Characterization of null curves in Lorentz-Minkowski spaces

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Abstract

We study null curves in Lorentz-Minkowski spaces by analysing the Frenet equations associated to different screen distributions. In particular, by using the theory of Duggal-Bejancu, we obtain the curvature functions of Bonnor, by finding the screen distribution and the null transversal bundle which gives us Bonnor's Frenet equations, [2].

Key words: null curve, Lorentz-Minkowski space, Frenet equations, curvature

MSC 2000: 53C50, 53

1. Introduction

The general theory of curves in a Riemannian manifold M have been developed a long time ago and now we have a deep knowledge of its local geometry as well as its global geometry. When M is a proper semi-Riemannian manifold (that is, the index ν of the metric of M satisfies $1 \leqslant \nu \leqslant \dim(M) - 1$) there exist three families of curves (spacelike, timelike, and null or lightlike curves) depending on their causal characters. It is well-known, [9], that the study of timelike curves has many analogies and similarities with that of spacelike curves. However, the fact that the induced metric on a null curve is degenerate leads to a study much more complicated and also different from the non-degenerate case.

In the geometry of null curves, difficulties arise because the arc length vanishes, so that it is not possible to normalize the tangent vector in the usual way. One method of proceeding is to introduce a new parameter called the pseudo-arc (already used by Vessiot, [13]) which normalize the derivative of the tangent vector. This was the point of view followed by W.B. Bonnor in [2] where he defined two curvatures K_2 and K_3 in terms of the pseudo-arc and a third curvature K_1 which takes only two values, 0 whether the null curve is a straight line, or 1 otherwise (see also the paper by M. Castagnino, [3]).

The importance of the study of null curves and its presence in the physical theories is clear from the fact, see [6], that the classical relativistic string is a surface or world-sheet in Minkowski space which satisfies the Lorentzian analogue of the minimal surface equations. The string equations simplify to the wave equation and a couple of extra simple equations, and by solving the 2-dimensional wave equation it turns out that strings are equivalent to pairs of null curves, or a single null curve in the case of an open string (see also [5], [7], [8], [10] and [12]).

Motivated by the growing importance of null curves in mathematical physics, A. Bejancu initiated in [1] an ambitious program for the general study of the differential geometry of null curves in Lorentz manifolds and, more generally, in semi-Riemannian manifolds. From a complementary vector subbundle to the tangent bundle of a null curve, he obtains the Frenet equations (with respect to a general Frenet frame) and proves the existence and uniqueness theorems for null curves in Lorentz manifolds.

In this note we study null curves in the Lorentz-Minkowskian spaces by analysing the Frenet equations associated to different screen distributions. We reformulate, following the theory of Duggal-Bejancu, the work of Bonnor since we find the screen distribution and the null transversal bundle which generates the Frenet equations introduced by Bonnor in [2].

2. Frenet equations of a null curve in $\mathbb{R}^4_1$

2.1. The Cartan tetrad

Following [4, p. 55] it is easy to see that the Frenet equations of a null curve α in a 4-dimensional Lorentzian manifold write down as follows:

$$\begin{cases} L' = hL + k_1 W_1 \\ N' = -hN + k_2 W_1 + k_3 W_2 \\ W_1' = -k_2 L - k_1 N + k_4 W_2 \\ W_2' = -k_3 L - k_4 W_1 \end{cases} \tag{1}$$

where $L = \alpha'$. If $h = 0$, then the parameter t is said to be a *distinguished parameter*. Moreover, if the last curvature k_4 vanishes, then $\{L, N, W_1, W_2\}$ is called a *distinguished Frenet frame*.

Let $\alpha : I \to \mathbb{R}^4_1$ be a null curve and choose a parameter t such that α' is arc-length parametrized, that is,

$$\langle \alpha'', \alpha'' \rangle = 1.$$

Then t is called the pseudo-arc parameter. Hence, the curve $\gamma = \alpha'$ is a unit spacelike curve. Now we are going to find a Frenet frame $\{L, N, W_1, W_2\}$ with $h = 0$ (showing that t is a distinguished parameter) and $k_1 = 1$.

Let us denote $L = \alpha'$. Then we can choose W_1 as the vector $W_1 = \alpha''$. It is not difficult to show that

$$\langle \alpha', \alpha'' \rangle = 0, \qquad \langle \alpha', \alpha''' \rangle = -1,$$
$$\langle \alpha'', \alpha''' \rangle = 0, \qquad \langle \alpha', \alpha'''' \rangle = 0.$$

Now we take

$$N = -\alpha''' - \frac{1}{2} \langle \alpha''', \alpha''' \rangle \alpha'.$$

Taking into account the Frenet equations we can calculate the second curvature as follows:

$$\begin{aligned} k_2 &= \left\langle \frac{DN}{dt}, W_1 \right\rangle \\ &= \left\langle -\alpha'''' - \langle \alpha'''', \alpha''' \rangle \alpha' - \frac{1}{2} \langle \alpha''', \alpha''' \rangle \alpha'', \alpha'' \right\rangle \\ &= \frac{1}{2} \langle \alpha''', \alpha''' \rangle . \end{aligned}$$

As for the third curvature k_3 and the second vector field of the screen distribution W_2 we have:

$$k_3 W_2 = \frac{DN}{dt} - k_2 W_1 = -\alpha'''' - \langle \alpha'''', \alpha''' \rangle \alpha' - \langle \alpha''', \alpha''' \rangle \alpha''.$$

Then an easy computation shows:

$$\begin{aligned} k_3 &= \sqrt{\langle \alpha'''', \alpha'''' \rangle - \langle \alpha''', \alpha''' \rangle^2}, \\ W_2 &= -\frac{1}{k_3}\left(\alpha'''' + \langle \alpha''', \alpha''' \rangle \alpha'' + \langle \alpha'''', \alpha''' \rangle \alpha'\right). \end{aligned}$$

Finally, a direct computation leads to

$$k_4 = \left\langle \frac{DW_1}{dt}, W_2 \right\rangle = \langle \alpha''', W_2 \rangle = 0.$$

Then we have shown the following

Proposition 2.1 *Let $\alpha : I \to \mathbb{R}_1^4$ be a null curve parametrized by the pseudo-arc. Then the curvature functions with respect to the screen vector bundle generated by*

$$\begin{cases} W_1 = \alpha'', \\ W_2 = -\frac{1}{k_3}\left(\alpha'''' + \langle \alpha''', \alpha''' \rangle \alpha'' + \langle \alpha'''', \alpha''' \rangle \alpha'\right), \end{cases}$$

are given by

$$\begin{aligned} k_1 &= 1, & k_2 &= \frac{1}{2}\|\alpha'''\|^2, \\ k_3 &= \sqrt{\langle \alpha'''', \alpha'''' \rangle - \langle \alpha''', \alpha''' \rangle^2}, & k_4 &= 0. \end{aligned}$$

The Frenet frame $\{L, N, W_1, W_2\}$ agrees with the Cartan tetrad introduced by Castagnino ([3]) and Bonnor ([2]).

Let us assume that $k_2 \neq 0$. Since $\gamma = \alpha'$ is a spacelike curve in $\mathbb{R}_1^4$, we can consider its proper Frenet frame $\{\ell, n, b_1, b_2\}$ and the corresponding curvature functions $\{\bar{k}_1, \bar{k}_2, \bar{k}_3\}$, related by the following Frenet equations:

$$\begin{cases} \frac{D\ell}{dt} = \varepsilon_1 \bar{k}_1 n \\ \frac{Dn}{dt} = -\bar{k}_1 \ell + \varepsilon_2 \bar{k}_2 b_1 \\ \frac{Db_1}{dt} = -\varepsilon_1 \bar{k}_2 n + \varepsilon_3 \bar{k}_3 b_2 \\ \frac{Db_2}{dt} = -\varepsilon_2 \bar{k}_3 b_1 \end{cases}$$

Then the following result can be easily obtained

Corollary 2.2 *Let $\alpha : I \to \mathbb{R}_1^4$ be a null curve parametrized by the pseudo-arc parameter with $k_2 \neq 0$. Then the following relations holds:*

$$\begin{aligned} k_2 &= \frac{1}{2}\varepsilon_1 \bar{k}_1^2, & k_3 &= \sqrt{\varepsilon_1 \bar{k}'^2_1 + \varepsilon_2 (\bar{k}_1 \bar{k}_2)^2}, \\ W_1 &= \ell, & W_2 &= \frac{1}{\sqrt{\varepsilon_1 \bar{k}'^2_1 + \varepsilon_2 (\bar{k}_1 \bar{k}_2)^2}}(-\varepsilon_1 \bar{k}'_1 n - \varepsilon_1 \varepsilon_2 \bar{k}_1 \bar{k}_2 b_1 - \bar{k}_1 \bar{k}'_1 \alpha'). \end{aligned}$$

2.2. Null curves from a space curve in $\mathbb{R}^3$

Let us split $\mathbb{R}^4_1$ as the orthogonal product $\mathbb{R}^4_1 = \mathbb{R}^1_1 \perp \mathbb{R}^3$ and let $P : \mathbb{R}^4_1 \to \mathbb{R}^3$ be the natural projection. Let $\gamma = P\alpha$ be the immersed curve in $\mathbb{R}^3$ obtained by projecting down α. Without loss of generality, let us assume that γ is arclength parametrized, so that we can write down

$$\alpha = (t, \gamma).$$

Now we are going to define a Frenet frame $\{L, N, W_1, W_2\}$ such that the curvature functions of α in this Frenet frame can be obtained from the curvature and torsion of the curve γ. Let $\{\ell_\gamma, n_\gamma, b_\gamma\}$ be the Frenet frame of γ satisfying

$$\left\{ \begin{array}{l} \frac{D\ell_\gamma}{dt} = k_\gamma n_\gamma \\ \frac{Dn_\gamma}{dt} = -k_\gamma \ell_\gamma + \tau_\gamma b_\gamma \\ \frac{Db_\gamma}{dt} = -\tau_\gamma n_\gamma. \end{array} \right.$$

Let us consider the transversal vector bundle field N given by

$$N = \left(-\frac{\tau_\gamma^2 + k_\gamma^2}{2k_\gamma^2}, \frac{k_\gamma^2 - \tau_\gamma^2}{2k_\gamma^2} \ell_\gamma + \frac{\tau_\gamma}{k_\gamma} b_\gamma \right).$$

Then span $\left\{ L = \frac{d}{dt}, N \right\}$ is a Lorentzian plane and so we can find the screen vector bundle as its complementary vector subbundle. By similar computations to those in the last section, we get the following result

Proposition 2.3 *Let $\alpha : I \to \mathbb{R}^4_1$ be a null curve with parameter t such that $P\alpha = \gamma$ is arclength parametrized. Let k_γ and τ_γ denote the curvature and torsion of γ, respectively. Then the curvature functions of α with respect to the screen vector bundle spanned by the sections*

$$\left\{ \begin{array}{l} W_1 = (0, n_\gamma), \\ W_2 = \left(-\frac{\tau_\gamma}{k_\gamma}, \frac{\tau_\gamma}{k_\gamma} \ell_\gamma + b_\gamma \right) \end{array} \right.$$

satisfy the following equalities:

$$k_1 = k_\gamma, \qquad k_2 = \frac{k_\gamma^2 + \tau_\gamma^2}{2k_\gamma},$$
$$k_3 = \frac{\tau_\gamma' k_\gamma - k_\gamma' \tau_\gamma}{k_\gamma^2}, \qquad k_4 = 0.$$

As a consequence we have

Corollary 2.4 *Let $\alpha : I \to \mathbb{R}^4_1$ be a null curve with curvature functions k_1, k_2 and k_3 with respect to the screen vector bundle given in the above proposition. The following statements are equivalent:*
(a) $\gamma = P\alpha$ is a generalized helix (that is, $\tau_\gamma = rk_\gamma$, r being a constant).
(b) α lies in a Lorentzian hyperplane.

As in non-degenerate case, we may consider the notion of null helices.

Definition 2.5 *A curve $\alpha : I \to \mathbb{R}^3_1$ is said to be a null helix (with respect to the screen distribution given in Proposition 2.3) if k_2 and k_3 are both constants.*

Proposition 2.6 *If $\alpha : I \to \mathbb{R}^4_1$ is a null helix, then the following relations hold:*

$$k_\gamma = \frac{2c_1}{1 + (c_2 t + c_3)^2} \quad \textit{and} \quad \tau_\gamma = \frac{2c_1(c_2 t + c_3)}{1 + (c_2 t + c_3)},$$

where c_1, c_2 and c_3 are constants.

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