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Decay of magnetic fields in de Sitter and FRW universes

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Abstract. Magnetic curvature effects, investigated by Barrow and Tsagas (Phys. Rev. D **77**, 107302 (2008)), as a mechanism for magnetic field decay in open Friedmann universes ($\Lambda < 0$), are applied to dynamo geometric Ricci flows in 3D curved substrate in laboratory. By simple derivation, a covariant three-dimensional magnetic self-induction equation is obtained. The presence of these curvature effects indicates that de Sitter cosmological constant ($\Lambda \geq 0$) leads to enhancement in the fast kinematic dynamo action which induces a stretching in plasma flows. From the magnetic growth rate, the strong shear case implies an anti-de Sitter case ($\Lambda < 0$) where BT magnetic decaying fields are possible. For weak shear, fast dynamos are possible. The self-induced equation in Ricci flows is similar to the equation derived by BT in the $(3+1)$-spacetime continuum. Lyapunov-de Sitter metric is obtained from Ricci flow eigenvalue problem. In the de Sitter analogue there is a decay rate of $\gamma \approx -\Lambda \approx -10^{-35}\,\mathrm{s}^{-2}$ from the corresponding cosmological constant Λ. This shows that, even in the dynamo case, the magnetic field growth is slower than de Sitter inflation, which renders strongly support to BT result.

1 Introduction

Recently Fields medalist Grisha Perelman [1] has used the concept of Ricci flows, proposed by Hamilton in 1982 [2], to prove long-standing unproved, Poincaré conjecture on two- and three-dimensional settings. Actually Perelman, argued that the Ricci flow could be immersed in a large spacetime structure, not necessarily relativistic. In these approaches Einstein and Jordan Brans-Dicke equations have been solved in this so-called, Ricci flow gravity. More recently, several attempts to generalize Ricci flows to gravity have been done mainly by Graf [3] and Letelier [4]. On the other hand, cosmology in the Laboratory (COSLAB) program developed mainly by Unruh, Visser, and Volovik [5–7] has produced several papers on the analog models in general relativity (GR) and cosmology, which mimic in superfluid and other optical and hydrodynamics labs, the classical and quantum conditions in the universe. Yet in another front, the fast dynamos operating in solar physics and other astrophysical and cosmic settings, have shown the importance of dynamo theory in explaining, the magnetic field grow locally in the universe such as in the inner cores of planets and stars. In this paper, the cosmological analog of laboratory dynamos has been proposed and derived from the Lyapunov metric exponents and Ricci flows equation in 3D. Dynamo stretching by Riemannian plasma curved substrates [8,9] have led us to this reasoning. The computation of the magnetic growth rate and covariant three-dimensional magnetic self-induced equation, shows that the presence of these curvature effects, indicates that de Sitter cosmological constant ($\Lambda \geq 0$), leads to enhancement of the fast dynamo action which amounts in the stretching of plasma flows. This result was proposed earlier by Barrow and Tsagas [10] in the form of a slow, decaying magnetic field. The magnetic field growth rate is computed in terms of eigenvalues of the Ricci tensor in Einstein spaces [11]. Besides reproducing the decaying magnetic result of BT in Ricci flows, fast dynamo action is also obtained when the real part of the magnetic growth rate is positive. Note that stretching dynamic by plasma flows have also been obtained by M. Nuñez [8]. In this paper techniques of Einstein gravity, called Ricci rotation coefficients, are used to obtain the a Ricci flows dynamos. Note that a most important framework in this derivation is the proof that the Lyapunov metric exponents obtained by the eigenvalue problem of the Ricci flows leads naturally to a 3D cross-section of the de Sitter spacetime, which support our conclusions. A detailed account of GR cosmological dynamos is contained in Widrow [12]. The importance of laboratory analogues stems from the fact that there is no apparently stringy dynamo [13], and since even if there

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were, high-energy physics at this level of energy is only in its infancy in CERN experiments. The resulting metric is called de Sitter-Lyapunov metric. Also recently Fedichev and Fischer [14] investigated a quasi-particle cosmic analogue, using also de Sitter equivalents to trapp particles. Actually BT attempt is not the first one to relate the magnetisation coupling to curvature invariants. A similar attempt has been done by Bassett *et al.* [15], which has related preheating phases of the universe to magnetic dynamo. In the de Sitter case a simple computation shows that, the growth rate of the magnetic field is slower than the expansion of the model, and shear is exactly the agent that slows down the dynamo growth. Thus one can say that cosmic fast dynamo is slower down by the shear of the cosmic Ricci flow. The paper is organized as follows: sect. 2 presents the Ricci flow as a de Sitter-Lyapunov metric and sect. 3, derivation of the self-induction equation in Ricci flows is given and the cosmic dynamo analogue is presented. Section 4 presents future prospects and conclusions.

2 de Sitter-Lyapunov analogue metric in Ricci flows

In this section, though it is more mathematical than the rest of the paper, it is fundamental to understand why the relation between the de Sitter space 3D sections appears so naturally in the context of the Lyapunov metric exponents which leads to chaotic dynamos, which are so interesting from the cosmological point of view. The Riemann metric given by the Ricci flow [1,2], is given in mathematical terms by

Definition 1.

$$\frac{\partial \mathbf{g}}{\partial t} = -2\mathbf{Ric}, \tag{2.1}$$

where here, $\mathbf{g}$ is the Riemann metric, over manifold $\mathcal{M}$, and the parameter t in the Riemann metric $\mathbf{g}(t)$, is given in the interval $t \in [a,b]$ in the field of real numbers $\mathbf{R}$. On a local chart $\mathcal{U}$ in $\mathcal{M}$, the expression (2.1) can be expressed as [16]

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} \tag{2.2}$$

where $\mathbf{Ric}$, is the Ricci tensor, whose components are R_{ij}. From this expression, one defines the eigenvalue problem as

$$R_{ij}\chi^j = \lambda\chi_i, \tag{2.3}$$

where $(i,j = 1,2,3)$. Substitution of the Ricci flow equation (2.2) into this eigenvalue expression and cancelling the eigendirection χ^i on both sides of the equation yields

$$\frac{\partial g_{ij}}{\partial t} = -2\lambda g_{ij}. \tag{2.4}$$

Solution of this equation yields the Lyapunov expression for the metric

$$g_{ij} = \exp[-2\lambda t]\delta_{ij}, \tag{2.5}$$

where δ_{ij} is the Kroenecker delta. Note that in principle if $\lambda \leq 0$ the metric grows without bounds, and in case it is negative it is bounded as $t \to \infty$. Recently Thiffeault has used a similar Lyapunov exponents expression in Riemannian manifolds to investigate chaotic flows, without attention to dynamos or Ricci flow. Thus one has proven the following lemma.

Lemma 1. *If* λ_i *is an eigenvalue spectra of the* $\mathbf{Ric}$ *tensor, the finite-time Lyapunov exponents spectra is given by*

$$\lambda_i = -\gamma_i \leq 0. \tag{2.6}$$

In the next section we shall use this argument to work with the de Sitter metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + e^{\Lambda t}(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \tag{2.7}$$

in the de Sitter-Lyapunov analogue 3D spacetime

$$\mathrm{d}s^2 = e^{\Lambda t}(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2), \tag{2.8}$$

which shows that the de Sitter-Lyapunov metric can be obtained from de Sitter metric by simply considering a constant slice $t = \text{const}$ of de Sitter spacetime. Nevertheless, it is important to recall that for practical uses small variations of the parameter t would be allowed, otherwise the metric should be Ricci flat since de Sitter-Lyapunov metric should be flat, if $t = \text{const}$.

3 Ricci dynamo flows as a 3D cosmic analogue

Now let us consider Lyapunov eigenvalues, which shall play an important role in the determination of the bounds of magnetic energy as a global dynamo action bound. Let us now consider the magnetic kinematic dynamo, most commonly known as an equation, with non-zero plasma resistivity η

$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} = \mathbf{B} \cdot \nabla \mathbf{v} + \eta \Delta \mathbf{B}, \tag{3.9}$$

where $\mathbf{B}$ is the magnetic field vector, and $\mathbf{v}$ is the flow velocity, where $\Delta := \nabla^2$ is the Laplacian operator. Here we also assume that the covariant flow derivative is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} = \mathbf{v} \cdot \nabla + \frac{\partial}{\partial t}\,. \tag{3.10}$$

Here we assume that the magnetic self-induction equation (3.9) is non-relativistic since we are not in the true GR but in laboratory analogue cosmology. A long but straightforward computation, yields the diffusion term as

$$\Delta \mathbf{B} = \frac{1}{\sqrt{g}} \partial_i [\sqrt{g} g^{ij} \partial_j \mathbf{B}], \tag{3.11}$$

which expanded using the frame $\mathbf{e}_i$ where $(i, j = 1, 2, 3)$ yields

$$\mathbf{B} = B^i \mathbf{e}_i, \tag{3.12}$$

which in turn yields

$$\Delta \mathbf{B} = [g^{ij} \partial_i \partial_j B^p + B^k [\partial_i \gamma^p{}_{jk} g^{ij} + \gamma^l{}_{jk} \gamma^p{}_{il} g^{ij}] + [\gamma^p{}_{jk} g^{ij} \partial_i B^k]] \mathbf{e}_p. \tag{3.13}$$

Here $\gamma^l{}_{jk}$ is the Ricci rotation coefficients (RRCs) analogous to the Riemann-Christoffel symbols. The RRCs is defined by

$$\partial_k \mathbf{e}_i = \gamma_{ki}{}^j \mathbf{e}_j. \tag{3.14}$$

The Christoffel symbols

$$\Gamma^i{}_{jk} = g^{il} [\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}] \tag{3.15}$$

do not appear in the computations, since we have assumed, that the trace of the Christoffel symbols vanish. To complete the derivation of the self-induction equation one needs to obtain the diffusion free part of the self-induced equation above, which in general curvilinear coordinates $x^i \in \mathcal{U}_i$, of the sub-chart U_i of the manifold, and in the rotating frame reference of the flow $\mathbf{e}_i$ reads

$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} = (\mathbf{B} \cdot \nabla) \mathbf{v}. \tag{3.16}$$

Before this derivation, let us now introduce the Ricci tensor into play, by considering the following trick:

$$\frac{\mathrm{d}}{\mathrm{d}t} [g^{il} g_{lk}] = \frac{\mathrm{d}}{\mathrm{d}t} [\delta^i{}_k] = 0, \tag{3.17}$$

which can be applied to the expression

$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} (g^{ik} B_k \mathbf{e}_i) \tag{3.18}$$

to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} (g^{ik} B_k) = \frac{\mathrm{d}}{\mathrm{d}t} (g^{ik}) B_k + g^{ik} \frac{\mathrm{d}}{\mathrm{d}t} B_k. \tag{3.19}$$

Now by making use of the Ricci flow equation above into this last expression, yields

$$\frac{\mathrm{d}}{\mathrm{d}t} g^{ik} B_i = -2R^{ik} B_k + g^{ik} \frac{\mathrm{d}}{\mathrm{d}t} B_k. \tag{3.20}$$

Note that for de Sitter spacetime the solenoidal magnetic field is also satisfied and no magnetic monopole is assumed in this phase. Thus

$$\nabla \cdot \mathbf{B} = 0. \tag{3.21}$$

From the evolution of the reference frame

$$\frac{\mathrm{d}\mathbf{e}_i}{\mathrm{d}t} = \omega_i{}^j \mathbf{e}_j \tag{3.22}$$

and the Ricci rotation coefficient, one obtains the magnetic curvature effect in dynamo theory, through the self-induction equation in Ricci flow as

$$[\gamma + 2\Lambda + \omega]B_i = B^p \left[v^l \gamma^i{}_{pl} + v_j \partial_p g^{ij} + g^{ij}\left[\sigma_{pj} + \Omega_{pj} - \frac{1}{3}\theta g_{pj}\right]\right], \tag{3.23}$$

where we have used the following expressions:

$$\partial_k \mathbf{e}_i = \Gamma_{ki}{}^j \mathbf{e}_j, \tag{3.24}$$

where ω_{ij} and $\Gamma_{ki}{}^j$ are, respectively, the vorticity, and the gradient of the Ricci flow, v^l is decomposed into its invariant format of vorticity Ω_{lp}, shear σ_{kl} tensors and expansion θ as

$$\partial_p v_l = \Omega_{pl} + \sigma_{pl} - \frac{1}{3}\theta g_{lk}. \tag{3.25}$$

These equations were also simplified by the assumption that the flow has a rigid rotation, or that the vorticity of the flow coincides with the vorticity of the frame or

$$\Omega_{pl} = \omega_{pl}. \tag{3.26}$$

This assumption is cosmologically reasonable, since the in inflationary cosmological models the vorticity and shear are smaller than the expansion, which represents the stretching in the language of dynamo theory. The fact that the expansion is the trace of the gradient strain $\theta = \mathrm{Tr}[\nabla \mathbf{v}]$, shows that the Ricci dynamo flows are compressible, which are more pathological dynamo flows than the ones that are compressible, or solenoidal

$$\nabla \cdot \mathbf{v} = 0. \tag{3.27}$$

To deduce expression (3.23) we still have used the eigenvalue expressions

$$\sigma_{ki} B^i = \sigma B_k, \tag{3.28}$$

$$\Omega_{ki} B^i = \Omega B_k, \tag{3.29}$$

$$\Theta_{ki} B^i = -\frac{1}{3}\theta B_k, \tag{3.30}$$

for the kinematical cosmological Ehlers-Sachs quantities. Besides one also uses the fact that the Ricci flows obey the Einstein manifold 3D condition

$$R_{lp} = \Lambda g_{lp}. \tag{3.31}$$

With all those simplifications the dynamo equation (3.23) allows us to compute the dynamo growth rate as

$$\gamma = \left[2\Lambda - \sigma + \frac{1}{3}\theta\right] + \frac{B^p B_i v_l}{B^2}[\gamma^i{}_p{}^l + \partial_p g^{il}]. \tag{3.32}$$

From this expression, one immediately notices that the stretching term contributes to enhance the dynamo action, while the positive γ de Sitter cosmological constant, also enhances the fast dynamo action. The anti-de Sitter or open Friedmann universe induces the decay of the magnetic field or the BT result [10]. The last expression comes from an expression with the Ricci tensor similar to the BT wave equation, which we repeat here for the readers convenience

$$\frac{\mathrm{d}^2 B_i}{\mathrm{d}\tau^2} - D^2 B_i = -5\Lambda \frac{\mathrm{d}}{\mathrm{d}\tau} B_i - 4\Lambda^2 B_i + \frac{1}{3}(\rho + 3p)B_i - R_{ij}B^j. \tag{3.33}$$

In their notation B_i is the magnetic vector field in the comoving frame, and $D^2 = h_{ij}\nabla^i\nabla^j$ is the 3D Laplacian, where $h_{ij} = g_{ij} + v_i v_j$ is the projection metric orthogonal to v^l. Note that in de Sitter spacetime, the growth rate of the dynamo action yields

$$\gamma = \left[2\Lambda - \sigma + \frac{1}{3}\theta\right] \tag{3.34}$$

since the terms $\gamma^i{}_p{}^l$ and $\partial_p g^{ij}$ vanish for the de Sitter metric components $g^{ij} = e^{-\Lambda t}\delta^{ij}$. Let us now compute the growth rate of the cosmic Ricci dynamo flow in the case of the 3D section of the Friedmann-Robertson-Walker (FRW) universe

$$\mathrm{d}l^2 = \frac{\mathrm{d}r^2}{(1 - \frac{\Lambda r^2}{2})} + r^2 \mathrm{d}\Omega^2, \tag{3.35}$$

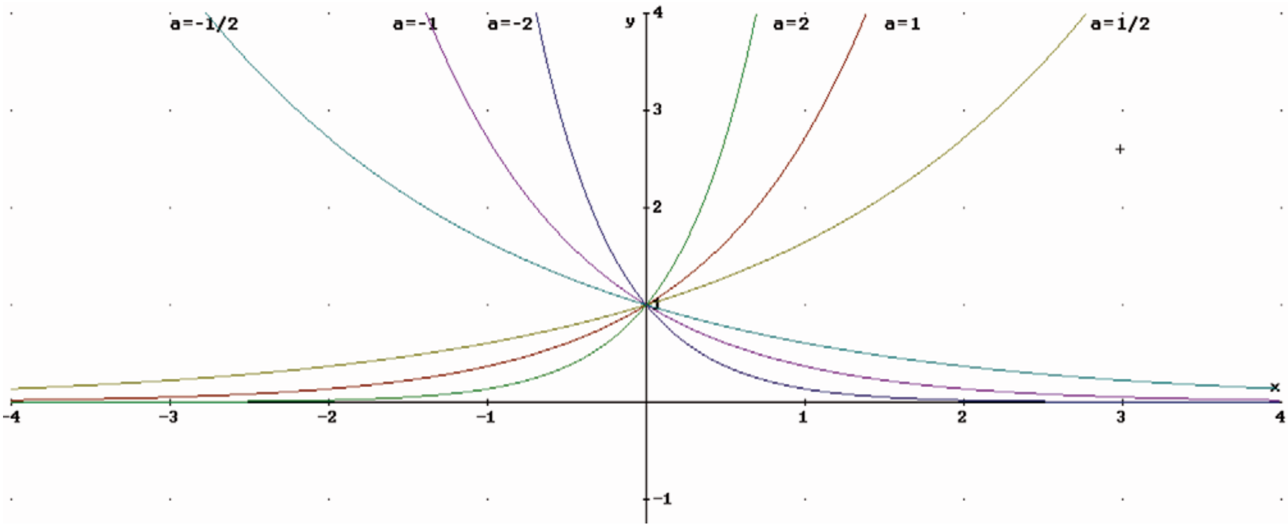

Fig. 1. Evolution of the graph of e^{at} for different values of a.

where $\mathrm{d}\Omega^2 = (\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$ is the solid angle. Computation of the Ricci rotation coefficients in the limit of $r \to 0$, yields

$$\gamma = \left[2\Lambda - \sigma + \frac{1}{3}\theta\right] - 2\left(1 + \frac{\Lambda r^2}{2}\right) v^r. \tag{3.36}$$

Note that not only shear slows down the dynamo action, but in FRW universe, since the cosmological constant is very small, the the last term does not enhance dynamo action and magnetic fields decay in this universe model. Taking into account the magnetic energy ϵ as

$$\epsilon = \int B^2 \mathrm{d}V, \tag{3.37}$$

which expressed in terms of the 3D Riemann metric components reads

$$\epsilon = \int B^i g_{ij} B^j \mathrm{d}V. \tag{3.38}$$

Since, by definition the fast dynamo action corresponds to the growth of magnetic energy in time as $\frac{\partial \epsilon}{\partial t} \geq 0$, this amount has to be computed by performing the partial time derivative of the expression (3.25). Actually the equal sign in the last condition represents the lower limit of marginal dynamos, where the magnetic energy integral remains constant. This computation yields

$$\frac{\partial \epsilon}{\partial t} = \frac{\partial[\int B^i g_{ij} B^j \mathrm{d}V]}{\partial t}. \tag{3.39}$$

Expansion of the RHS of this expression shows clearly where the Ricci flow eigenvalue effect is going to appear. A simple computation, shows that the energy integral confirms the dynamo action. Throughout the paper the diffusion term was not explicitly computed since because we use the limit of diffusion free to check for the presence of slow dynamos, which seems not to exist globally in the universe. Note that the in the de Sitter case the magnetic field can be written as

$$B^i = B^0 e^{[2\Lambda - \sigma + \frac{1}{3}\theta]t}, \tag{3.40}$$

which shows that the shear eigenvalue σ, slows down the dynamo action, while the cosmological constant in de Sitter space enhances it. The anti-de Sitter effective [3] spacetime of course contributes to slow down the magnetic field as the negative exponents contribute to the decay of the magnetic field in the effective universe (see fig. 1).

4 Conclusions

By making use of mathematical tools from Riemannian geometry, so popularised in Einstein general relativity, called Ricci Rotation Coefficients, one obtains a fast dynamo action in stretching magnetic field lines endowed with shear in de Sitter-Lyapunov analogue spacetime metric. Besides the fast dynamo action for de Sitter or closed (3+1)-spacetime Ricci flows, where the cosmological constant $\Lambda > 0$, which is a new result, one is able to reproduce the BT magnetic field decay in the $(3+1)$ real spacetime of GR cosmology. This seems to shed some light on the implications of Ricci flow in more generalised settings and take it out from the pure mathematical applications. Note that several applications of Ricci flows to physics have been considered so far, but this is the first time, to our knowledge, that it is applied to cosmological analogues. An interesting panorama of the applications of Ricci flows manifolds in Physics maybe found in the paper by Woolgar [18]. Ricci flows are mainly applied in solitons and since this is an important subject to cosmology [18] we may address the relation between our cosmic analogues Ricci flows to solitons. This may appear elsewhere.

After we finished this paper, it came to our knowledge that Marklund and Clarkson [16] have presented a general GR covariant formalism for the dynamo magnetohydrodynamics equation, where however, no Ricci flows are present and only gravitational waves applications in diffusive plasma are given. Vortex dynamos in analogue models can be also treated elsewhere, based on a non-Riemannian vortex acoustics model presented earlier by the author [19]. Recent magnetic flux tubes in Riemannian manifolds [20] may also be addressed in the Ricci flow dynamo context.

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References

1. G. Perelmann, The entropy formula for the Ricci flow and its geometric applications, arXiv:math/0211159.
2. R.S. Hamilton, J. Differen. Geom. **17**, 255 (1982) B. Chow, D. Knopf, *Ricci Flows: An introduction* (AMS, New York, 2004).
3. W. Graf, *Ricci Flow Gravity*, arXiv:gr-qc/0602054.
4. P.S. Letelier, Int. J. Theor. Phys. A **43**, 4281 (1991).
5. G. Volovik, *Universe in a Helium Droplet* (Oxford University Press, 2003).
6. W. Unruh, Phys. Rev. Lett. **46**, 1351 (1981).
7. U. Fischer, M. Visser, Phys. Rev. Lett. **88**, 110201 (2002).
8. J.L. Thiffeault, Differential constraints in Chaotic Flows on Curved Manifolds (2002) arXiv:nlin/0105101.
9. M. Nuñez, J. Phys. A, 8903 (2003).
10. J.D. Barrow, C. Tsagas, Phys. Rev. D **77**, 107302 (2008).
11. A. Besse, *Einstein Manifolds* (Springer-Verlag, 2007).
12. L. Widrow, Rev. Mod. Phys. **74**, 775 (2002).
13. A. Vilenkin, A. Shellard, *Topological Defects in Spacetime* (Cambridge University Press, 2000).
14. Yu. Fedichev, U. Fischer, Phys. Rev. A **69**, 0303063 (2004).
15. B. Bassett, G. Pollifrone, S. Tsujikawa, F. Viniegra, Preheating-cosmic magnetic dynamo? arXiv:astro-ph/0010628.
16. M. Marklund, C.A. Clarkson, Mon. Not. R. Astron. Soc. **358**, 892 (2005).
17. Yu. Fedichev, U. Fischer, Phys. Rev. A **69**, 0303063 (2004).
18. E. Woolgar, Can. J. Phys. **86**, 645 (2008) arXiv:0708.2144.
19. L.C. Garcia de Andrade, Phys. Rev. D **70**, 064004 (2004).
20. L.C. Garcia de Andrade, Phys. Plasmas **15**, 122106 (2008).