Equivariant Willmore Surfaces in Conformal Homogeneous three Spaces

Manuel Barros[1], Angel Ferrández[2*] and Óscar J. Garay[3]

Abstract

The complete classification of homogeneous three spaces is well known for some time. Of special interest are those with rigidity four which appear as Riemannian submersions with geodesic fibers over surfaces with constant curvature. Consequently their geometries are completely encoded in two values, the constant curvature, c, of the base space and the so called bundle curvature, r. In this paper, we obtain the complete classification of equivariant Willmore surfaces in homogeneous three spaces with rigidity four. All these surfaces appear by lifting elastic curves of the base space. Once more, the qualitative behavior of these surfaces is encoded in the above mentioned parameters (c, r). The case where the fibres are compact is obtained as a special case of a more general result that works, via the principle of symmetric criticality, for bundle-like conformal structures in circle bundles. However, if the fibres are not compact, a different approach is necessary. We compute the differential equation satisfied by the equivariant Willmore surfaces in conformal homogeneous spaces with rigidity of order four and then we reduce directly the symmetry to obtain the Euler Lagrange equation of $4r^2$-elasticae in surfaces with constant curvature, c. We also work out the *solving natural equations* and the *closed curve problem* for elasticae in surfaces with constant curvature. It allows us to give explicit parametrizations of Willmore surfaces and Willmore tori in those conformal homogeneous 3-spaces.

Keywords: Equivariant Willmore surface, homogeneous 3-space, Riemannian submersion, bundle-like conformal metric, elastic curve, Berger sphere, Heisenberg manifold.

1 Introduction

Problems related with the Willmore energy are classical in the mathematical literature. In dimension two, they are concerned with the analysis of surfaces whose behavior is governed

[1]Departamento de Geometría y Topología, Facultad de Ciencias, Universidad de Granada, 1807 Granada, Spain. E-mail address: `mbarros@ugr.es`.
[2]Departamento de Matemáticas, Universidad de Murcia, Campus de Espinardo, 30100 Murcia, Spain. E-mail address: `aferr@um.es`.
[3]Departamento de Matemáticas, Facultad de Ciencia y Tecnología, Universidad del País Vasco, Aptdo 644, 48014, Bilbao, Spain. E-mail address: `oscarj.garay@ehu.es`
(*) Corresponding author A. Ferrández.
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by the Willmore energy, which in physical terms, measures the total tension that surfaces receive from a conformal structure in the background where they lie. Critical points of this action are known as Willmore surfaces and they are of special interest in three dimensional ambient spaces not only in their own right, but also for their many interesting applications (see [5] and references therein). Important families of Willmore surfaces are known when the ambient space is conformal to a homogeneous structure with the maximum order of rigidity, that is the case when the background space is a three dimensional real space form (see, for example, [20, 24] for boundary free compact surfaces and [3, 4] for surfaces with boundary).

On the other hand, the complete classification of homogeneous 3-structures is well known for sometime. There are three possibilities for the degree of rigidity of these spaces since they may have an isometry group of dimension 6, 4 or 3. The maximum rigidity, 6, corresponds to the spaces with constant curvature. However, of special interest are the homogeneous 3-dimensional spaces with isometry group of dimension four. This family includes, besides the Berger spheres, important spaces of the canonical Thurston geometries. During the last years surfaces in these backgrounds are being deeply studied, specially those with constant mean curvature (see [1, 12, 13] as important references in this sense).

In this paper, we study Willmore surfaces in homogeneous 3-spaces with isometry group of dimension four, though our method can be also applied equally well to the case of maximum degree of rigidity. Therefore, in our study the families of Willmore surfaces obtained in [3, 4, 20, 24] will appear as very special cases.

The main point in our study of homogeneous 3-spaces with rigidity of order four is that they appear as Riemannian submersions with geodesic fibers over surfaces with constant curvature. In particular, they posses bundle-like metrics and their classes of congruence are completely determined, up to topology, for a pair of constants: the curvature of the base, c; and the bundle curvature (the mixed curvature of the bundle-like metric), r. Using the notation of [12] a 3-dimensional homogeneous space with rigidity of order four will be denoted by $\mathbf{E}(c, r)$ and, as we have said before, its metric provides a Riemannian submersion, $\mathfrak{p} : \mathbf{E}(c, r) \to B(c)$, with geodesic fibers and bundle curvature r over a surface with constant curvature c.

In the general context provided by the Riemannian submersion, $\mathfrak{p} : M \to B$, associated to a three dimensional fibre bundle M on a surface B with structure group G, we consider the following:

Problem. *How should we choose a curve, γ, in the base space B so that its complete lifting $\mathfrak{p}^{-1}(\gamma)$ be a Willmore surface in M?*

In connection with this problem, we have the following:

Conjecture. *The equivariant surface $\mathfrak{p}^{-1}(\gamma)$ is Willmore in M if and only if the curve γ is an elastica in the base space B relative to a suitable metric which is conformal to the original inner product it was carrying. Said otherwise, it is a critical point, in* *that conformal metric, of the total squared curvature energy constrained by a potential. Furthermore, that potential measures the obstruction to the integrability of the horizontal distribution.*

We will give an affirmative answer to this conjecture when the fibres of the Riemannian submersion are compact, that is, if the structure group G is compact (a circle). This fact allows us to ensure the existence of Willmore surfaces in the conformal class of any three dimensional bundle-like metric. Even more, we will show that *there exist Willmore tori in any three dimensional bundle-like conformal class.* The main ingredient that we use to obtain the answer is the Palais principle of symmetric criticality, [23], which enable us to reduce symmetry under the compactness of G.

If G is not compact, we cannot solve the problem in all its generality. However, we use a direct variational approach involving the computation of the field equation for Willmore surfaces in a general setting, which allows us to get a positive answer in some special frameworks, including those associated with a homogeneous space with four dimensional isometry group.

As a consequence, we give the complete classification of equivariant Willmore surfaces in three dimensional conformal homogeneous spaces having 4-dimensional isometry group, no matter if the fibres are compact or not. In both cases, the original problem becomes one about elastic curves in $B(c)$, for which we use the machinery developed in [19, 25]. The field equation for these curves, and so their qualitative behavior, is completely encoded in the parameters (c, r) that determine the homogeneous structure as we have described in the section 6. Our main results can be summarized as follows:

(1) The family of equivariant Willmore surfaces in the conformal $\mathbf{E}(c, r)$ *with* $c \geq 2r^2$ *is made up of the following surfaces:*

- *(1.1) Minimal surfaces obtained by lifting geodesics.*
- *(1.2) A one-parameter class of surfaces obtained by lifting wavelike elastic curves.*

(2) The family of equivariant Willmore surfaces in the conformal $\mathbf{E}(c, r)$ *with* $c < 2r^2$ *is made up of the following surfaces:*

- *(2.1) Minimal surfaces obtained by lifting geodesics.*
- *(2.2) Surfaces with constant mean curvature* $\sqrt{2(2r^2 - c)}/2$ *shaped on circles with curvature* $\sqrt{2(2r^2 - c)}$.
- *(2.3) A one-parameter class of surfaces built on orbitlike elastic curves.*
- *(2.4) A one-parameter class of surfaces built on wavelike elastic curves.*
- *(2.5) A surface shaped on a borderlike elastic curve.*

In the first case, $c \geq 2r^2$, besides the Riemannian product $\mathbb{R}^2 \times \mathbb{S}^1$, we find some Berger spheres including the round one ($c = 4r^2$). However, in the second case, $c < 2r^2$, in addition to the remaining Berger spheres, we obtain the Heisenberg geometries associated with the group Nil_3 and the geometries associated with the group $\mathrm{SL}(2,\mathbb{R})$.

In the last section, we include a detailed study of elastic curves in a surface with constant curvature. It is done from the point of view of two classical problems in the theory of curves. On the one hand, we consider the *solving natural equation* problem. In general, it can be theoretically solved using quadratures whenever one knows the curvature function. These integrations can not be done explicitly for a general curvature function, but may for elastic curves. It allows us to exhibit explicit parametrizations of elastic curves and then use the previous results to obtain explicit parametrizations of Willmore surfaces in homogeneous three spaces with order four rigidity. However, our study goes further. Indeed, we also consider the so called *closed curve problem.* Therefore, for elastic curves with periodic curvature function, we find necessary and sufficient conditions that determine when elasticae are closed. It allows us to exhibit explicit examples of Willmore tori in homogeneous three spaces, with rigidity of order four, where previously we make a suitable quotient in order to ensure the compactness of the fibres.

2 The Willmore problem on bundle-like conformal classes

Let M be a three dimensional principal fibre bundle on a surface B. We have then a structural one dimensional group, $G=\{\varphi_t : t\in\mathbb{R}\}$, and a natural projection $\mathfrak{p}: M\to B$. Let dt^2 be an invariant metric on G and f a positive smooth function on the basis B. For any Riemannian metric, g, on B and any principal connection, ω, we can define a *generalized Kaluza-Klein metric* on M, say $\bar{g}=\mathfrak{p}^*(g)+(f\circ\mathfrak{p})^2\,\omega^*(dt^2)$. In particular, when f is chosen to be constant, then $\bar{g}$ is called a *Kaluza-Klein* or *bundle-like metric.* Let us recall a few important properties of this class of metrics:

(1) The action of G on M is carried out by isometries of $(M,\bar{g})$.

(2) The projection $\mathfrak{p}$ is a Riemannian submersion whose fibres are geodesics in $(M,\bar{g})$ if and only if $\bar{g}$ is a Kaluza-Klein metric.

(3) Given a generalized Kaluza-Klein metric, we can find another Kaluza-Klein metric which is conformal to the original one. In fact, one just needs to take $\tilde{g}=\frac{1}{(f\circ\mathfrak{p})^2}\,\bar{g}=\mathfrak{p}^*\left(\frac{1}{f^2}\,g\right)+\omega^*(dt^2)$.

Let γ be an immersed curve in B, then $S_\gamma=\mathfrak{p}^{-1}(\gamma)$ is a surface immersed in M which is invariant under the G-action. Certainly, S_γ is embedded if γ is simple and it is compact when the curve is closed. Conversely, all of G-invariant surfaces in M are obtained in this way: they are complete liftings of curves in B. Topologically the surface S_γ is $\gamma\times G$. From now on, we will use the following terminology: S_γ will be called the *tube* shaped on the curve γ if G is compact, while it will be referred to as the *sheet* on γ if G is not compact. Moreover, we will use the terms *torus* (respectively, *cylinder*) for a tube (respectively, *sheet*) shaped on a closed curve. If γ is parametrized by its arc length in (B, g), then any horizontal lift, $\bar{\gamma}$, is also a unit speed curve in $(M, \bar{g})$. Then, the surface S_γ can be parametrized by taking as coordinate curves the horizontal lifs of γ and the fibres of the submersion $\phi(s,t) = \varphi_t(\bar{\gamma}(s))$. As a consequence, these surfaces are flat when f is constant, that is, when $\bar{g}$ is a bundle-like metric. From now on, these surfaces will be called *equivariant surfaces*.

Let Γ be a union of regular curves in $(M, \bar{g})$ and N_o a unit normal vector field along Γ which is orthogonal to Γ. For a surface, S, with boundary ∂S, let $\mathbf{I}_\Gamma(S, M)$ be the space of immersions, $\phi : S \to M$, that satisfy the following first order boundary conditions

$$\phi(\partial S) = \Gamma, \qquad N_\phi / \Gamma = N_o,$$

where N_ϕ denotes the Gauss map associated with the immersion ϕ. Roughly speaking, if we identify each immersion $\phi \in \mathbf{I}_\Gamma(S, M)$ with its graph, $\phi(S)$, viewed as a surface with boundary in M, then $\mathbf{I}_\Gamma(S, M)$ can be regarded as the space of immersed surfaces in M having the same boundary and being tangent along the common boundary.

The Willmore problem deals with the dynamics of the boundary value problem associated to the above boundary conditions and governed by the following Willmore energy

$$\mathcal{W} : \mathbf{I}_\Gamma(S, M) \to \mathbb{R}, \qquad \mathcal{W}(\phi) = \int_S \left(H_\phi^2 + R_\phi\right) dA_\phi + \int_{\partial S} \kappa_\phi \, ds.$$

where H_ϕ stands for the mean curvature of the immersion $\phi(S)$, R_ϕ is the sectional curvature of the target space restricted to the tangent bundle of $\phi(S)$ and κ_ϕ is the geodesic curvature of $\phi(\partial S)$ in $\phi(S)$. In some sense, we are measuring the total extrinsic curvature of the pair $(\phi(S), \phi(\partial S))$. Critical points of the above stated problem are called Willmore surfaces for the prescribed boundary conditions, and from now on they will be referred to as Willmore surfaces with boundary or, simply, as Willmore surfaces. Here by a critical surface we mean, as usual, that any reasonable compact piece or polygon of the surface is a critical point for the induced problem. More precisely, a connected, simply connected, compact domain with non-empty interior, $\Omega \subset S$, is said to be a polygon if it has a piecewise smooth boundary, $\partial\Omega$, which is made up of a finite number of regular curves. Now, we say that $\phi \in \mathbf{I}_\Gamma(S, M)$ is a critical point of $(\mathbf{I}_\Gamma(S, M), \mathcal{W})$, if for any polygon $\Omega \subseteq S$, the restriction $\phi|_\Omega$ is a critical point of the Willmore energy acting on the space $\mathbf{I}_{\phi(\partial\Omega)}(\Omega, M))$ of immersions $\psi : \Omega \to M$ that satisfy the induced boundary conditions $\psi(\partial\Omega) = \phi(\partial\Omega)$, $N_\psi|_{\partial\Omega} = N_\phi|_{\partial\Omega}$.

This problem is invariant under conformal changes in the background metric. Thus, it is actually a variational problem which is defined on the conformal class, $[\bar{g}]$. However, since we already know that there exists a bundle-like metric representative within any generalized Kaluza-Klein conformal class, we can restrict ourselves to the case of bundle-like metrics using suitable conformal changes (for more details on this variational problem see [4, 5, 11] and references therein).

In this paper, we deal with the Willmore problem for surfaces in $(M, \bar{g})$ that satisfy G-invariant, first order boundary conditions. Therefore, we may assume that Γ is formed by a pair of fibres of the submersion and that N_o is a G-invariant unit normal vector field along Γ, which is orthogonal to Γ. In this setting, the fundamental problem we are concerned with here can be stated as follows

How should we choose a curve γ in (B, g) so that S_γ be a Willmore surface in $(M, [\bar{g}])$?

In other words, we have to determine the Willmore surfaces which are invariant under the G-action on M, that we call *equivariant Willmore surfaces.*

3 Symmetry reduction under compactness of the fibres, Willmore tubes and early applications

In this section, we give an answer to the above problem when the structure group is compact, that is, $G = \mathbb{S}^1$. The main tool that we use is the principle of symmetric criticality (see [23]) which allows us to reduce the symmetry in the following sense: an equivariant tube, S_γ, is Willmore if and only if it is a critical point of the Willmore energy restricted to the space of equivariant tubes.

To compute the Willmore density on equivariant immersions, we will need some machinery from the theory of Riemannian submersions (see [10], [22] and references therein). The geometry of Riemannian submersions is mainly governed by two invariants which are known as the O'Neill invariants. The first, T, is defined in terms of the second fundamental form of the fibres and it vanishes when fibres are totally geodesic, what in our case means that the metric is bundle-like or Kaluza-Klein. The second invariant, A, measures the obstruction to integrability of the horizontal distribution and so it vanishes when the principal connection, ω, is flat. Making the natural conformal change in the metric, $\bar{g}$, we can assume that fibres are geodesic and so we have the following relationship between the mean curvature, H_γ, of S_γ, in $(M, \tilde{g})$, and the curvature function, κ_γ, of γ in $\left(B, \frac{1}{f^2}\, g\right)$ (see [3])

$$H_\gamma^2 = \frac{1}{4}\,\kappa_\gamma^2 \circ \mathfrak{p}. \tag{1}$$

The second term appearing in the two-dimensional Willmore energy is a sectional curvature of a mixed section, that is a plane spanned by a horizontal vector and a vertical vector. In general, under the assumption of geodesic fibres ($T = 0$), if $\bar{X}$ is the horizontal lift of a vector field X, V is vertical and both are of unit length, then the sectional curvature of the corresponding mixed section is given by

$$R(\bar{X}, V) = |A_{\bar{X}} V|^2 = \frac{1}{2}\left(r(X, X) \circ \mathfrak{p} - \tilde{r}(\bar{X}, \bar{X})\right), \tag{2}$$

where r and $\tilde{r}$ are the Ricci curvatures of the metrics $\frac{1}{f^2}\, g$ and $\tilde{g}$, respectively. Now, in the unit tangent bundle, $\mathbb{S}^1(B)$, of $\left(B, \frac{1}{f^2}\, g\right)$, we define the potential $\Psi : \mathbb{S}^1(B) \to \mathbb{R}$ by

$$\Psi(X) \circ \mathfrak{p} = 2\left(r(X,X) \circ \mathfrak{p} - \tilde{r}(\bar{X},\bar{X})\right).$$

Consequently, the Willmore energy, computed with the metric $\tilde{g}$, on a symmetric immersion is given by

$$\mathcal{W}(S_\alpha) = \frac{1}{4}\int_{\mathbf{S}_\alpha} \left((\kappa_\alpha^2 + \Psi(\alpha')) \circ \mathfrak{p}\right)\, dA_\alpha = \frac{\pi}{2}\int_\alpha (\kappa_\alpha^2 + \Psi(\alpha'))\, ds.$$

Hence, the searching for critical equivariant tubes is reduced to that of curves that are critical points of the following elastic energy functional

$$\mathcal{E}(\beta) = \int_\beta (\kappa_\beta^2 + \Psi(\beta'))\, ds,$$

acting on the space of curves which are clamped with respect to the projected boundary conditions, that is, curves connecting two fixed points and being tangent to two fixed unit vectors at that points. These curves are known as elasticae with potential Ψ (see [25] as a main reference for elastica with constant potential), so that we give a simple and interesting answer to the stated problem as follows:

Theorem 3.1 *S_α is a Willmore tube if and only if α is an elastic curve with potential in $\left(B, \frac{1}{f^2}\, g\right)$.*

Now, the existence of elastic curves with arbitrary potential, in any Riemannian manifold, is theoretically known. In particular, in compact spaces, the existence of closed elastic curves is well known for arbitrary potentials Ψ (see for example [18]). Therefore, as a consequence, we obtain the following

Corollary 3.2 *There exist equivariant Willmore tori in any generalized Kaluza-Klein conformal structure defined on any $\mathbb{S}^1$ principal fibre bundle with compact base.*

We point out that the Euler-Lagrange equation associated to an elastic energy action has been computed (see [19]) when the potential Ψ is a constant, say λ, which works as a Lagrange multiplier. In particular, in a surface with Gaussian curvature function K, this equation is given in terms of the curvature function of curves as follows

$$2\kappa'' + \kappa^3 + (2K - \lambda)\kappa = 0. \tag{3}$$

Some applications. The previous theorem has early applications, some of them correspond with well known results. Let us indicate the following.

(1) The simplest circle bundle that one can construct over a surface B is the trivial one, $M = B \times \mathbb{S}^1$. In this case generalized Kaluza-Klein metrics correspond with those known as warped product metrics and bundle-like metrics are nothing but Riemannian products. In the latter case, it should be noted that both O'Neill's invariants vanish, in fact $T = 0$ because fibers are geodesics and $A = 0$ because the horizontal distribution is integrable. Therefore, the previous theorem can be applied in the following way. The equivariant surface $S_\gamma = \gamma \times \mathbb{S}^1$ is a Willmore tube in the conformal class associated with the metric $\bar{g} = g + dt^2$ if and only if γ is free elastica in (B, g), that is a critical point of the total squared curvature functional ($\Psi = 0$) acting on a suitable space of curves with no penalty on the length. Now, we can take advantage of the study of free elastica in surfaces with constant curvature say c, [19], to obtain one-parameter families of Willmore tubes in $B(c) \times \mathbb{R}$ and rational one-parameter subfamilies of Willmore tori when $c \neq 0$. A more detailed description of these familes of Willmore surfaces can be checked as a very special case of the discussion that we will do later.

(2) On the other hand, by removing an axis in $\mathbb{R}^3$, say the z-axis, and using cylindrical coordinates in $M = \mathbb{R}^3 - \{z\text{−axis}\}$, we can regard $(M, \bar{g})$ endowed with the Euclidean metric $\bar{g}$, as a warped product of the half Euclidean plane $P = \{(x, 0, z) : x > 0\}$ and the unit circle. This warped product being associated with the function on P that measures the distance to the removed axis. Therefore, we have a trivial circle bundle endowed with a generalized Kaluza-Klein metric. In other words, after removing and axis, the Euclidean metric can be viewed as a generalized Kaluza-Klein one. In addition, by making a suitable conformal change in the metric one can see that the corresponding conformal bundle-like metric is precisely the Riemannian product of a hyperbolic half plane and a circle. In this way, we can construct a one-parameter family of Willmore tubes in the Euclidean space, which contains in turn a rational one-parameter subfamily of Willmore tori (see [4, 20] for details).

(3) As it is known, the trivial circle bundle over $\mathbb{S}^2$ corresponds to a monopole with charge zero. Apart from this circle bundle, probably the most popular circle bundle over $\mathbb{S}^2$ is given by the Hopf map, $\mathfrak{p} : \mathbb{S}^3 \to \mathbb{S}^2$ (see [27] for a survey on its applications). It corresponds with a charge one Dirac monopole. This fibration becomes into a Riemannian submersion with geodesic fibres, if we assume that both spheres are round with radii one and one half, respectively. Now the potential which provides the Willmore tubes works as a Lagrange multiplier that constraint the length of elasticae in $\mathbb{S}^2$ and it can be easily computed to be four. Consequently, we obtain a one-parameter class of Willmore tubes in the three sphere which contains a rational one-parameter subclass of Willmore tori that were first obtained by U. Pinkall, [24]. If we use a global positive scaling factor (also called a constant squashing parameter) on the fibres of the previous Hopf map, then we obtain a nice representation of the Berger spheres endowed with bundle-like metrics associated with the round two sphere. Consequently, the previous theorem applies to provide Willmore surfaces, in particular Willmore tori, as equivariant ones built on elastic curves, with a suitable

constant potential, in the round two sphere. This result was first obtained in [3] (compare also with [6]) and it will appear later as a special case in our detailed discussion.

(4) The class of principal fibre bundles, with structure group G, over a certain space, B, which admit a flat connection can be briefly described as follows (see [17], vol. I, for details). One starts from a regular covering space, $\tilde{B}$ of B, which can be viewed as a principal fibre bundle with the group of deck transformations, D, as structure group, admiting a natural flat connection. Then, one chooses a monomorphism from D into G to extend the transition functions and so obtaining a principal fibre bundle with structure group G. Furthermore, that monomorphism can be extended to a monomorphism of principal fibre bundles which provides a flat connection in the new bundle. In this construction, the original covering space appears as the holonomy bundle, through any point, of the so obtained flat connection. Certainly, to obtain a non trivial construction, the space B should be non simply connected. In this context and according to the setting of this paper, we wish to describe the class of three dimensional principal fibre bundles, $M(B, G)$, over an embedded surface of revolution in $\mathbb{R}^3$, which admit a flat connection. The fundamental group, $\pi_1(B)$, is free abelian with one or two generators according to whether the profile curve of B, δ, is not closed or it is closed, respectively. From now on, we will write $B = R(\delta)$. In the last case, the deck transformation group, D, of any covering space is, up to isomorphisms, either $\mathbb{Z}_n \otimes \mathbb{Z}_m$, or $\mathbb{Z} \otimes \mathbb{Z}_m$ or $\mathbb{Z} \otimes \mathbb{Z}$. Since the structure group has dimension one, one does not dispose of monomorphisms from D into G. In other words, there exist no three dimensional principal fibre bundles, with flat connection, over a compact embedded surface of revolution in $\mathbb{R}^3$. Consequently, we can restrict ourselves to the case where δ is not closed. In this case $\pi_1(R(\delta)) \cong \mathbb{Z}$ and consequently, the deck transformation group, D, of any covering space is, up to isomorphisms, either $\mathbb{Z}_n$ for some $n \in \mathbb{N}$, or $\mathbb{Z}$ (this case occurring when the covering space is the universal one). On the other hand, the structure group, G, must be either $(\mathbb{R}, +)$ or the multiplicative group $\mathbb{S}^1$. Then, the class of three dimensional principal fibre bundles over $R(\delta)$, which admit a flat connection, can be describe as follows

(i) For any $r \in \mathbb{R}$, the monomorphism $\phi_r : \mathbb{Z} \to \mathbb{R}$, defined by $\phi_r(k) = rk$, provides a principal bundle $M_r(R(\delta), \mathbb{R})$ which admit a flat connection whose holonomy subbundle is isomorphic to $\mathbb{R}^2(R(\delta), \mathbb{Z})$.

(ii) For any $n \in \mathbb{N}$, the monomorphism, $\psi_n : \mathbb{Z}_n \to \mathbb{S}^1$, that identifies $\mathbb{Z}_n$ with the group of primitive n-roots of unity, provides the principal bundle $\bar{M}_n(R(\delta), \mathbb{S}^1)$ which admit a flat connection whose holonomy subbundle is isomorphic to a suitable regular covering space of $R(\delta)$.

(iii) For any real number q which is not a rational multiple of π, the monomorphism, $\varphi_q : \mathbb{Z} \to \mathbb{S}^1$ defined by $\varphi_q(k) = e^{ikq}$, provides the principal fibre bundle

$\breve{M}_q(R(\delta), \mathbb{S}^1)$ which admit a flat connection whose holonomy subbundle is isomorphic to $\mathbb{R}^2(R(\delta), \mathbb{Z})$.

The last two cases correspond with circle principal bundles so that the fibres are compact and then Theorem 4.1 applies. On the other hand, the flatness of the principal connection automatically implies that the potential, constraining the elastic curves in $R(\delta)$, vanishes identically. Therefore, *the equivariant Willmore tubes in these two one-parameter classes of conformal bundle-like metrics correspond to the class of free elasticae in* $R(\delta)$ (this result was first shown in [7]).

4 A direct variational approach under non compactness of the fibres and Willmore sheets

To study the case where the group G is not compact, we will use a direct approach to avoid the use of the symmetric criticality principle which might be unclear in this case. Therefore, we need to compute the first variation formula associated with the Willmore functional acting on surfaces S in a Riemannian three space, $(M, \bar{g})$ (not necessarily bundle-like), which have a fixed boundary and are tangent along the common boundary. As usual, a variation of $\phi \in \mathbf{I}_\Gamma(S, M)$ is a map, $\Phi : S \times (-\delta, \delta) \to M$, such that the mappings $\phi_v(m) = \Phi(m, v)$ belong to $\mathbf{I}_\Gamma(S, M)$ and $\phi_0 = \phi$. We define the vector field $Z(m, v) = \Phi_* \left(\frac{\partial}{\partial v}(m, v)\right)$ along Φ which vanishes identically along the boundary. In particular, $Z(m) = Z(m, 0)$, is a vector field along ϕ vanishing on ∂S which we call the variational field associated with the above variation. This allows us to identify the tangent space $T_\phi(\mathbf{I}_\Gamma(S, M))$ with that of vector fields along ϕ vanishing on ∂S. To compute the first variation of the Willmore enery $\partial\mathcal{W}(\phi) : T_\phi(\mathbf{I}_\Gamma(S, M)) \to \mathbb{R}$, we pick $Z \in T_\phi(\mathbf{I}_\Gamma(S, M))$ and choose a variation, Φ, of ϕ with variational field Z, then, with the obvious meaning, we have

$$\partial\mathcal{W}(\phi)[Z] = \left\{ \frac{\partial}{\partial v} \left[\int_S \left(H_v^2 + R_v\right) \, dA_v + \int_{\partial S} \kappa_v \, ds \right] \right\}_{v=0}.$$

However, under the boundary conditions that we are considering along this paper (surfaces with the same boundary and being tangent along the common boundary) the total curvature of the boundary is a constant under variations and so we only need to pay attention to the two-dimensional integral term. Using standard variational arguments which involve several integrations by parts (see [3, 28] for details), we obtain that Willmore surfaces satisfy

$$\partial\mathcal{W}(\phi)[Z] = \int_S \left[\bar{g}([\mathcal{S}(\phi) + N_\Phi(R_\Phi)]N_\Phi, Z^\perp)\right] \, dA_\phi = 0, \tag{4}$$

for all $Z \in T_\phi(\mathbf{I}_\Gamma(S, M))$, where $Z^\perp$ denotes the normal component of Z and $\mathcal{S}$ is a Schrödingerlike operator defined on $\mathbf{I}_\Gamma(S, M)$ by

$$\mathcal{S}(\phi) = \Delta_\phi H_\phi + H_\phi \left(2H_\phi^2 - 2K_\phi + \mathrm{Ric}(N_\phi, N_\phi)\right).$$

Certainly, the applications of the formula (4) take place in those contexts where it reduces to a differential equation involving terms that depend only on the surfaces, but not on the variational fields. For example, if $(M, \bar{g})$ has constant curvature, then $N_\Phi(R_\Phi) = 0$ and consequently the Willmore surfaces correspond with immersions in the kernel of $\mathcal{S}$, that is, they are solutions of the following, well known, field equation (compare with [28])

$$\Delta H + 2H(H^2 - K) = 0, \tag{5}$$

where, of course, K stands for the Gaussian curvature of the surface endowed with the induced metric $\phi^*(\bar{g})$.

As far as we know, apart from the constant curvature backgrounds, the only setting where (4) turns out to be a treatable differential equation is that provided by the equivariant surfaces in semi Riemannian products of surfaces and one dimensional Lie groups (see [5] for details). Now, we want to use a similar idea in order to derive the Euler-Lagrange equation in the context of a bundle-like metric associated to a flat normal connection. As a consequence, we will obtain an extension of a result in [7] that was worked out by using Palais' symmetric criticality principle under the compactness of the structure group.

Proposition 4.1 *Let $\mathfrak{p} : M \to B$ be a three dimensional principal fibre bundle over a Riemannian surface, which is endowed with the bundle-like metric associated with a flat connection. Then, $\mathfrak{p}^{-1}(\gamma)$ is Willmore in the corresponding bundle-like conformal class if and only if γ is a free elastica in B.*

Proof. First, note that the bundle-like nature of the metric implies that the fibres are geodesics and so $T = 0$. On the other hand, the flatness of the connection provides the integrability of the horizontal distribution and so $A = 0$. Using this information, we can do a similar proof to that made in [5] for semi Riemannian products or, alternatively, follow the proof of the theorem in the next section with $A = 0$, to conclude that the term which corresponds to the transversal derivative appearing in (4) vanishes identically. More precisely, for any variation of $\phi(S) = \mathfrak{p}^{-1}(\gamma)$ in M, we get $N_\Phi(R_\Phi) = 0$. Consequently, $\mathfrak{p}^{-1}(\gamma)$ is Willmore in the bundle-like conformal class if and only if it is immersed through an immersion, ϕ, belonging to the kernel of the differential operator $\mathcal{S}$. Equivalently, its mean curvature function is a solution of the following field equation

$$\Delta_\phi H_\phi + H_\phi \left(2H_\phi^2 - 2K_\phi + \mathrm{Ric}(N_\phi, N_\phi)\right) = 0.$$

However, the equivariant surfaces are flat so $K_\phi = 0$. On the other hand, from (1) and (2) we obtain that the curvature function of γ in B satisfies

$$2\kappa'' + \kappa^3 + 2K\kappa = 0,$$

K standing for the Gaussian curvature of B. Now, the above equation is nothing but the Euler Lagrange equation for free elastica in B ([19], see also (3) with $\lambda = 0$), what concludes the proof.

4.1 Some applications

In Section 3 we have described the class of three dimensional principal bundles over a surface of revolution which admit a flat connection. The profile curve needs not be closed and so the whole class is made up of three one-parameter subclasses of bundles. One corresponds to principal bundles with non compact structure group and the other two subclasses correspond with circle principal bundles. It should be noted that the same conclusion is obtained for any surface whose fundamental group is free abelian with one generator.

Now, each curve γ in the surface of revolution $R(\delta)$ provides a one-parameter family of sheets and two one-parameter families of tubes. All these surfaces are Willmore in the corresponding conformal bundle-like structures, provided that γ is a free elastica in $R(\delta)$. For example, suppose that $R(\delta)$ is a circular right cylinder, then all of parallels are geodesics, which are minima for the total squared curvature and so trivial free elasticae. Consequently, every three dimensional conformal bundle-like metric associated with a flat connection over a circular right cylinder admits a foliation by either tori (if the structure group is compact) or cylinders (if the structure group is not compact) which are minimal and so Willmore in the corresponding bundle-like metric. However, circular right cylinders are not the only revolution surfaces providing this kind of Willmore foliations. Besides circular cylinders, the only surface of revolution all of whose parallels are free elasticae is described as follows (see [8] for details). In the open half-plane $x > 0$, $y = 0$, consider the following curve which is parametrized by its arclength

$$\delta(s) = \left(\frac{c}{4}\, s^2, 0, \frac{s}{2}\sqrt{1 - \frac{c^2}{4}s^2} - \frac{1}{c}\arccos\frac{c}{2}s + b \right), \qquad s \in (-2/c, 0) \cup (0, 2/c),$$

where $b, c \in \mathbb{R}$ with $c > 0$. Then, all of parallels of the surface of revolution $R(\delta)$, obtained when rotating δ around the z-axis, are free elasticae.

Consequently, with the notation of Section 3, we have

(1) $M_r(R(\delta), \mathbb{R})$, $r \in \mathbb{R}$, admits a Willmore foliation whose fibres are Willmore cylinders with constant mean curvature in the corresponding bundle-like metric.

(2) $\bar{M}_n(R(\delta), \mathbb{S}^1)$, $n \in \mathbb{N}$, admits a Willmore foliation whose fibres are Willmore tori with constant mean curvature in the corresponding bundle-like metric.

(3) $\breve{M}_q(R(\delta), \mathbb{S}^1)$, with q is a real number which is not a rational multiple of π, admits a Willmore foliation whose fibres are Willmore tori with constant mean curvature in the corresponding bundle-like metric.

5 Equivariant Willmore surfaces in homogeneous three spaces

A natural breeding ground to apply our main theorem is provided by the homogeneous three spaces. It is well known that if $(M, \bar{g})$ denotes a homogeneous oriented three space, then its isometry group, $\mathbf{I}(M)$, is a Lie group with dimension 6, 4 or 3. From now on, we will say that it has rigidity of order 6, 4 or 3, respectively. In the first case, the space has constant curvature and so it is covered by either $\mathbb{S}^3$ (positive curvature), $\mathbb{R}^3$ (curvature zero) or $\mathbb{H}^3$ (negative curvature). In the last section, we have constructed wide families of Willmore tubes and tori in the corresponding conformal class.

The second degree of rigidity correspond to those homogeneous three spaces with $\dim(\mathbf{I}(M)) = 4$. This is certainly the widest and interesting family of homogeneous three spaces. Among its simply connected members, one can find besides a couple of Riemannian product, $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$, the Berger spheres, the Heisenberg group, Nil_3, and the universal covering of the special lineal group $\mathrm{SL}(2, \mathbb{R})$. Obviously, the family also includes quotient of these spaces by suitable isometry subgroups, namely

- Lens spaces, $\mathrm{L}_n = \mathbb{S}^3/\mathbb{Z}_n$, $n \geq 2$, including the projective space, $\mathbb{RP}^3 = \mathrm{L}_2$, with the corresponding induced Berger metrics;
- Heisenberg bundles, including those over flat tori; and
- The projective special linear group $\mathrm{PSL}(2, \mathbb{R}) = \widetilde{\mathrm{PSL}}(2, \mathbb{R})/\mathbb{Z}_2$ and other quotients of $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$.

Any homogeneous three space with rigidity of order four, $(M, \bar{g})$, can be viewed as a bundle over a surface with constant curvature. To exploit this point of view, we will follow the notation of [12]. This fibration provides a Riemannian submersion, $\mathfrak{p} : M \to B(c)$, with geodesic fibers ($T = 0$), so that the homogeneous metric is a bundle-like one over a constant curvature (say c) metric, g. In addition, the vertical flow is generated by a unit Killing vector field V and it allows one to compute the second O'Neill invariant as

$$A_X V = r\,(X \times V), \tag{6}$$

where X is a horizontal vector field and r is a constant, the bundle curvature ([12]), though actually it is the mixed curvature of this bundle. Both constants, c (curvature of the base) and r (bundle curvature), classify the homogeneous space up to isometries and topology. In other words, each pair of real numbers, (c, r), determines, up to topology, a congruence class $\mathbf{E}(c, r)$ of homogeneous three spaces whose isometry group has either dimension 4, if $c \neq 4r^2$, or dimension 6 (constant curvature), if $c = 4r^2$. If the fibres are circles, then an interesting and immediate consequence of our first theorem reduces the searching for invariant Willmore tubes in the conformal class of $\mathbf{E}(c, r)$ to that of elastic curves in $B(c)$ associated with the constant potential $4r^2$. More precisely, we have the following

Corollary 5.1 $\mathbf{S}_\gamma = \mathfrak{p}^{-1}(\gamma)$ *is a Willmore tube in* $\mathbf{E}(c,r)$ *if and only if* γ *is a critical point, in* $B(c)$*, of the elastic energy*

$$\mathcal{E}(\beta) = \int_\beta (\kappa_\beta^2 + 4r^2)\, ds.$$

This result only holds when the fibres are compact, i. e., the structure group is a circle. However, we can extend that statement to every homogeneous three space $\mathbf{E}(c,r)$, no matter if the structure group is compact or not. To do that, the main machinery is provided by the direct variational approach that we have developed in the last section. More precisely, the formula (4) can be applied when $M = \mathbf{E}(c,r)$ to characterize those equivariant surfaces that are Willmore. In this case, that formula becomes into a differential equation and so the family of equivariant Willmore surfaces appears as the kernel of the differential operator $\mathcal{S}$. More precisely, we have the following

Theorem 5.2 *Let* $\phi \in \mathbf{I}(S, \mathbf{E}(c,r))$ *be an equivariant immersion, that is,* $\phi(S) = \mathfrak{p}^{-1}(\gamma)$ *for a curve* $\gamma \subset B(c)$*. Then, it is a Willmore surface if and only if* $\mathcal{S}(\phi) = 0$*.*

Proof. We will take first a look to the Fermi coordinate systems on a three dimensional Riemannian space, M. Given a point $p \in M$, the notion of normal coordinate system, in a neighborhood of p, associated with an orthonormal basis of T_pM is well known. The Fermi coordinates arose as the natural extension of normal coordinates when one replaces the point p by a surface S of M (see [15] and references therein for details). Let $\xi = T^\perp M$ be the normal bundle of S in M, which is a differentiable manifold of dimension three. Now, we can define the exponential map associated with ξ as follows

$$\mathbf{exp}_\xi : \xi \to M, \qquad \mathbf{exp}_\xi(q, \mathrm{x}) = \mathbf{exp}_q(\mathrm{x}), \quad \forall (q, \mathrm{x}) \in \xi,$$

although strictly speaking it could be defined only in a neighborhood of the zero section of ξ. It should be noted that we may identify S with the zero section of ξ, so that S is regarded as a surface of ξ as well as a surface of M. Under this identification, it is clear that $T_{(q,0)}\xi = T_qS \oplus T_q^\perp S$. As a consequence of the inverse function theorem, the mapping $\mathbf{exp}_\xi : \xi \to M$ maps a neighborhood of $S \subset \xi$ onto a neighborhood of $S \subset M$. So denote by $\mathcal{P}_S$ the largest neighborhood of the zero section in ξ for which the exponential mapping provides a diffeomorphism onto its image. This property is exploited to define the Fermi coordinates. To do it, we need an arbitrary system of coordinates (s,t) defined in a certain neighborhood, say $U \subset S$ together with a unitary section, N, of the normal bundle, ξ, restricted to U. In this setting, we can define the Fermi coordinate system as follows

$$\mathrm{x}_1\left(\mathbf{exp}_\xi(q, v\mathbf{N}(q)\right) = s(q), \quad \mathrm{x}_2\left(\mathbf{exp}_\xi(q, v\mathbf{N}(q)\right) = t(q), \quad \mathrm{x}_3\left(\mathbf{exp}_\xi(q, v\mathbf{N}(q)\right) = v.$$

In other words, if we parametrize U by $\phi(s,t)$ then the Fermi coordinates, defined in $\mathbf{exp}_\xi(\mathcal{P}_U)$, of $\mathbf{exp}_\xi(\phi(s,t), v\, N(s,t))$ are just the cylindrical coordinates (s,t,v) of

$(\phi(s,t), v\, N(s,t))$ in the neighborhood $\mathcal{P}_U$ of the zero section in the normal bundle, ξ, restricted to U.

Coming back to the statement, certainly ϕ provides a Willmore surface if and only if $\partial\mathcal{W}(\phi)[Z] = 0$ for all $Z \in T_\phi(\mathbf{I}_\Gamma(S, \mathbf{E}(c,r))$. So, to prove this statement, we only need to show that, in this case, the term $N_\Phi(R_\Phi)$ appearing in the above first variation vanishes identically. Choosing a horizontal lift, $\bar\gamma(s)$, of $\gamma(s)$, which we assume to be parametrized by its arclength, the equivariant immersion ϕ can be expressed by

$$\phi(s,t) = \varphi_t(\bar\gamma(s)).$$

Along this map, we have the following orthonormal frame $\phi_s = d\varphi_t(\bar\gamma'(s))$, $\phi_t = V$ and $N = V \wedge \phi_s$. Moreover, it is clear that we can restrict ourselves to variations which are normal to the surfaces, that is, associated with variational vector field $Z(s,t) = f(s,t)\, N(s,t)$ and suitable functions f along the surface. Now, we have the variation

$$\Phi(v,s,t) = \mathbf{exp}_{\phi(s,t)}\,(v\, Z(s,t)).$$

Then, we have

$$\Phi_s = \phi_s + v f_s\, N + v f\, \nabla_s\, N,$$
$$\Phi_t = \phi_t + v f_t\, N + v f\, \nabla_t\, N,$$

where ∇ stands for the Levi Civita connection in $\mathbf{E}(c,r)$. Now, we have

$$N_\Phi(R_\Phi) = N\left(\frac{R(\Phi_s,\Phi_t,\Phi_t,\Phi_s)}{\langle\Phi_s,\Phi_s\rangle\langle\Phi_t,\Phi_t\rangle - \langle\Phi_s,\Phi_t\rangle^2}\right)_{v=0}.$$

We express both terms of the above quotient as polynomials in v as follows

$$\begin{aligned} R(\Phi_s,\Phi_t,\Phi_t,\Phi_s) &= r^2 + 2v[f_s\, R(\phi_s,\phi_t,\phi_t,N) + f\, R(\phi_s,\phi_t,\phi_t,\nabla_s\, N) \\ &+ f_t\, R(\phi_s,\phi_t,N,\phi_s) + f\, R(\phi_s,\phi_t,\nabla_t\, N,\phi_s)] + v^2 ... \end{aligned}$$

$$\begin{aligned} \Delta(v,s,t) &= \langle\Phi_s,\Phi_s\rangle\langle\Phi_t,\Phi_t\rangle - \langle\Phi_s,\Phi_t\rangle^2 \\ &= 1 + 2vf\,(\langle\phi_s,\nabla_s\, N\rangle + \langle\phi_t,\nabla_t\, N\rangle) + v^2 ... \end{aligned}$$

Then the transversal derivative is given by

$$\begin{aligned} N_\Phi(R_\Phi) &= 2\,\{f_s\, R(\phi_s,\phi_t,\phi_t,N) + f_t\, R(\phi_s,\phi_t,N,\phi_s)\} \\ &+ 2f\,\{R(\phi_s,\phi_t,\phi_t,\nabla_s N) - \langle\phi_s,\nabla_s N\rangle\, R(\phi_s,\phi_t,\phi_t,\phi_s)\} \\ &+ 2f\,\{R(\phi_s,\phi_t,\nabla_t N,\phi_s) - \langle\phi_t,\nabla_t N\rangle\, R(\phi_s,\phi_t,\phi_t,\phi_s)\}\,. \end{aligned}$$

However, $\nabla_s N = \langle\nabla_s N,\phi_s\rangle\phi_s + \langle\nabla_s N,\phi_t\rangle\phi_t$ which automatically implies that

$$R(\phi_s,\phi_t,\phi_t,\nabla_s N) - \langle\phi_s,\nabla_s N\rangle\, R(\phi_s,\phi_t,\phi_t,\phi_s) = 0,$$

and the same argument works to see that the third term of the right hand also vanishes. Therefore,

$$N_\Phi(R_\Phi) = 2\left\{f_s\, R(\phi_s, \phi_t, \phi_t, N) + f_t\, R(\phi_s, \phi_t, N, \phi_s)\right\}.$$

On the other hand, we use all the machinery associated with the Riemannian submersions (see [10]). Remember that, in our case, the fibres are geodesics so $T = 0$ and so

$$R(\phi_s, \phi_t, \phi_t, N) = -\langle (\nabla_{\phi_t} A)_{\phi_s}\; N, \phi_t\rangle - \langle A_{\phi_s}\phi_t, A_N\phi_t\rangle.$$

The first term of the right hand side vanishes due to the alternating properties of ∇A, so

$$R(\phi_s, \phi_t, \phi_t, N) = -r^2\,\langle \phi_s \times \phi_t, N \times \phi_t\rangle = 0.$$

The last term we wish to delete can be written as follows

$$R(\phi_s, \phi_t, N, \phi_s) = \langle (\nabla_{\phi_s} A)_{\phi_s}\; N, \phi_t\rangle.$$

Now, observe that $A_{\phi_s}N = -r\phi_t$ and so

$$(\nabla_s A)_{\phi_s} N + A_{\nabla_s\phi_s}N + A_{\phi_s}\nabla_s N = -r\,\nabla_s\phi_t.$$

However, $\nabla_s\phi_s$ has the direction of N and consequently $A_{\nabla_s\phi_s}N = 0$. Finally, we have

$$\langle A_{\phi_s}\nabla_s N, \phi_t\rangle = \langle \nabla_s N, \phi_s\rangle\,\langle A_{\phi_s}\phi_s, \phi_t\rangle + r\langle \nabla_s N, \phi_t\rangle\,\langle \phi_s \times \phi_t, \phi_t\rangle = 0,$$

which finishes the proof.

Corollary 5.3 *The equivariant surface $S_\gamma = \mathfrak{p}^{-1}(\gamma)$ is a Willmore one in $\mathbf{E}(c,r)$ if and only if γ it is a critical point, in $B(c)$, of the elastic energy*

$$\mathcal{E}(\beta) = \int_\beta (\kappa_\beta^2 + 4r^2)\, ds.$$

Proof. This result has been shown when the fibres are compact, a circle. Now, we can give a simple proof which works anytime, no matter if the fibres are compact or not. In fact, according to the last theorem, $S_\gamma = \mathfrak{p}^{-1}(\gamma)$ is Willmore in $\mathbf{E}(c,r)$ if and only if the corresponding immersion belongs to the kernel of the operator $\mathcal{S}$. However, the equation $\mathcal{S}(\phi) = 0$ for equivariant surfaces can be projected down onto the base $B(c)$, getting the following differential equation for the curvature function of the curve γ in $B(c)$

$$2\kappa'' + \kappa^3 + 2(c - 2r^2)\kappa = 0, \tag{7}$$

which is nothing but the Euler Lagrange equation, in $B(c)$, associated with the quoted elastic energy.

6 Qualitative study of elasticae to callibrate the size of the equivariant Willmore surfaces families

Once we have reduced the study of equivariant Willmore surfaces in homogeneous three spaces $\mathbf{E}(c,r)$ to that of elastic curves in $B(c)$, we wish to give some details on the families of Willmore surfaces that we obtain in each one of the homogeneous three geometries. However, we have to do a couple of considerations. First, in our computations we can include the case $c = 4r^2$, though it corresponds with a real space form (rigidity of order six) and so we will describe the class of Willmore tubes obtained by Pinkall ([24]). Secondly, in the next discussion we can also consider the cases where $r = 0$, though these geometries correspond to Riemannian products of surfaces with constant curvature and fibres.

The elastic curves in a surface $B(c)$, with constant curvature c, have been intensively studied (see for example [19, 21]). Let us briefly describe the qualitative behavior of these curves. In our case, the curvature function, $\kappa(s)$, of elastic curves, in $\mathbf{N}(c)$, satisfies the Euler Lagrange equation (7). Multiplying it by κ' and writing $u = \kappa^2$, one sees that a first integral of (7) is of the form $(u')^2 = P(u)$, for a certain third degree polynomial $P(u)$. Now, the non constant solutions of this equation will appear whenever $P(u) > 0$ and consequently the polynomial has three real roots satisfying $-a_1 \leq 0 \leq a_2 \leq a_3$ and

$$a_1 - a_2 - a_3 = 4(c - 2r^2), \qquad a_1\, a_2\, a_3 = 0. \tag{8}$$

The general solution is given in terms of the elliptic functions as follows

$$\kappa^2(s) = a_3(1 - q^2\,\mathrm{sn}^2(ms, p)),$$

where $\mathrm{sn}(x,p)$ is the elliptic sinus function and other parameters are given by

$$p^2 = \frac{a_3 - a_2}{a_3 + a_1}, \quad q^2 = \frac{a_3 - a_2}{a_3}, \quad m^2 = \frac{a_3\, q^2}{4p^2} = \frac{a_3 + a_1}{4}.$$

It is clear that $a_3 > 0$, otherwise the elastica is a geodesic. Furthermore, from (8), we see that one of the other two roots should vanish. Therefore, for convenience, we will split the discussion in two cases:

(1) If $c - 2r^2 \geq 0$, then $a_2 = 0$. Certainly, the geodesics of $B(c)$ are the only elastic curves with constant curvature. To obtain non trivial solutions, we note that the parameters defining their curvature functions satisfy

$$0 < p^2 = \frac{a_3}{a_3 + a_1} < 1, \quad q^2 = 1, \quad m = \sqrt{\frac{a_3}{4p^2}} = \frac{1}{2}\sqrt{a_3 + a_1}.$$

Moreover, we can compute the maximum squared curvature, a_3, in terms of p and the homogeneous structure data (c,r) to obtain

$$(1 - 2p^2)\, a_3 = 4p^2(c - 2r^2),$$

which shows that $1-2p^2$ is positive when $c-2r^2>0$, while $p^2=1/2$ if $c=2r^2$. Then we get the maximum curvature of the non trivial solutions

$$\kappa_o=\sqrt{a_3}=2p\sqrt{\frac{c-2r^2}{1-2p^2}}, \qquad 0<p\leq\frac{\sqrt{2}}{2}.$$

The solutions are summarized as follows:

(i) If $c=2r^2$, then we have the following elastic curves

- Geodesics, $\kappa=0$
- The one-parameter class of *wavelike* elastic curves with curvature functions

$$\kappa(s)=\kappa_o\,\mathrm{cn}\left(\frac{\sqrt{2}\,\kappa_o}{2}\,s,\frac{\sqrt{2}}{2}\right), \qquad \kappa_o>0. \tag{9}$$

(ii) If $c>2r^2$, then we have the following elastic curves

- Geodesics, $\kappa=0$
- The one-parameter class of *wavelike* elastic curves whose curvature functions are given by

$$\kappa(s)=2p\sqrt{\frac{c-2r^2}{1-2p^2}}\,\mathrm{cn}\left(\sqrt{\frac{c-2r^2}{1-2p^2}}\,s,p\right), \qquad 0<p<\frac{\sqrt{2}}{2}. \tag{10}$$

As a consequence, we obtain the following:

Corollary 6.1 *The class of Willmore tubes in either a Berger sphere with $c=2r^2>0$ or $\mathbb{R}^2\times\mathbb{S}^1$ $(c=r=0)$ is made up of the following surfaces:*

1. *Minimal surfaces shaped on geodesics in $B(c)$.*
2. *A one-parameter class of tubes built on the wavelike elastic curves whose curvature function is given in* (9)*.*

Corollary 6.2 *The class of Willmore tubes in either a Berger sphere with $c>2r^2>0$ or $\mathbb{S}^2\times\mathbb{S}^1$ $(c>0,\ r=0)$ consists of the following surfaces:*

1. *Minimal surfaces shaped on geodesics in $B(c)$.*
2. *A one-parameter class of tubes built on the wavelike elastic curves whose curvature function is given in* (10)*.*

It should be noted that Willmore surfaces, in the conformal round three sphere, obtained by U. Pinkall, [24], are also obtained, as a special case, in the last corollary just choosing $c = 4$ and $r = 1$.

(2) If $c - 2r^2 < 0$, then it is clear that, besides geodesics, those curves with constant curvature function $\sqrt{2(2r^2 - c)}$ are elastic curves. To investigate the non trivial solutions of (7), we proceed as follows. We will study separately two cases depending on which root vanishes.

(2.1) If $a_1 = 0$, then $0 < p^2 = q^2 = \frac{a_3 - a_2}{a_3} < 1$. Then, we combine this information with (8) to obtain the value of the maximum squared curvature, a_3, in terms of p, and the homogeneous structure parameters (c, r)

$$2(2r^2 - c) < a_3 = \frac{4(2r^2 - c)}{2 - p^2} < 4(2r^2 - c).$$

Therefore, we get a one-parameter family of *orbitlike* elastic curves whose curvature functions are given by

$$\kappa(s) = 2\sqrt{\frac{2r^2 - c}{2 - p^2}}\, \mathrm{dn}\left(\sqrt{\frac{2r^2 - c}{2 - p^2}}\, s, p\right), \qquad 0 < p < 1. \tag{11}$$

(2.2) If $a_2 = 0$, then we obtain a one-parameter family of *wavelike* elastic curves with curvature functions given by

$$\kappa(s) = 2p\sqrt{\frac{2r^2 - c}{2p^2 - 1}}\, \mathrm{cn}\left(\sqrt{\frac{2r^2 - c}{2p^2 - 1}}\, s, p\right), \qquad \frac{\sqrt{2}}{2} < p < 1. \tag{12}$$

and the maximum squared curvature satisfying $a_3 > 4(2r^2 - c)$.

Two details should be pointed out. First, we obtain another elastica when $a_1 = a_2 = 0$. In this case the maximum squared curvature is $a_3 = 4(2r^2 - c)$ and the curvature function is given by

$$\kappa(s) = 2\sqrt{2r^2 - c}\, \mathrm{sech}(\sqrt{2r^2 - c}\, s). \tag{13}$$

This elastica is called *borderline* and it is strongly related with the tractrix as we will see in the next section. Secondly, there exists a gap separating geodesics from the main continuum of elasticae.

Now, those families of elasticae generate corresponding Willmore tubes according to the following:

Corollary 6.3 *The class of Willmore tubes in* $\mathbf{E}(c, r)$*, with* $c < 2r^2$*, is made up of the following surfaces:*

1. *Minimal sheets (or tubes) shaped on geodesics of* $B(c)$*.*

2. *Sheets (or tubes) with constant mean curvature* $\sqrt{2(2r^2 - c)}/2$*.*

3. *A one-parameter family of sheets (or tubes) built on orbitlike elasticae with curvature functions given in* (11).

4. *A one-parameter family of sheets (or tubes) built on wavelike elasticae with curvature functions given in* (12).

5. *A sheet (or tube) shaped on a borderline elastica with curvature function given in* (13).

It should be observed that, as a consequence of this result, just choosing $c = -1$ and $r = 0$, we find the Willmore surfaces in the conformal Euclidean three space, viewed as a Riemannian product of a hyperbolic plane and a circle, obtained by J. Langer and D. A. Singer in [20].

7 Explicit examples of Willmore sheets, tubes and tori.

In this section, we wish to illustrate the previous discussion by constructing explicit examples of Willmore surfaces of any kind: sheets, tubes and tori. Therefore, we will be mainly interested in Willmore surfaces of homogeneous spaces with 4-dimensional isometry group. For simplicity, in this part we will omit most of the long computations and, for the reader convenience, we will analyze separately the three cases: $c = 0$, $c < 0$, and $c > 0$.

7.1 Nilmanifolds or geometries associated with the Heisenberg group

It is known that a symplectic vector space with dimension $2n$, say (V, ω), determines an associated Heisenberg group on $V \times \mathbb{R}$. In particular, the three dimensional Heisenberg group is defined on $\mathbb{R}^3$ when starting from the symplectic structure $(\mathbb{R}^2, \omega)$, where

$$\omega((x, y), (\bar{x}, \bar{y})) = \det \begin{pmatrix} x & y \\ \bar{x} & \bar{y} \end{pmatrix},$$

and defining the group operation as follows

$$(x, y, z) \star (\bar{x}, \bar{y}, \bar{z}) = \left(x + \bar{x}, y + \bar{y}, z + \bar{z} + \frac{1}{2}\, \omega((x, y), (\bar{x}, \bar{y})) \right).$$

From a classical point of view, the three dimensional Heisenberg group, Nil_3, appeared as the nilpotent Lie subgroup in $\mathrm{GL}(3, \mathbb{R})$. However, the following map provides an isomorphism between $(\mathbb{R}^3, \star)$ and Nil_3

$$(x,y,z) \mapsto \begin{pmatrix} 1 & x & z+\frac{xy}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

The three dimensional Heisenberg group is one of the eight canonical Thurston three dimensional geometries. In fact, it can be endowed with a one-parameter class of left invariant metrics that, in the usual coordinates of $\mathbb{R}^3$, can be written by

$$g_r = dx^2 + dy^2 + (dz + r(y\,dx - x\,dy))^2, \qquad r > 0.$$

The Heisenberg three space (Nil_3, g_r) which we will denote by $\mathrm{Nil}_3(r)$ is a homogeneous space whose isometry group has dimension four. According to the used terminology, it is identified with $\mathbf{E}(0,r)$ and it appears as a fibre bundle over the Euclidean plane with projection $\mathfrak{p} : \mathrm{Nil}_3(r) \to \mathbb{R}^2$, $\mathfrak{p}(x,y,z) = (x,y)$. Consequently, the discussion in Section 6 can be applied. In particular, regardless of the squashing parameter r, we get that $\mathrm{Nil}_3(r)$ admits the following Willmore surfaces:

(1) Minimal sheets constructed over straight lines.

(2) Cylinders with constant mean curvature r built over circles with radius $\frac{1}{2r}$.

(3) A one-parameter class of sheets constructed on orbitlike elasticae in the Euclidean plane and curvature function given by (11) (with $c = 0$).

(4) A one-parameter class of sheets constructed on wavelike elasticae in the Euclidean plane and curvature function given by (12) (with $c = 0$).

(5) A sheet built over a borderlike elastica in the Euclidean plane with curvature function (13) (with $c = 0$).

The goal now is to find explicit parametrizations of the above surfaces. To start with we observe that a first integral of (7) with $c = 0$ is

$$\dot{\kappa}^2 + \frac{1}{4}\kappa^4 - 2r^2\kappa^2 = d, \quad d \in \mathbb{R}\,. \tag{14}$$

On the other hand, along any elastica $\gamma(s)$, which is a solution of (7), we can define the vector field

$$\mathcal{J} = (\kappa^2 - 4r^2)T + 2\kappa_s N\,, \tag{15}$$

which can be extended to a Killing field on $\mathbb{R}^2$ (see [19]). From (7) and (15), one sees that the length of $\mathcal{J}$ is constant and then its integral curves are straight lines. Take an orthonormal parametrization $x(u,v)$ of $\mathbb{R}^2$ such that $x_u = b\mathcal{J}$, $b \in \mathbb{R}$ and write the elastica as $\gamma(s) = x(u(s), v(s))$. Let $T(s)$ be the unit tangent to $\gamma(s)$ and choose a unit normal $N(s)$, so that $\{T(s), N(s)\}$ is positively oriented.

Combining (7), (15) and $b\mathcal{J} = x_u$, we find $b^2 = \frac{1}{4d+16r^4}$. Moreover, using the definition of $\mathcal{J}$ given in (15) and since $\|T(s)\| = 1$, we get

$$u'(s)^2 + v'(s)^2 = 1\,, \tag{16}$$

$$u'(s) = \langle x_u, T\rangle = \langle b\mathcal{J}, T\rangle = b\left(\kappa^2 - 4r^2\right)\,, \tag{17}$$

$$v'(s) = \langle x_u, N\rangle = \langle b\mathcal{J}, N\rangle = 2\,b\,\kappa_s\,. \tag{18}$$

Integrating (18) we get, without loss of generality,

$$v(s) = \frac{1}{\sqrt{d+4r^4}}\,\kappa(s)\,. \tag{19}$$

Now, we discuss the different possibilities which appear in the item (2) in Section 6 when $c = 0$. If $a_1 = 0$, then $0 < p^2 = q^2 = \frac{a_3-a_2}{a_3} < 1$ and $4r^2 < a_3 = \frac{8r^2}{2-p^2} < 8r^2$. Therefore, we get a one-parameter family of orbitlike elastic curves whose curvature functions are given by

$$\kappa(s) = \frac{2\sqrt{2}\,r}{\sqrt{2-p^2}}\,\mathrm{dn}\left(\frac{\sqrt{2}\,r\,s}{\sqrt{2-p^2}}, p\right), \qquad 0 < p < 1, \tag{20}$$

which combined with (19) gives

$$v(s) \;=\; \frac{2\sqrt{2}\,r}{\sqrt{(d+4r^4)(2-p^2)}}\,\mathrm{dn}\left(\frac{\sqrt{2}\,r\,s}{\sqrt{2-p^2}}, p\right), \qquad 0 < p < 1. \tag{21}$$

After substitution of (20) in (17) and integrating we obtain

$$u(s) \;=\; \frac{2r}{\sqrt{d+4r^4}}\left(r\,s - \frac{\sqrt{2}}{\sqrt{2-p^2}}E\left(\mathrm{am}\left(\frac{\sqrt{2}r\,s}{\sqrt{2-p^2}}, p\right), p\right)\right), \tag{22}$$

where $0 < p < 1$, $E(-, p)$ is the elliptic integral of the second kind of modulus p, $\mathrm{am}(-, p)$ is the Jacobi amplitude and $d = -\frac{a_2 a_3}{4} = \frac{16(p^2-1)r^4}{(p^2-2)^2}$.

If $a_2 = 0$, then we have

$$\kappa(s) = \frac{2\sqrt{2}\,rp}{\sqrt{2p^2-1}}\,\mathrm{cn}\left(\frac{\sqrt{2}r}{\sqrt{2p^2-1}}\,s, p\right), \qquad \frac{\sqrt{2}}{2} < p < 1, \tag{23}$$

and the maximum squared curvature satisfies $a_3 > 4(2r^2 - c)$. Proceeding similarly to the previous case we get

$$u(s) \;=\; \frac{2r}{\sqrt{d+4r^4}}\left(\frac{r\,s}{2p^2-1} - \frac{\sqrt{2}}{\sqrt{2p^2-1}}\,E\left(\mathrm{am}\left(\frac{\sqrt{2}\,r\,s}{\sqrt{2p^2-1}}, p\right), p\right)\right), \tag{24}$$

$$v(s) \;=\; \frac{2\sqrt{2}\,rp}{\sqrt{(d+4r^4)(2p^2-1)}}\,\mathrm{cn}\left(\frac{\sqrt{2}\,r\,s}{\sqrt{2p^2-1}}, p\right), \qquad \frac{\sqrt{2}}{2} < p < 1, \tag{25}$$

where, as before, $E\,(-,p)$ is the elliptic integral of the second kind of modulus p, $\mathrm{am}\,(-,p)$ is the Jacobi amplitude and $d=\frac{a_1a_3}{4}=\frac{16p^2(1-p^2)r^4}{(1-2p^2)^2}$. The limiting case $p=\frac{\sqrt{2}}{2}$ gives the so called free elastica which corresponds to the choice $r=0$, that is, the background is a Euclidean three space.

Finally, when $a_1=0=a_2$, the maximum squared curvature is $a_3=8r^2$, the curvature function is given by

$$\kappa(s)=2\sqrt{2}r\ \mathrm{sech}(\sqrt{2}\,rs), \tag{26}$$

and the associated borderline elastica is a member of the so called *Poleni's curves family* ([16]). In fact, in 1729 Giovanni Poleni studied a family of curves related to the tractrix, which are known as syntractrices. A syntractrix is the locus of a point on the tangent to a tractrix at a constant distance, L, from its intersection with the axis. When L is twice the constant length of the segment generating the tractrix, one obtains the so called *courbe des forçats* (see [16] and references therein).

The natural equations for (26) can be solved obtaining the Poleni's curve (borderline elastica) coordinate functions

$$u(s)=s-\frac{\sqrt{2}}{r}\tanh(\sqrt{2}\,rs), \qquad v(s)=\frac{\sqrt{2}}{r}\,\mathrm{sech}(\sqrt{2}\,rs)\,, \tag{27}$$

up to plane motions.

Therefore, using the parametrizations we found in (21)-(22), (24)-(25) and (27) for the planar elastica with potential $4r^2$, we obtain explicit parametrizations of the Willmore sheets in the conformal Heisenberg three space. For instance, choosing the simplest parametrization, i. e., that corresponding to the Poleni's curve (27), we would obtain the following Willmore sheet in $\mathrm{Nil}_3(r)$ (see Fig. 1)

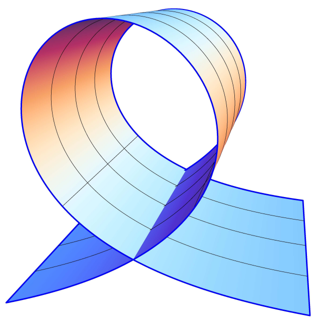

Fig. 1: *Willmore sheet in* $Nil_3(r)$ *over a Poleni's curve.*

parametrized by

$$\mathrm{X}(s,z)=\begin{pmatrix} 1 & s-\frac{\sqrt{2}}{r}\tanh(\sqrt{2}\,rs) & z+\frac{\sqrt{2}}{2r}\mathrm{sech}(\sqrt{2}\,rs)(s-\frac{\sqrt{2}}{r}\tanh(\sqrt{2}\,rs)) \\ 0 & 1 & \frac{\sqrt{2}}{r}\,\mathrm{sech}(\sqrt{2}\,rs) \\ 0 & 0 & 1 \end{pmatrix}.$$

Once we have explicitly described the whole class of equivariant Willmore sheets in the three dimensional Heisenberg group $\mathrm{Nil}_3(r)$, we should find out if this family contains cylinders. Since they are obtained by lifting closed elastic curves, to answer this question we have to check the *closed curve problem* for elasticae in the Euclidean plane. In other words, we will look for closed elastic curves in the Euclidean plane. We already know that circles with radii $\frac{1}{2r}$ are closed elasticae and so they provide equivariant Willmore cylinders with constant mean curvature in $\mathrm{Nil}_3(r)$. It is also clear that the curvature function of a closed elastica should be a periodic function. Then the borderline elastica will be discarded as a candidate and the searching for closed elasticae with non constant curvature is reduced to the two big families of wavelike and orbitlike elastic curves. The condition for a planar elastica with curvature $\kappa(s)$ to close up is $\int_0^\rho (\kappa^2(s) - 4r^2)\, ds = 0$, where ρ is the period of κ, [2]. Now, we can prove that orbitlike elasticae in the Euclidean plane never close. For wavelike elastic curves, we use (23) and integrate to obtain that closed wavelike elasticae in the Euclidean plane correspond with the solutions of the equation

$$2E(p) = K(p)\,, \tag{28}$$

where $K(p)$, $E(p)$ are the complete elliptic integrals of the first and second kind, respectively. It provides us a unique solution, the eight-shaped elastica. This discussion can be summarized in the following:

Corollary 7.1 *The family of equivariant Willmore cylinders in $Nil_3(r)$ consists, up to congruences, of the following surfaces*

1. *An embedded cylinder with constant mean curvature r built by lifting a circle with radius $\frac{1}{2r}$.*
2. *An immersed cylinder constructed by lifting the eight-shaped elastica.*

Let us recall that for any lattice Γ in $\mathrm{Nil}_3(r)$ the compact quotient $\Gamma\backslash\mathrm{Nil}_3(r)$ is a nilmanifold (sometimes called Heisenberg three space). However, it is well known that, up to congruences in $\mathrm{Nil}_3(r)$, the lattices can be indexed by the natural numbers $\{\Gamma_n : n \in \mathbb{N}\}$, where

$$\Gamma_n = \left\{ \begin{pmatrix} 1 & a & c/n \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}.$$

The line bundle, $\mathfrak{p} : \mathrm{Nil}_3(r) \to \mathbb{R}^2$ induces a Seifert fibration, $\mathfrak{p}_n : M_n = \Gamma_n\backslash\mathrm{Nil}_3(r) \to \mathrm{T}^2 \approx \mathbb{Z}^2\backslash\mathbb{R}^2$, which is a circle bundle over the torus T^2 whose fibres are flow orbits by right translations associated with the central one-parameter subgroup

$$Z(\mathrm{Nil}_3(r)) = \left\{ \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}.$$

This construction provides the complete class of circle bundles over a torus (see for example [14]). Furthermore, it is compatible with the Kaluza-Klein mechanism, which allows us to consider, in the Seifert three spaces M_n, the bundle-like metric projected from the corresponding in the Heisenberg group. That Seifert fibration factorizes as $\mathfrak{p}_n = \pi \circ \tilde{\mathfrak{p}}_n$, where $\tilde{\mathfrak{p}}_n : M_n = \Gamma_n \backslash \mathrm{Nil}_3(r) \to \mathbb{R}^2$ and $\pi : \mathbb{R}^2 \to \mathrm{T}^2 \approx \mathbb{Z}^2 \backslash \mathbb{R}^2$ stands for the natural projection. Therefore, we can apply our main result to obtain the whole class of equivariant Willmore tori in $M_n = \Gamma_n \backslash \mathrm{Nil}_3(r)$. Summarizing we have:

Corollary 7.2 *The complete list of equivariant Willmore tori in $M_n = \Gamma_n \backslash Nil_3(r)$ is*

1. *An embedded torus with constant mean curvature r built by lifting a circle with radius $\frac{1}{2r}$ in the Euclidean plane.*
2. *An immersed torus built by lifting the eight-shaped elastica in the Euclidean plane.*

However, the class of equivariant Willmore tori in the Heisenberg three spaces can be made larger by using directly the Seifert fibration $\mathfrak{p}_n : M_n = \Gamma_n \backslash \mathrm{Nil}_3(r) \to \mathrm{T}^2 \approx \mathbb{Z}^2 \backslash \mathbb{R}^2$ by lifting closed elastic curves in the flat torus $\mathrm{T}^2 \approx \mathbb{Z}^2 \backslash \mathbb{R}^2$.

7.2 Geometries associated with the projective special linear group

The Thurston list of 3-dimensional canonical geometries consists of eight simply connected homogeneous spaces. One of them is the universal cover, $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$, of $\mathrm{PSL}(2,\mathbb{R})$. However, for any natural number, $n \geq 2$, the space $\widetilde{\mathrm{PSL}}(2,\mathbb{R})/\mathbb{Z}_n$ is still a homogeneous three space whose isometry group has dimension four and so it is a circle bundle over the hyperbolic two plane. In particular, that happens with $\mathrm{PSL}(2,\mathbb{R}) = \widetilde{\mathrm{PSL}}(2,\mathbb{R})/\mathbb{Z}_2$. Therefore, we turn our attention to the group $\mathrm{PSL}(2,\mathbb{R})$. There are several equivalent ways to see this group in geometry. For example,

$$\mathrm{PSL}(2,\mathbb{R}) = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,:\, \det(M) = ad - bc = 1 \right\},$$

can be identified with the group of Möbius transformations with real coefficients. These mappings preserve the open half plane, $\mathrm{Im}(z) > 0$, of the complex plane $\mathbb{C}$ and then $\mathrm{PSL}(2,\mathbb{R})$ is isomorphic to the isometry group of the hyperbolic plane. Moreover, $\mathrm{PSL}(2,\mathbb{R})$ can be naturally identified with the following quadric of $\mathbb{C}^2$

$$\mathrm{PSL}(2,\mathbb{R}) = \{(z_1, z_2) \in \mathbb{C}^2 \,:\, |z_1|^2 - |z_2|^2 = 1\}, \tag{29}$$

and, even more, it can be also regarded as a suitable tube built over a complex hyperplane in the complex hyperbolic plane $\mathbb{CH}^2$.

Let $\mathbb{H}^2(c)$ be the hyperbolic plane with curvature $c < 0$ regarded as one of the two sheets of a suitable hyperboloid, namely

$$\mathbb{H}^2(c) = \left\{ (\rho\, e^{i\eta}, a) \in \mathbb{C} \times \mathbb{R} \,:\, \rho^2 - a^2 = \frac{1}{c},\, a > 0 \right\}. \tag{30}$$

Then, we use the model described in (29) to define the map

$$\mathfrak{p} : \mathrm{PSL}(2,\mathbb{R}) \to \mathbb{H}^2(c)\,, \qquad \mathfrak{p}(z_1,z_2) = \frac{1}{\sqrt{-c}}(2z_1\,\bar{z}_2, |z_1|^2 + |z_2|^2),$$

which provides a circle principal bundle. The fibre over $(\rho\, e^{i\eta}, a) \in \mathbb{H}^2(c)$ is the following circle

$$\mathfrak{p}^{-1}(\rho\, e^{i\eta}, a) = \left\{ \frac{\sqrt{2}}{2}\left(\sqrt{1 + a\sqrt{-c}}\, e^{i\eta}, \sqrt{-1 + a\sqrt{-c}}\right) e^{i\theta} \;:\; \theta \in \mathbb{R} \right\}.$$

It is obvious that the vector field generating the fibre flow is given by $V(z_1,z_2) = i(z_1,z_2)$, so that if we consider the following vector fields in $\mathrm{PSL}(2,\mathbb{R})$

$$X_1(z_1,z_2) = (\bar{z}_2, \bar{z}_1), \qquad X_2(z_1,z_2) = i\,X_1(z_1,z_2) = i(\bar{z}_2,\bar{z}_1),$$

we get a global frame $\{X_1, X_2, V\}$ on the whole $\mathrm{PSL}(2,\mathbb{R})$. We define the one form $\omega \in \Lambda^1(\mathrm{PSL}(2,\mathbb{R}))$ by

$$\omega(X_1) = \omega(X_2) = 0, \qquad \omega(V) = 1,$$

to obtain a principal connection on this circle principal bundle. If g_o denotes the metric of $\mathbb{H}^2(c)$, then we can use ω to define the one-parameter class of bundle-like metrics

$$\bar{g}_r = \mathfrak{p}^*(g_o) + r^2\,\omega^*(dt^2), \qquad r \neq 0,$$

which, obviously, are left invariant in $\mathrm{PSL}(2,\mathbb{R})$. In this way $(\mathrm{PSL}(2,\mathbb{R}), \bar{g}_r)$ becomes into a homogeneous three space whose isometry group has dimension four and, consequently, we can apply again the previous discussion in Section 6 to get the complete family of equivariant Willmore tubes in the aforementioned conformal classes.

In order to find explicit examples, we have to investigate elastica in the hyperbolic plane $\mathbb{H}^2(c)$. Without loss of generality we may assume that $c = -1$. Then, a first integral of (7) is

$$\dot{\kappa}^2 + \frac{1}{4}\kappa^4 - (2r^2+1)\kappa^2 = d, \quad d \in \mathbb{R}\,. \tag{31}$$

For later purposes we denote by $Q(x)$ the polynomial $Q(x) := d - \frac{1}{4}x^4 + (2r^2+1)x^2$. As before, the following vector field

$$\mathcal{J} = (\kappa^2 - 4r^2)T + 2\kappa_s N\,, \tag{32}$$

defined along any elastica $\gamma(s)$ solution of (7), can be extended to a Killing field on $\mathbb{H}^2(-1)$, which is also denoted by the same letter (Proposition 2.1, [19]). Let $T(s)$ be the unit tangent to $\gamma(s)$ and take the unit normal $N(s)$, so that $\{T(s), N(s)\}$ is positively oriented. Imagine that $\kappa(s)$, the curvature of $\gamma(s)$, reaches a local maximum at s_o and let β be the integral curve of $\mathcal{J}$ through $\gamma(s_o)$. From Proposition 2.2 in [19], we have

$$\kappa_\beta = \frac{2\kappa(s_o)}{\kappa^2(s_o) - 4r^2}\,. \tag{33}$$

Now, we distinguish several cases according to the value of the constant of integration d in (31). For simplicity and without loss of generality, we assume that $r = 1$.

Case 1: $-9 < d < -4$. We take $d \neq -8$, otherwise we will find a singularity in the integration process. In this case, $a_1 = 0$ and we have two solutions of (31) for any value of d. By analyzing the roots of $Q(x)$ and using (33), we can see that $\kappa_\beta > 1$ and therefore the integral curves of $\mathcal{J}$ are circles. Take a parametrization

$$x(u, \theta) = (\sinh u \cos \theta, \sinh u \sin \theta, \cosh u) \tag{34}$$

of the hyperboloid's upper sheet model of $\mathbb{H}^2(-1)$, whose metric coefficients are $g_{11} = 1$, $g_{12} = 0$, $g_{22} = \sinh^2 u$, and choose $\mathcal{J}$ so that $b\mathcal{J} = x_\theta$. Then, as $\|T(s)\| = 1$, we can use the definition of $\mathcal{J}$ given in (32) to get the following relations

$$(u')^2 + (\theta')^2 \sinh^2 u = 1\,, \tag{35}$$

$$\theta' \sinh^2 u = \langle x_\theta, T\rangle = \langle b\mathcal{J}, T\rangle = b\left(\kappa^2 - 4\right)\,, \tag{36}$$

$$u' \sinh u = \langle x_\theta, N\rangle = \langle b\mathcal{J}, N\rangle = 2\, b\, \kappa_s\,. \tag{37}$$

A direct integration of (37) yields

$$\cosh u = 2b\kappa + \mu\,, \tag{38}$$

where $\mu \in \mathbb{R}$. Furthermore, from (31) and (32), we find $4d = \frac{1}{b^2}\sinh^2 u - 4(4+\kappa^2)$, which combined with (38) gives

$$\mu = 0, \quad 4b^2(d+4) = -1\,. \tag{39}$$

From (11), (38) and (39) we have

$$u(s) = \operatorname{arccosh}\left(\frac{2\sqrt{3}}{\sqrt{-(4+d)(2-p^2)}}\,\mathrm{dn}\left(\frac{\sqrt{3}\,s}{\sqrt{2-p^2}}, p\right)\right). \tag{40}$$

Substituting (38) and (39) in (36) gives

$$\theta' = \frac{b\left(\kappa^2 - 4\right)}{4b^2\kappa^2 - 1}\,. \tag{41}$$

Finally, using (11) in (41) and integrating one gets

$$\begin{aligned}\theta(s) \quad &= \quad \frac{\sqrt{-(d+4)}}{6}\left(3s + \frac{\sqrt{3}(2-p^2)^{3/2}(8+d)}{p^2(4+d) - 2(10+d)}\times\right.\\ &\times \quad \left.\Pi\left(\frac{12p^2}{2(10+d) - p^2(4+d)}, \mathrm{am}\left(\frac{\sqrt{3}s}{\sqrt{2-p^2}}, p\right), p\right)\right),\end{aligned} \tag{42}$$

where $0 < p < -5 + 3\sqrt{5}$, $\Pi(n, -, p)$ is the elliptic integral of the third kind of characteristic n and modulus p, $\mathrm{am}(-, p)$ is the Jacobi amplitude and $d = -\frac{a_2 a_3}{4} = \frac{36(p^2-1)}{(p^2-2)^2}$.

Thus (40) and (42) give us the coordinate functions of the elastica with potential 4 with respect to the system (34).

The remaining cases can be worked out similarly and we will not go into much detail. The main features of them can be summarized as follows:

Case 2: $d = -4$. Again $a_1 = 0$, and for any value of d we have a solution of (31) of type (11). But now, the analysis of the roots of $Q(x)$ and using (33) forces $\kappa_\beta = 1$ and, therefore, the integral curves of $\mathcal{J}$ are horocycles. By choosing a suitable coordinate system it would be also possible to obtain an explicit parametrization of the elastica as in the previous case.

Case 3: $-4 < d < 0$. Once more $a_1 = 0$ and we have a solution of (31) of type (11) for any value of d. Now, the roots of $Q(x)$ and (33) would give that $\kappa_\beta < 1$ and so the integral curves of $\mathcal{J}$ are equidistant curves. By choosing again a suitable coordinate system we could to obtain an explicit parametrization of the elastica in this case too.

Case 4: $0 < d$. Geometrically this case is similar to the previous one in the sense that the integral curves of $\mathcal{J}$ are equidistant curves, but now $a_2 = 0$ and we have one solution of (31) of type (12) for any value of d. One can proceed similarly to obtain the explicit parametrization of the elastica for positive d.

Hence, (34), (40) and (42) (and the corresponding equations which we would have obtained in the remaining cases proceeding similarly as in case 1) give us explicit parametrizations of our elastica in $\mathbb{H}^2(-1)$, what can be used in turn to get explicit parametrizations of Willmore tubes in $\mathrm{PSL}(2,\mathbb{R})$.

As an illustration, we consider the limiting case $d = 0$ and obtain an explicit parametrization for the Willmore tube built over the borderline elastica of the hyperbolic plane. This elastica is nothing but the Poleni's curve in $\mathbb{H}^2(c)$ whose curvature function is given in (13). If we use the half plane Poincaré model of the hyperbolic plane, $\mathbb{H}^2(c) = \{(u,v) \in \mathbb{R}^2 : v > 0\}$, then the natural equations for Poleni's curve read as follows

$$\begin{aligned} u''(s) - \frac{2\,u'(s)\,v'(s)}{v(s)} &= -2m\,v'(s)\,\mathrm{sech}(ms), \\ v''(s) + \frac{(u'(s))^2 - (v'(s))^2}{v(s)} &= 2m\,u'(s)\,\mathrm{sech}(ms), \end{aligned}$$

where we have used $\sqrt{2r^2 - c} = m$. Now these equations can be easily integrated to obtain the following solution for the hyperbolic Poleni's curve

$$u(s) = ms, \qquad v(s) = \cosh(ms)\,,$$

which is a catenary. These coordinates for the Poleni's curve can be brought to the hyperboloid model of $\mathbb{H}^2(c)$ (30), by using the map which identifies both models

$$\Psi(u,v) = (\rho\, e^{i\eta}, a) = \left(\frac{u}{v} + \frac{u^2+v^2-1}{2v}\,i, \frac{u^2+v^2+1}{2v}\right).$$

Therefore, the hyperbolic Poleni's curve in the hyperboloid model is given by

$$\delta(s) = \Psi(ms, \cosh(ms)) = \left(\frac{ms}{\cosh(ms)} + \frac{m^2s^2 + \sinh^2(ms)}{2\cosh(ms)}\, i, \frac{m^2s^2 + \sinh^2(ms) + 1}{2\cosh(ms)} \right).$$

Consequently, we obtain the following parametrization for the Willmore tube, in $(\mathrm{PSL}(2,\mathbb{R}), \bar{g}_r)$, built on the above hyperbolic Poleni's curve (see Fig. 2)

$$\mathrm{X}(s,\theta) = \frac{\sqrt{2}}{2} \left(\sqrt{1 + a(s)\sqrt{-c}}\, e^{i\eta(s)}, \sqrt{-1 + a(s)\sqrt{-c}} \right) e^{i\theta},$$

where

$$\eta(s) = \arctan\left(\frac{m^2s^2 + \sinh^2(ms)}{2ms} \right), \qquad a(s) = \frac{m^2s^2 + \sinh^2(ms) + 1}{2\cosh(ms)}.$$

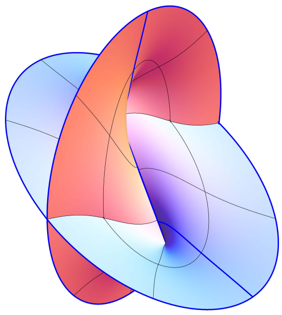

Fig. 2: *Projection, in a coordinate hyperplane, of a piece of a Willmore tube in* $\mathrm{PSL}(2,\mathbb{R})$ *over a hyperbolic Poleni's curve.*

Finally, note that there are plenty of examples of closed elasticae of rotational type in case 1 which can be used to construct examples of Willmore tori in $\mathrm{PSL}(2,\mathbb{R})$. This can be seen in the following way. The curvature of our elastica in case 1 is a periodic function of period $2\sqrt{\frac{2-p^2}{3}}K(p)$, where $K(p)$ is the complete elliptic integral of the first kind. Then, from (40) and (42), the corresponding elastica will close up provided $\theta(2\sqrt{\frac{2-p^2}{3}}K(p))$ is a rational multiple of 2π. But $\theta(2\sqrt{\frac{2-p^2}{3}}K(p))$ is a non-constant function as p moves in $(0, -5+3\sqrt{5})$, which means that there are infinitely many values of p in the interval $(0, -5+3\sqrt{5})$ providing closed elasticae with potential 4 in $\mathbb{H}^2(-1)$. Similar computations can be made for any choice of r, so that we summarize this remarkable achievement in the following:

Corollary 7.3 *There exists a rational one-parameter family of equivariant Willmore tori in* $(\mathrm{PSL}(2,\mathbb{R}), \bar{g}_r)$.

It is worth pointing out the following remarks:

(i) Corollary 7.3 also holds for $\widetilde{\mathrm{PSL}}(2,\mathbb{R})/\mathbb{Z}_n$, with $n \geq 2$. However, for $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ we obtain a rational one-parameter class of equivariant Willmore cylinders.

(ii) As the potential is $4r^2 > 0$, the corresponding closed elastica have self-intersections (see [26]) and, therefore, none of the tori obtained in Corollary 7.3 are embedded.

(iii) The Kaluza-Klein mechanism, used to construct bundle-like metrics, also works for Lorentzian metrics. In particular, we have the following one-parameter class of Lorentzian bundle-like metrics on $\mathrm{PSL}(2,\mathbb{R})$:

$$\tilde{g}_r = \mathfrak{p}^*(g_o) - r^2\,\omega^*(dt^2), \qquad r \neq 0.$$

Now $(\mathrm{PSL}(2,\mathbb{R}), \tilde{g}_r) \to \mathbb{H}^2(c)$ becomes into a semi-Riemannian submersion and we can still reduce symmetry. In this sense, equivariant Willmore tubes, which are timelike, correspond with elastic curves in the hyperbolic plane $\mathbb{H}^2(c)$ associated with a potencial $\lambda = -4r^2$. In contrast with the Riemannian case, we now have simple (without self-intersections) elastic curves in the hyperbolic plane for certain values of r. To be precise, if we consider, for simplicity, $c = -1$, then the whole family of simple elastic curves, according to the values of r, can be described as follows (see [26]):

- For $r^2 < \frac{1}{4}$, the circle with radius $\sinh^{-1}\left(\sqrt{\frac{1}{1-4r^2}}\right)$.
- For any natural number $n \geq 2$ and each r, with $r^2 < \frac{1}{4} - \frac{1}{4(n^2-1)}$, there is a simple elastic curve that closes after n laps.

7.3 Berger spherical geometries

The Berger spheres first appeared in [9], where M. Berger obtained the classification of simply connected normal homogeneous Riemannian spaces with positive sectional curvature. These spheres can be geometrically realized as geodesic spheres of either a complex projective plane or a complex hyperbolic plane, and consequently they have constant scalar curvature. However, using the usual Hopf map, the Berger spheres can be also viewed as three spheres endowed with bundle-like metrics. Although there exists a two-parameter class of Berger spheres, up to homotheties, it can be reduced to a one-parameter class which, obviously, is enough for our purposes. Therefore, we start with, the unit sphere, $\mathbb{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 \,:\, |z_1|^2 + |z_2|^2 = 1\}$. Consider the round 2-sphere $\mathbb{S}^2\left(\frac{1}{2}\right)$, with radius $1/2$ so it has curvature $c = 4$, and the Hopf map

$$\mathfrak{p} : \mathbb{S}^3 \to \mathbb{S}^2\left(\frac{1}{2}\right), \qquad \mathfrak{p}(z_1, z_2) = \left(z_1\bar{z}_2, \frac{1}{2}(|z_1|^2 - |z_2|^2)\right), \tag{43}$$

which provides a circle principal bundle whose fibre flow is generated by the vector field $V(z_1, z_2) = i(z_1, z_2)$. The following vector fields in $\mathbb{S}^3$

$$X_1(z_1, z_2) = (-\bar{z}_2, \bar{z}_1), \qquad X_2(z_1, z_2) = i\,X_1(z_1, z_2) = i(-\bar{z}_2, \bar{z}_1)\,,$$

along with the vertical one, V, determine a global frame on the whole $\mathbb{S}^3$. Now, one can define a principal connection, ω, by

$$\omega(X_1) = \omega(X_2) = 0, \qquad \omega(V) = 1.$$

We now introduce the Berger metrics, associated with ω and the round metric g_o of $\mathbb{S}^2\left(\frac{1}{2}\right)$, as

$$\bar{g}_r = \mathfrak{p}^*(g_o) + r^2\,\omega^*(dt^2), \qquad r \neq 0.$$

The discussion in Section 5 holds also here, so that we obtain the complete class of Willmore tubes and tori in the conformal Berger spheres $(\mathbb{S}^3, \bar{g}_r)$. However, in contrast with the situation previously analyzed for $\mathbb{R}^2$ and $\mathbb{H}^2(-1)$, now the value of the squashing parameter, r, should be carefully considered. Thus, if $r^2 \leq 2$, the class of Willmore tubes consists of minimal tubes (shaped on geodesics in the base space $\mathbb{S}^2\left(\frac{1}{2}\right)$) and a one-parameter class of Willmore tubes built on a corresponding class of wavelike elasticae in the two sphere. While if $r^2 > 2$, besides the Willmore tubes with constant mean curvature, we get two one-parameter classes of Willmore tubes built on wavelike and orbitlike elasticae, respectively, and a borderlike elasticae whose curvature function is given in (13). This curve is also called the spherical Poleni's curve and apart from the elastic circles, it is the only elastica that can be parametrized by elementary functions. In order to do that, we follow the notations of [16]. Choose geographical coordinates in $\mathbb{S}^2\left(\frac{1}{2}\right)$, ϕ for the longitude and θ for the colatitude, and write the natural equations for the Poleni's curve in these coordinates

$$\begin{aligned}
\phi'' + 2\frac{\cos(\theta)}{\sin(\theta)}\,\phi'\theta' &= 2m\,\mathrm{sech}(ms)\,\frac{\theta'}{\sin(\theta)},\\
\theta'' - \sin(\theta)\cos(\theta)\,(\phi')^2 &= -2m\,\mathrm{sech}(ms)\,\phi'\sin(\theta),
\end{aligned}$$

where $m = \sqrt{2(r^2-2)}$. By solving this system, we have that the Poleni's curve can be expressed in geographical coordinates as follows

$$\phi(s) = ms, \qquad \theta(s) = \arccos(\mathrm{sech}(ms)).$$

Thus, in cartesian coordinates, the Poleni's curves in $\mathbb{S}^2\left(\frac{1}{2}\right)$ are given by

$$\gamma_r(s) = (x(s), y(s), z(s)) = \frac{1}{\sqrt{c}}\left(\tanh(ms)\cos(ms), \tanh(ms)\sin(ms), \mathrm{sech}(ms)\right),$$

with $m = \sqrt{2(r^2-2)}$ and $r^2 > 2$.

Consequently, by using the Hopf mapping described previously in (43), we obtain the following parametrization of the Willmore tubes shaped on spherical Poleni's curves (see Fig. 3)

$$\mathrm{X}(s,\tau) = \frac{\sqrt{2}}{2}\left(\sqrt{1+\mathrm{sech}(ms)}\,e^{ims}, \sqrt{1-\mathrm{sech}(ms)}\right)e^{i\tau}, \quad m = \sqrt{2(r^2-2)}.$$

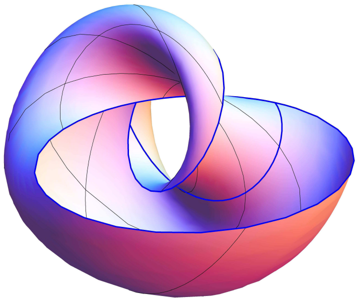

Fig. 3: *Stereographic projection of a piece of a Willmore tube in a Berger sphere built over a spherical Poleni's curve.*

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