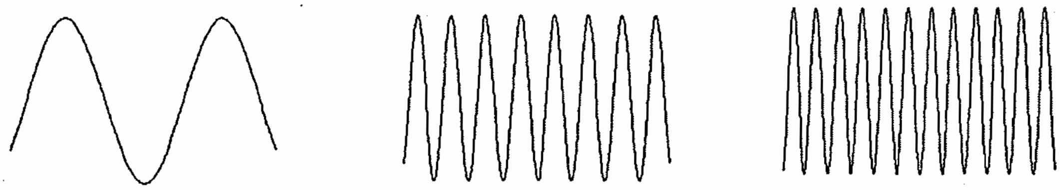
where $\delta \in \text{span}\{T, \xi_2, \xi_3\}^\perp$, $\langle \xi_i, \xi_i \rangle = \varepsilon_i$ and τ is the torsion function (the second curvature if $n > 3$). Assume now that M^n_ν is of constant curvature c. Then the

Euler-Lagrange equation can be rewritten as follows

$$\begin{aligned}
&2\varepsilon_2\kappa'' + \varepsilon_1\kappa^3 - 2\varepsilon_3\kappa\tau^2 + \varepsilon_1\varepsilon_2(2c-\lambda)\kappa = 0,\\
&2\kappa'\tau + \kappa\tau' = 0,\\
&\kappa\tau\delta = 0.
\end{aligned}$$

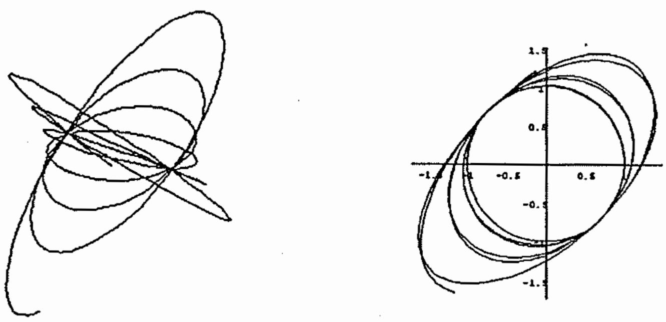
Taking $u = \kappa^2$ these equations can be solved by standard techniques in terms of elliptic functions.

For instance, a qualitative description of elasticae in the Lorentz-Minkowski plane $\mathbb{L}^2$ is given as follows. In general, the elasticae in $\mathbb{L}^2$ are curves which oscillates around a geodesic, so that the parameter λ, in some sense, could be viewed as the wavelength. That length increases or decreases according to $\varepsilon_1\lambda$ does. In the following we skecht some of these curves.



$\varepsilon_1\lambda > 0$ $\lambda = 0$ $\varepsilon_1\lambda < 0$

As for the pseudo-hyperbolic plane $\mathbb{H}_1^2(-1)$ the behaviour of the elastic curves is essentially the same as in $\mathbb{L}^2$, they also oscillate around geodesics. In particular, we can draw a free elastica oscillating around the central circle in $\mathbb{H}_1^2(-1)$.



Free elastica Projection on xy-plane

Let M_s^2 be a surface in an indefinite 3-space $\widetilde{M}_\mu^3$ of constant curvature c.

We define the operator W over sections of the normal bundle of M_s^2 into $\widetilde{M}_\mu^3$ as follows

$$W : \mathfrak{N}M \to \mathfrak{N}M, \qquad W(\xi) = (\Delta^D + 2\langle H, H\rangle I - \widetilde{A})\xi,$$

$\widetilde{A}$ standing for Simons operator.

A cross section ξ will be called a Willmore section if $W(\xi) = 0$. Suppose that M is compact and consider the Willmore functional

$$\mathcal{W}(M) = \int_M (\langle H, H\rangle + c)dv.$$

Then the operator W naturally appears provided that one computes the first variation formula for $\mathcal{W}$.

Now Willmore surfaces are nothing but the extremal points of the Willmore functional and they are characterized from the fact that their mean curvature vector fields are Willmore fields.

Proposition

Let $\pi_s : \mathbb{H}^3_1(-1) \to \mathbb{H}^2_s(-1/4)$ and $\beta : I \to \mathbb{H}^2_s(-1/4)$ be as before. Then the Hopf cylinder M_β satisfies $WH = \mu H$, $\mu \in \mathbb{R}$, if and only if β is an elastica in $\mathbb{H}^2_s(-1/4)$.

We know that the fibers of $\pi_0 : \mathbb{H}^3_1(-1) \to \mathbb{H}^2(-1/4)$ are circles, and so compact, whereas the fibers of $\pi_1 : \mathbb{H}^3_1(-1) \to \mathbb{H}^2_1(-1/4)$ are not compact. Therefore to find compact Hopf surfaces we have to consider Hopf torus shaped on closed curves in $\mathbb{H}^2(-1/4)$

In the anti-De Sitter world we have known that a Hopf torus M_β is a Willmore surface in $\mathbb{H}^3_1(-1)$ if and only if β is an elastica in $\mathbb{H}^2(-1/4)$ with $\lambda = -4$. However we have recently known from D. Singer (private communication) that cannot be hold. Thus we have to say that *there is no (Lorentzian) Willmore Hopf torus in $\mathbb{H}^3_1(-1)$.*

4 The Betchov-Da Rios equation ([9])

The Betchov-Da Rios (BDR) equation $u' \wedge u'' = \dot{u}$, also called localized induction equation in 3-dimensional hydrodynamics, is a soliton equation for space curves $u(t, s)$, where $u' = \partial u/\partial t$ and $\dot{u} = \partial u/\partial s$.

It is a straigthforward computation that, in general, the standard parametrization $X(t, z)$ of M_β is not a solution of BDR.

We ask for the classification of $h \in \mathrm{Diff}(\mathbb{R}^2)$ in order to $Y = X \circ h$ be a solution of BDR equation in $\mathbb{H}^3_1(-1)$.

We completely solve this problem.

Let η be a unit normal vector field to M_β in $\mathbb{H}^3_1(-1)$. Then η can be written as follows:

$$\eta = \begin{cases} -\sin(z)\overline{T}(t) + \varepsilon_1 \cos(z)\xi^*_2(t), & s = 0, \\ -\sinh(z)\overline{T}(t) + \varepsilon_1 \cosh(z)\xi^*_2(t), & s = 1. \end{cases}$$

A straightforward computation yields $Y(u,v)$ is a solution of BDR equation if and only if the following PDE system holds:

$$\begin{aligned} t_v &= (-1)^s \varepsilon_1 t_u z_u (t_u \overline{\kappa} + 2z_u), \\ z_v &= t_u^2 (t_u \overline{\kappa} + 2z_u), \\ 0 &= t_u z_{uu} - z_u t_{uu}. \end{aligned}$$

Solving we get

Theorem

Let β be an arc length parametrized curve in $\mathbb{H}_s^2(-1/4)$ and M_β its Hopf cylinder in $\mathbb{H}_1^3(-1)$. For any $h \in Diff(\mathbb{R}^2)$, take $Y = X \circ h : \mathbb{R}^2 \to M_\beta$, X being the standard covering of $\mathbb{R}^2$ over M_β. Then Y is a solution of BDR soliton equation in $\mathbb{H}_1^3(-1)$ if and only if the following statements hold:

(i) *β has constant curvature, say κ, in $\mathbb{H}_s^2(-1/4)$;*

(ii) *$h(u,v) = (t(u,v), z(u,v))$ is given by*

$$\begin{aligned} t(u,v) &= au + (-1)^s ag\rho v + c_1, \\ z(u,v) &= agu + \varepsilon_1 a\rho v + c_2, \end{aligned}$$

where $(\varepsilon_1 - (-1)^s g^2)a^2 = \varepsilon$, ε_1 being the causal character of β, ε the causal character of the u-curves, $g \in \mathbb{R} - \{-\kappa/2\}$, $\rho = \varepsilon_1(\kappa + 2g)a^2$ is the curvature of the u-curves in $\mathbb{H}_1^3(-1)$ and a, c_1, c_2 are arbitrary constants.

Corollary 1

Let M_β be a Lorentzian Hopf cylinder in $\mathbb{H}_1^3(-1)$ of constant mean curvature. Then the only soliton solutions of BDR equation in $\mathbb{H}_1^3(-1)$ lying in M_β are the null geodesics of M_β.

Corollary 2

Let β be a closed curve of constant curvature in $\mathbb{H}^2(-1/4)$ with length L enclosing an oriented area A. Then for any rational number q, the slope

$$g = \frac{2\pi}{L}\left(q + \frac{A}{\pi}\right)$$

defines a unique closed helix in $\mathbb{H}_1^3(-1)$ and therefore a closed solution of BDR equation in $\mathbb{H}_1^3(-1)$ living in the Hopf torus M_β. Furthermore, the closed solution is either spacelike, or timelike or null according to $q \in (q_1, q_2)$, $q \in \mathbb{R} - (q_1, q_2)$, $q \in \{q_1, q_2\}$, respectively, where $q_1 = -\frac{A}{\pi} - \frac{L}{2\pi}$ and $q_2 = -\frac{A}{\pi} + \frac{L}{2\pi}$.

5 General helices in 3-dimensional Lorentzian space forms ([10])

Helices got as solutions of BDR brought us to mind a Barros' idea: look out general helices.

A curve of constant slope or **general helix** in Euclidean space $\mathbb{R}^3$ is defined by the property that its tangent indicatrix is a planar curve. The straight line perpendicular to this plane is called the axis of the general helix.

A classical result stated by M.A. Lancret in 1802 and first proved by B. de Saint Venant in 1845 is: "A necessary and sufficient condition in order to a curve be a general helix is that the ratio of curvature to torsion be constant".

For a given couple of one variable functions (eventually curvature and torsion parametrized by arclength) one might like to get an arclength parametrized curve for which the couple works as the curvature and torsion functions. This problem is usually referred as "the solving natural equations problem"

The natural equations for general helices can be integrated, not only in $\mathbb{R}^3$, but also in the 3-sphere $\mathbb{S}^3$ (the hyperbolic space is poor in this kind of curves and only helices are general helices). Indeed Barros, [6], has shown that general helices are geodesics either of right general cylinders or of Hopf cylinders, according to the curve lies in $\mathbb{R}^3$ or $\mathbb{S}^3$.

What about general helices in the 3-dimensional Lorentzian space forms?

A non-null curve γ immersed in $\mathbb{L}^3$ is called a general helix if its tangent indicatrix is contained in some plane, say π, of $\mathbb{L}^3$. Since π can be either degenerate or non-degenerate, then both cases are distinguished by calling degenerate and non-degenerate general helices, respectively.

We will point out a remarquable and deep difference between the behaviour of general helices in Euclidean and Lorentzian geometries:

While in $\mathbb{R}^3$ general helices are geodesics in right general cylinders, as classically is shown, we will prove that general helices in $\mathbb{L}^3$ are geodesics in either right general cylinders or flat B-scrolls, according to the general helix is non-degenerate or degenerate.

This nice difference between Euclidean and Lorentzian geometries (from the point of view of the behaviour of general helices) confirms once more the important role of the notion of B-scroll in Lorentzian geometries.

General helices in 3-dimensional De Sitter $\mathbb{S}^3_1$ and anti De Sitter $\mathbb{H}^3_1$ spaces are considered with the help of the idea of Langer and Singer (see [21]): use Killing vector field along a curve in a 3-dimensional real space form.

The Lancret theorem in $\mathbb{S}^3_1$ and $\mathbb{H}^3_1$ underlines deep differences between the pseudospherical and pseudohyperbolic spaces. The pseudohyperbolic case is nicely analogous to the Lorentz-Minkowskian case, whereas in the pseudospherical case there are no nontrivial general helices. From this point of view, *the roles played by the non flat Lorentzian space forms $\mathbb{H}^3_1$ and $\mathbb{S}^3_1$ correspond with those played by the non flat Riemannian space forms $\mathbb{S}^3$ and $\mathbb{H}^3$, respectively.*

Let $\gamma(t)$ be a non-null immersed curve in a 3-dimensional Lorentzian space form M with sectional curvature c and let $v(t) = |\gamma'(t)|$ be the speed of γ.

Let us consider a variation of γ, $\Gamma = \Gamma(t,z): I \times (-\varepsilon, \varepsilon) \to M$ with $\Gamma(t,0) = \gamma(t)$. In particular one can choose $\varepsilon > 0$ in such a way that all t-curves of the variation have the same causal character as that of γ. Associated with Γ there are two vector fields along Γ, $V(t,z) = \frac{\partial \Gamma}{\partial z}(t,z)$ and $W(t,z) = \frac{\partial \Gamma}{\partial t}(t,z)$. In particular $V(t) = V(t,0)$ is the variational vector field along γ and $W(t,z)$ is the tangent vector fields of the t-curves. We will use the notation $V = V(t,z)$, $v = v(t,z)$, $\kappa = \kappa(t,z)$, etc. with the obvious meanings. Also, if s denotes the arclength parameter of the t-curves, we will write $v(s,z)$, $V(s,z)$, $\kappa(s,z)$, etc. for the corresponding reparametrizations.

A straightforward but long computation allows us to obtain formulas for $\frac{\partial v}{\partial z}(t,0)$, $\frac{\partial \kappa^2}{\partial z}(t,0)$ and $\frac{\partial \tau^2}{\partial z}(t,0)$ which we collect, along with another standard identity, in the following lemma.

Lemma

(1) $[V,W] = 0$;

(2) $\dfrac{\partial v}{\partial z}(t,0) = -\varepsilon_1 g v$, *with* $g = \langle \overline{\nabla}_T V, T \rangle$;

(3) $\dfrac{\partial \kappa^2}{\partial z}(t,0) = 2\varepsilon_2 \langle \overline{\nabla}_T^2 V, \overline{\nabla}_T T \rangle + 4\varepsilon_1 g \kappa^2 + 2\varepsilon_2 \langle R(V,T)T, \overline{\nabla}_T T \rangle$;

(4) $\dfrac{\partial \tau^2}{\partial z}(t,0) = -2\varepsilon_2 \langle \dfrac{1}{\kappa} \overline{\nabla}_T^3 V - \dfrac{\kappa'}{\kappa^2} \overline{\nabla}_T^2 V + \varepsilon_1(\varepsilon_2 \kappa + \dfrac{c}{\kappa}) \overline{\nabla}_T V - \varepsilon_1 \dfrac{c\kappa'}{\kappa^2} V, \tau B \rangle$,

where $\langle , \rangle$ *denotes the Lorentzian metric of* M *and* $\kappa' = \dfrac{\partial \kappa}{\partial t}(t,0)$.

Without loss of generality we can assume γ to be arclength parametrized.

A vector field $V(s)$ along γ, which infinitesimally preserves unit speed parametrization (that means $\frac{\partial v}{\partial z}(t,0) = 0$ for a V-variation of γ) is said to be a Killing vector field along γ if this evolves in the direction of V whithout changing shape, only position. In other words, the curvature and torsion functions of γ remain unchanged as the curve evolves.

Hence Killing vector fields along γ are characterized by the equations

$$\frac{\partial v}{\partial z}(t,0) = \frac{\partial \kappa^2}{\partial z}(t,0) = \frac{\partial \tau^2}{\partial z}(t,0) = 0.$$

Then V is a Killing vector field along γ if and only if it satisfies the following conditions:

a) $\langle \overline{\nabla}_T V, T \rangle = 0$,

b) $\langle \overline{\nabla}_T^2 V, N \rangle + \varepsilon_1 c \langle V, N \rangle = 0$,

c) $\langle \dfrac{1}{\kappa} \overline{\nabla}_T^3 V - \dfrac{\kappa'}{\kappa^2} \overline{\nabla}_T^2 V + \varepsilon_1(\varepsilon_2 \kappa + \dfrac{c}{\kappa}) \overline{\nabla}_T V - \varepsilon_1 c \dfrac{\kappa'}{\kappa^2} V, \tau B \rangle = 0$.

Now when M is simply connected, since the restriction to γ of any Killing field $\widetilde{V}$ of M is a Killing vector field along γ, one concludes from a well known dimension argument, the following lemma.

Lemma

Let M be a complete, simply connected, Lorentzian space form and γ a non-null immersed curve in M. A vector field V on γ is a Killing vector field along γ if and only if it extends to a Killing field $\widetilde{V}$ on M.

The Lancret theorem in $\mathbb{L}^3$

Let γ be a non-null immersed curve in $\mathbb{L}^3$ with curvature and torsion functions κ and τ, respectively. Then the following statements are equivalent:

(a) *γ is a general helix in $\mathbb{L}^3$;*

(b) *There exists a constant length Killing vector field V along γ which is orthogonal to the acceleration vector field of γ;*

(c) *There exists a constant r such that $\tau = r\kappa$.*

Moreover a general helix γ is degenerate if and only if $r = \pm 1$ and its normal vector field is spacelike. The Killing vector field V in (b) is not uniquely determined if γ is a helix (κ and τ both are constant); however, in this case, V can be uniquely determined, up to constants, once it is chosen parallel along γ (say otherwise, its extended Killing vector field in $\mathbb{L}^3$ is a translation vector field).

Solving natural equation for non-degenerate general helices

Let β be a non-null immersed curve in $\mathbb{L}^3$. Then β is a non-degenerate general helix if and only if it is a geodesic in some right cylinder whose directrix and generatrix are both non-null.

Solving natural equation for degenerate general helices

Let β be a non-null immersed curve in $\mathbb{L}^3$. Then β is a degenerate general helix if and only if it is a geodesic in some flat B-scroll in $\mathbb{L}^3$.

How to define general helices in non-flat 3-dimensional Lorentzian spaces forms?

Definition

A curve γ in M is said to be a general helix if there exists a Killing vector field V along γ with constant length and orthogonal to the acceleration vector field of γ.

We will say that V is **an axis** of the general helix γ.

Obvious examples of general helices in M are the following. Curves with torsion vanishing anywhere, where the unit binormal works as an axis. Helices are also general helices, where any vector field chosen in the rectifying plane having constant coordinates relative to T and B runs as an axis.

We can follow notation and terminology used in $\mathbb{L}^3$ to say that zero torsion curves are non-degenerate general helices, because the axis B is obviously non-null. As for curves with both constant curvature and torsion we know that for any pair of constants a and b the vector field along γ given by $V(s) = aT + bB$ is always a Killing vector field. Of course, when $\varepsilon_2 = -1$, i.e., the rectifying plane is positive definite at any point, all Killing vector fields $V(s)$ are non-null and we will say that the general helix is non-degenerate. However, if $\varepsilon_2 = 1$, i.e., the rectifying plane is Lorentzian, we have Killing vector fields along γ being either spacelike, or timelike, or null. It does not allow us to decide if such a general helix is degenerate or not. However, we can determine a unique Killing vector field along the helix by forcing it to be parallel along γ. The helix is said to be **degenerate** or **non-degenerate** according to V is null or non-null, respectively.

The Lancret theorem in the De Sitter space

A non-null immersed curve γ in $\mathbb{S}^3_1$ is a general helix if and only if either

(1) $\tau \equiv 0$ *and γ is a curve in some totally geodesic surface of $\mathbb{S}^3_1$; or*

(2) γ *is a helix in $\mathbb{S}^3_1$ (i.e. curvature κ and torsion τ constants).*

The Lancret theorem in the anti De Sitter space

A non-null immersed curve γ in $\mathbb{H}^3_1$ is a general helix if and only if either

(1) $\tau \equiv 0$ *and γ is a curve in some totally geodesic surface of $\mathbb{H}^3_1$. The curve admits only one axis which agrees with its binormal, being parallel along the curve and non-null. The general helix is non-degenerate; or*

(2) γ *is a helix in $\mathbb{H}^3_1$. It admits a plane (the rectifying plane) of axes but only one is parallel along γ. This parallel axis is null, and so γ is degenerate, if and only if $\varepsilon_2 = +1$ and $\tau = \pm\kappa$. Otherwise γ is non-degenerate; or*

(3) *there exists a certain constant b such that the curvature κ and the torsion τ functions of γ are related by $\tau = b\kappa \pm 1$. The curve admits a unique axis which can not be parallel along γ. It is null, and so γ is degenerate, if and only if $b = \pm 1$ and γ has spacelike normal vector ($\varepsilon_2 = +1$).*

Solving natural equation for non-degenerate general helices in $\mathbb{H}^3_1(-1)$

Let β a non-null immersed curve in $\mathbb{H}^3_1$. Then β is a non-degenerate general helix if and only if it is a geodesic in some Hopf cylinder M_γ.

Solving natural equation for degenerate general helices in $\mathbb{H}^3_1(-1)$

Let β a non-null immersed curve in $\mathbb{H}^3_1$. Then β is a degenerate general helix if and only if it is a geodesic in some flat B-scroll over a null curve.

6 Willmore tori and Willmore-Chen submanifolds into pseudo-Riemannian spaces ([11])

Two problems

(i) Find examples of Willmore surfaces in the anti De Sitter space.

(ii) Find examples of Willmore-Chen submanifolds in pseudo-Riemannian spaces (with non zero index).

6.1 Willmore tori in non standard anti de Sitter 3-space

Let $\pi: (M, g) \to (B, h)$ be a pseudo-Riemannian submersion.

A very interesting deformation of the metric g by changing the relative scales of B and the fibres (see [14]).

The canonical variation g_t, $t > 0$, of g by setting

$$\begin{aligned} g_t|_{\mathcal{V}} &= t^2\, g|_{\mathcal{V}}, \\ g_t|_{\mathcal{H}} &= g|_{\mathcal{H}}, \\ g_t(\mathcal{V}, \mathcal{H}) &= 0, \end{aligned}$$

where $\mathcal{V}$ and $\mathcal{H}$ stand for vertical and horizontal distributions, respectively, associated with the submersion. Thus we obtain a one-parameter family of pseudo-Riemannian submersions $\pi_t: (M, g_t) \to (B, h)$ with the same horizontal distribution $\mathcal{H}$, for all $t > 0$.

Let us consider the canonical variation of the indefinite Hopf fibration

$$\pi = \pi_0 : \mathbb{H}^3_1 \to \mathbb{H}^2(-1/2)$$

to get a one-parameter family of pseudo-Riemannian submersions

$$\pi_t: (\mathbb{H}^3_1, g_t) \to (\mathbb{H}^2(-1/2), g_0).$$

Let γ be a unit speed curve immersed in $\mathbb{H}^2(-1/2)$. Set $\mathcal{T}_{\gamma,t} = \pi_t^{-1}(\gamma)$. Then $\mathcal{T}_{\gamma,t}$ is a Lorentzian flat surface immersed in $\mathbb{H}^3_1$, that will be called **the Lorentzian Hopf cylinder over γ.**

As the fibres of π_t are $\mathbb{H}^1_1$, which topologically are circles, then $\mathcal{T}_{\gamma,t}$ is a Hopf torus in $(\mathbb{H}^3_1, g_t)$, provided that γ is a closed curve.

Proposition

Let S be an immersed surface into $(\mathbb{H}^3_1, g_t)$. Then S is G-invariant if and only if S is a Lorentzian Hopf cylinder $\mathcal{T}_{\gamma,t} = \pi_t(\gamma)$ over a certain curve γ immersed in the hyperbolic 2-plane $(\mathbb{H}^2(-1/2), g_0)$.

Theorem

Let $\pi_t: (\mathbb{H}^3_1, g_t) \to (\mathbb{H}^2(-1/2), g_0)$, $t > 0$, be the canonical variation of the pseudo-Riemannian Hopf fibration. Let γ be a closed immersed curve in $(\mathbb{H}^2(-1/2), g_0)$ and $\mathcal{T}_{\gamma,t} = \pi_t^{-1}(\gamma)$ its Lorentzian Hopf torus. Then $\mathcal{T}_{\gamma,t}$ is a Willmore surface in $(\mathbb{H}^3_1, g_t)$ if and only if γ is an elastica in $(\mathbb{H}^2(-1/2), g_0)$ with Lagrange multiplier $\lambda = -4t^2$.

Proof

The Willmore functional on $\mathcal{M} = \{\phi: T \to (\mathbb{H}^3_1, g_t) : \phi \text{ is an immersion}\}$ is

$$\Omega(\phi) = \int_T (\langle H, H\rangle + R^t)\, dv,$$

H and R^t standing for the mean curvature vector field of T and the sectional curvature of $(\mathbb{H}_1^3, g_t)$, measured with respect to the tangent plane to (T, ϕ), respectively. It is clear that, for any $e^{i\theta} \in \mathbb{S}^1$, we have that $\Omega(\phi) = \Omega(e^{i\theta} \cdot \phi)$. Now let us denote by $\mathcal{C}$ the set of critical points of Ω in $\mathcal{M}$, i.e., $\mathcal{C}$ is the set of genus one Willmore surfaces. Let $\mathcal{M}_G$ be the submanifold of $\mathcal{M}$ made up by those immersions of T which are $(G = \mathbb{S}^1)$-invariant and let $\mathcal{C}_G$ be the set of critical points of Ω restricted to $\mathcal{M}_G$. The principle of symmetric criticality of Palais, [23], can be used here to find that $\mathcal{C} \cap \mathcal{M}_G = \mathcal{C}_G$. Now from the above Proposition we obtain that $\mathcal{C}_G = \{\mathcal{T}_{\gamma,t} = \pi_t^{-1}(\gamma) : \gamma \text{ is an immersed closed curve in } (\mathbb{H}^2(-1/2), g_0)\}$. To compute $\Omega(\mathcal{T}_{\gamma,t})$, i.e., the Willmore functional on $\mathcal{C}_G$, we first notice that $\alpha = \frac{1}{2}\kappa$, κ being the curvature function of γ in $(\mathbb{H}^2(-1/2), g_0)$.

On the other hand

$$R^t = -g_t(ti\overline{X}, ti\overline{X}) = -t^2.$$

Let L be the length of γ. As the fibres of g_t are circles of radii t, we have

$$\Omega(\mathcal{T}_{\gamma,t}) = \int_{\pi_t^{-1}(\gamma)} (\alpha^2 + R^t) dv = \int_0^L \int_0^{2\pi t} \left(\frac{1}{4}\kappa^2 - t^2\right) ds dr = \frac{\pi t}{4} \int_0^L (\kappa^2 - 4t^2) ds$$

and the proof finishes.

Then we give, for $t \in (0,1)$, infinitely many Willmore tori in $(\mathbb{H}_1^3, g_t)$.

6.2 Willmore-Chen submanifolds in the hyperbolic space

We give a new method to construct critical points of the Willmore-Chen functional in the pseudo-hyperbolic space $\mathbb{H}_r^n = \mathbb{H}_r^n(-1)$.

First step: write $\mathbb{H}_r^n$ as a warped product with base space the standard hyperbolic space $\mathbb{H}^{n-r}$.

Second step: use the conformal invariance of the Willmore-Chen variational problem to make a conformal change of the canonical metric of $\mathbb{H}_r^n$.

Third step: use the principle of symmetric criticality of R. Palais to reduce the problem to a variational one for closed curves in the once punctured standard $(n-r)$-sphere.

Given $0 < r < n$, let

$$\mathbb{H}^{n-r} = \{(x_0, x) \in \mathbb{R} \times \mathbb{R}^{n-r} : -x_0^2 + \langle x, x\rangle = -1 \text{ and } x_0 > 0\}$$

the hyperbolic $(n-r)$-space and

$$\mathbb{H}_r^n = \{(\xi, \eta) \in \mathbb{R}^{r+1} \times \mathbb{R}^{n-r} : -\langle \xi, \xi\rangle + \langle \eta, \eta\rangle = -1\}$$

the pseudo-hyperbolic n-space. They are hypersurfaces in $\mathbb{R}_1^{n-r+1}$ and $\mathbb{R}_{r+1}^{n+1}$, respectively. The induced metrics on these spaces, from those in the corresponding pseudo-Euclidean spaces, define standard metrics h_0 on $\mathbb{H}_r^n$ and g_0 on $\mathbb{H}^{n-r}$, both with constant curvature -1.

Let $\mathbb{S}^r$ be the standard unit r-sphere endowed with its canonical metric $d\sigma^2$. Consider the mapping

$$\Phi: \mathbb{H}^{n-r} \times \mathbb{S}^r \to \mathbb{H}^n_r$$

defined by

$$\Phi((x_0, x), u) = (x_0 u, x).$$

It is not difficult to see that Φ defines a diffeomorphism whose inverse is $\Phi^{-1}(\xi, \eta) = ((|\xi|, \eta), \xi/|\xi|)$. For any curve $\beta(t) = ((x_0(t), x(t)), u(t))$ in $\mathbb{H}^{n-r} \times \mathbb{S}^r$ we have

$$\left|\mathrm{d}\Phi_{\beta(t)}(\beta'(t))\right|^2 = -x_0'(t)^2 + |x'(t)|^2 - x_0(t)^2|u'(t)|^2.$$

Let $f : \mathbb{H}^{n-r} \to \mathbb{R}$ be the positive function given by $f(x_0, x) = x_0$ and consider the metric $g = g_0 - f^2 d\sigma^2$ on $\mathbb{H}^{n-r} \times \mathbb{S}^r$. The pseudo-Riemannian manifold $(\mathbb{H}^{n-r} \times \mathbb{S}^r, g)$ is called the **warped product** of base $(\mathbb{H}^{n-r}, g_0)$ and fibre $(\mathbb{S}^r, -d\sigma^2)$ with warping function f.

It is usually denoted by $(\mathbb{H}^{n-r}, g_0) \times_f (\mathbb{S}^r, -d\sigma^2)$ or $\mathbb{H}^{n-r} \times_f (-\mathbb{S}^r)$ when the metrics on the base and fibre are understood (see [14] and [22]).

Φ is an isometry between $\mathbb{H}^{n-r} \times_f (-\mathbb{S}^r)$ and $(\mathbb{H}^n_r, h_0)$.

A new metric h on $\mathbb{H}^n_r$ is defined by

$$h = \frac{1}{f^2} h_0 = \frac{1}{f^2} g_0 - d\sigma^2,$$

with the obvious meaning by removing the pulling back via Φ.

Thus $(\mathbb{H}^n_r, h)$ is the pseudo-Riemannian product of $(\mathbb{H}^{n-r}, \frac{1}{f^2} g_0)$ and $(\mathbb{S}^r, -d\sigma^2)$.

Finally it is not difficult to see that $(\mathbb{H}^{n-r}, \frac{1}{f^2} g_0)$ has constant sectional curvature 1, so that it can be identified, up to isometries, with the once punctured standard $(n-r)$-sphere $(\Sigma^{n-r}, d\sigma^2)$.

Consequently, $(\mathbb{H}^n_r, h)$ is nothing but $(\Sigma^{n-r}, d\sigma^2) \times (\mathbb{S}^r, -d\sigma^2)$, up to isometries.

$SO(r+1)$-invariant submanifolds in $\mathbb{H}^n_r$

For any immersed curve $\gamma : [0, L] \to \mathbb{H}^{n-r}$, we define the semi-Riemannian $(r+1)$-submanifold $\Upsilon_\gamma = \Phi(\gamma \times \mathbb{S}^r)$. It is clear that Υ_γ has index r and we will refer to Υ_γ as the cylinder over γ.

Now let $G = SO(r+1)$ be the group of isometries of $(\mathbb{S}^r, -d\sigma^2)$.

Then G acts transitively on $(\mathbb{S}^r, -d\sigma^2)$.

So we define an action of G on $\mathbb{H}^n_r$ as follows

$$a \cdot (\xi, \eta) = \Phi(a \cdot \Phi^{-1}(\xi, \eta)) = (a(\xi), \eta),$$

for any $a \in G$.

This action is realized through isometries of $(\mathbb{H}^n_r, h_0)$. The following statement characterizes the cylinders over curves in $\mathbb{H}^{n-r}$ as symmetric points of the above mentioned G-action.

Proposition

Let M be an $(r+1)$-dimensional submanifold in $\mathbb{H}^n_r$. Then M is G-invariant if and only if M is a cylinder Υ_γ over a certain curve γ in $\mathbb{H}^{n-r}$.

Critical points of $\mathcal{F}^r(\gamma) = \int_\gamma (\kappa^2)^{\frac{r+1}{2}} ds$

Now we deal with the functional

$$\mathcal{F}^r(\gamma) = \int_\gamma (\kappa^2)^{\frac{r+1}{2}} ds,$$

defined on the manifold of regular closed curves (or curves satisfying given first order boundary data) in a given pseudo-Riemannian manifold, where r stands for any natural number (even though all computations also hold if r is a real number). Notice that we write the integrand in that form to point out that it is an even function of the curvature κ. Also $\mathcal{F}^1$ agrees with $\mathcal{G}$, which is the elastic energy functional for free elasticae.

Let $\gamma: I \subset \mathbb{R} \to \mathbb{S}^m$ be a unit speed curve in the unit m-sphere with curvatures $\{\kappa, \tau, \ldots\}$ and Frenet frame $\{T = \gamma', \xi_2, \ldots, \xi_m\}$. Given a variation $\Gamma := \Gamma(s,t) : I \times (-\varepsilon, \varepsilon) \to \mathbb{S}^m$ of γ, with $\Gamma(s,0) = \gamma(s)$, we have the associated variation vector field $W(s) = \frac{\partial \Gamma}{\partial t}(s,0)$ along γ. We will use the notation and terminology of Langer-Singer. Set $V(s,t) = \frac{\partial \Gamma}{\partial s}$, $W(s,t) = \frac{\partial \Gamma}{\partial t}$, $v(s,t) = |V(s,t)|$, $T(s,t) = \frac{1}{v}V(s,t)$, $\kappa(s,t) = |\nabla_T T|^2$, ∇ being the Levi-Civita connection of $\mathbb{S}^m$. The following lemma collects some basic facts which we will use to find the Euler-Lagrange equations relative to $\mathcal{F}^r$.

Langer and Singer Lemma ([21])

With the above notation, the following assertions hold:

$$\begin{aligned}
[V, W] &= 0; \\
\frac{\partial v}{\partial t} &= \langle \nabla_T W, T \rangle v; \\
[W, T] &= -\langle \nabla_T W, T \rangle T; \\
[[W, T], T] &= T(\langle \nabla_T W, T \rangle) T; \\
\frac{\partial \kappa^2}{\partial t} &= 2\langle \nabla_T^2 W, \nabla_T T \rangle - 4\langle \nabla_T W, T \rangle \kappa^2 + 2\langle R(W, T)T, \nabla_T T \rangle,
\end{aligned}$$

R being the Riemann curvature tensor of $\mathbb{S}^m$.

Now $\frac{\partial}{\partial t}\Big|_{t=0} \mathcal{F}^r(\Gamma(s,t)) = 0$ allows us to get the following Euler equation, which characterizes the critical points of $\mathcal{F}^r$ on the quoted manifolds of curves:

$$\begin{aligned}
&(\kappa^2)^{(r-1)/2} \nabla_T^3 T \\
&+2\frac{\mathrm{d}}{\mathrm{ds}}((\kappa^2)^{(r-1)/2}) \nabla_T^2 T
\end{aligned}$$

$$+\{(\kappa^2)^{(r-1)/2}+\frac{\mathrm{d}^2}{\mathrm{ds}^2}((\kappa^2)^{(r-1)/2})+\frac{2r+1}{r+1}(\kappa^2)^{(r+1)/2}\}\nabla_T T$$
$$+\frac{2r+1}{r+1}\frac{\mathrm{d}}{\mathrm{ds}}((\kappa^2)^{(r+1)/2})T=0.$$

From here and the Frenet equations for γ, we find the following characterization of the critical points of $\mathcal{F}^r$.

Proposition

Let γ be a regular curve in $\mathbb{S}^m$ with curvatures $\{\kappa,\tau,\delta,\ldots\}$. Then γ is a critical point of

$$\mathcal{F}^r(\gamma)=\int_\gamma(\kappa^2)^{(r+1)/2}ds$$

if and only if the following equations hold:

$$\begin{aligned} r\kappa''+\frac{r}{r+1}\kappa^3-\kappa\tau^2+\kappa+\frac{r(r-1)}{\kappa}(\kappa')^2 &= 0,\\ (\kappa^2)^r\tau &= 0,\\ \delta &= 0. \end{aligned}$$

In particular, γ lies in some $\mathbb{S}^2$ or $\mathbb{S}^3$ totally geodesic in $\mathbb{S}^m$.

From now on we will call r-**generalized elasticae** to the critical points of $\mathcal{F}^r$. In particular, free elasticae are nothing but 1-generalized elasticae.

A key result

Characterize the cylinders in $(\mathbb{H}^n_r,h_0)$ which are Willmore-Chen submanifolds.

Theorem

Let γ be a fully immersed closed curve in the hyperbolic space $\mathbb{H}^{n-r}$. The cylinder $\Upsilon_\gamma=\Phi(\gamma\times\mathbb{S}^r)$ in $(\mathbb{H}^n_r,h_0)$ is a Willmore-Chen submanifold if and only if γ is a generalized free elastica in the once punctured unit sphere $(\Sigma^{n-r},d\sigma^2)$. In particular, $n-r\leq 3$.

The proof is mainly based on the symmetric criticality principle of Palais.

Some examples

To find examples of non trivial Willmore-Chen submanifolds in the pseudo-hyperbolic space $(\mathbb{H}^n_r,h_0)$ we apply the latter Theorem.

Example 1.1

Let γ be an immersed closed curve in the hyperbolic 2-plane. The Lorentzian cylinder $\Upsilon_\gamma=\Phi(\gamma\times\mathbb{S}^1)$ is a Willmore torus in the 3-dimensional anti De Sitter space $(\mathbb{H}^3_1,h_0)$ if and only if γ is a free elastica in the once punctured unit 2-sphere $(\Sigma^2,d\sigma^2)$.

The complete classification of closed free elasticae in the standard 2-sphere was achieved by J.L. Langer and D.A. Singer, which can be briefly and geometrically described as follows ([21]): Up to rigid motions in the unit 2-sphere, the family of closed free elasticae consists of a geodesic γ_0, say the equator, and an integer two parameter family $\{\gamma_{m,n} : 0 < m < n,\ m,n \in \mathbb{Z}\}$, where $\gamma_{m,n}$ means that it closes up after n periods and m trips around the equator γ_0.

As a consequence we have

Example 1.2

There exist infinitely many Lorentzian Willmore tori in the 3*-dimensional anti De Sitter space. This family includes* $\{\Upsilon_{\gamma_{m,n}} : 0 < m < n,\ m,n \in \mathbb{Z}\}$ *and* Υ_{γ_0}.

A second case we will consider is $n - r = 3$. Then we look for critical points of $\mathcal{F}^r(\gamma)$ inside the family of helices in the standard once punctured 3-sphere $(\Sigma^3, d\sigma^2)$.

Let γ be a helix in $(\Sigma^3, d\sigma^2)$ with curvature κ and torsion τ. Assume that γ is a not a geodesic; otherwise, it is a trivial solution. Then γ is an r-generalized free elastica if and only if

$$\frac{r}{r+1}\kappa^2 - \tau^2 + 1 = 0.$$

A long and messy computation leads to

Theorem

For any natural number r*, there exists a one parameter family* $\{\gamma_q\}_{q\in\mathbb{Q}\setminus\{0\}}$ *of closed helices in* $(\Sigma^3, d\sigma^2)$ *which are* r*-generalized free elastica.*

As a consequence we obtain

Example 2

Let r *be any natural number. For any non zero rational number* q*, there exists an* $(r+1)$*-dimensional Willmore-Chen submanifold* $\Upsilon_\gamma = \Phi(\gamma\times\mathbb{S}^r)$ *in the pseudo-hyperbolic space* $(\mathbb{H}_r^{r+3}, h_0)$*,* γ *being an* r*-generalized free elastic closed helix in the once punctured unit* 3*-sphere* $(\Sigma^3, d\sigma^2)$ *whose slope* ℓ *is computed as above.*

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Depto. de Matemáticas, Universidad de Murcia
Campus de Espinardo, 30100 Murcia, Spain

E-mail address:
`aferr@fcu.um.es`