NULL HELICES AND DEGENERATE CURVES IN LORENTZIAN SPACES

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Dedicated to A. M. Naveira

In this paper we introduce a reference along a degenerate curve (null or non null) in an n-dimensional Lorentzian space with the minimun number of curvatures. That reference is called the Cartan frame of the curve. In the Lorentzian space forms case, we obtain a complete classification of helices (that is, curves with constant Cartan curvatures) in low dimensions. In all cases we present existence, uniqueness and congruence theorems.

1 Introduction

In a proper semi-Riemannian manifold there exist three families of curves depending on their causal characters. It is well-known[1] that the study of timelike curves has many analogies and similarities with that of spacelike curves. However, the fact that the induced metric on a null curve is degenerate leads to a much more complicated study and also different from the non-degenerate case. Even more, a timelike or spacelike curve can have a null higher order derivative and then its study is also different from that of Riemannian case.

In the geometry of null curves difficulties arise because the arc length vanishes, so that it is not possible to normalize the tangent vector in the usual way. A solution is to introduce the pseudo-arc parameter (already used by Vessiot[2]) which normalizes the derivative of the tangent vector (see the papers by W.B. Bonnor[3] and M. Castagnino[4]).

The importance of the study of null curves and its presence in the physic theories is clear[5,6,7,8,9]. Recently, Nersessian and Ramos[10] show that there exists a geometrical particle model based enterely on the geometry of the null curves in Minkowskian 4-dimensional spacetime which under quantization yields the wave equations corresponding to massive spinning particles of arbitrary spin. The same authors[11] study the simplest geometrical particle model which is associated with null curves in 3-dimensional Lorentz-Minkowski space.

Motivated by the growing importance of null curves in mathematical physics, A. Bejancu[12] initiated an ambitious program for the general study of the differential geometry of null curves in Lorentzian manifolds and, more generally, in semi-Riemannian manifolds (see also his book[13]).

This paper is organized as follows. In Section 2 we introduce the Cartan frame and the Cartan curvatures of a null curve and compute the ordinary differential equation that helices satisfy. In the next section we obtain the existence and uniqueness theorems relative to Cartan curves (see Theorems 2 and 3) and determine the families of null helices in $\mathbb{R}_1^5$, $\mathbb{S}_1^4$ and $\mathbb{H}_1^4$ (Theorems 4, 5 and 6). In Section 4 we introduce the s-degenerate curves as an extension of the null curves (that are 1-degenerate curves) and, in Section 5, we present existence and congruence theorems for that kind of curves. Finally, in the last section we classify 2-degenerate Cartan helices in 4-dimensional Lorentzian space forms.

The results of this paper can be extended to degenerate curves in pseudo-Euclidean spaces of index two[14] and following the same ideas they can be also stated in $\mathbb{S}_2^n$ and $\mathbb{H}_2^n$. With some extra effort they can be extended to pseudo-Riemannian space forms of higher indices.

2 The Cartan frame of a null curve

Let M_1^n be an orientable Lorentzian manifold and consider C a null curve locally parametrized by $\gamma : I \subset \mathbb{R} \to M_1^n$. Assume that $\{\gamma', \gamma'', \ldots, \gamma^{(n)}\}$ is a linearly independent family and define $E_i = \operatorname{span}\{\gamma', \gamma'', \ldots, \gamma^{(i)}\}$, $i = 1, \ldots, n$. Let $L \in E_1$, so that $\gamma' = \bar{k}_1 L$, for a certain function $\bar{k}_1$. Since $E_2 = E_1 \oplus \operatorname{span}\{\gamma''\}$ we can choose a unit spacelike vector W_1 satisfying $E_2 = \operatorname{span}\{L, W_1\}$. Now, since $E_3 = E_2 \oplus \operatorname{span}\{\gamma^{(3)}\}$ we obtain that E_3 is a Lorentzian subspace of E_n, then there exists only one null vector N such that $\langle L, N \rangle = \varepsilon = \pm 1$, $\langle W_1, N \rangle = 0$ and $E_3 = \operatorname{span}\{L, W_1, N\}$. In general, for $i = 2, \ldots, n-3$, we can find orthonormal spacelike vectors $\{W_1, \ldots, W_i\}$ such that $E_{i+2} = \operatorname{span}\{L, W_1, N, \ldots, W_i\}$ and the basis $\{\gamma', \gamma'', \ldots, \gamma^{(i+2)}\}$ and $\{L, W_1, N, \ldots, W_i\}$ have the same orientation. Finally, the vector W_m, $m = n - 2$, is chosen in order that the basis $\{L, W_1, N, \ldots, W_m\}$ is positively oriented. An easy computation shows that there exist functions $\{\bar{k}_1, \ldots, \bar{k}_{m+3}\}$ such that the following equations hold (compare with Ref.

12)

$$\gamma' = \bar{k}_1 L,$$
$$L' = \varepsilon \bar{k}_2 L + \bar{k}_3 W_1,$$
$$W_1' = \varepsilon \bar{k}_4 L - \varepsilon \bar{k}_3 N,$$
$$N' = -\varepsilon \bar{k}_2 N - \bar{k}_4 W_1 + \bar{k}_5 W_2, \tag{1}$$
$$W_2' = -\varepsilon \bar{k}_5 L + \bar{k}_6 W_3,$$
$$W_i' = -\bar{k}_{i+3} W_{i-1} + \bar{k}_{i+4} W_{i+1}, \quad i = 3, \ldots, m-1$$
$$W_m' = -\bar{k}_{m+3} W_{m-1}.$$

Without loss of generality we may assume that γ is parametrized by the pseudo-arc parameter, that is, $\langle \gamma'', \gamma'' \rangle = 1$. Now choose $L = \gamma'$ and $W_1 = \gamma''$, so that $\bar{k}_1 = 1$, $\bar{k}_2 = 0$ and $\bar{k}_3 = 1$. Let us take N given by $N = -\varepsilon \gamma^{(3)} - \varepsilon \frac{1}{2} \langle \gamma^{(3)}, \gamma^{(3)} \rangle \gamma'$, then the forth curvature is given by $\bar{k}_4 = -\langle N', W_1 \rangle = \varepsilon \frac{1}{2} \langle \gamma^{(3)}, \gamma^{(3)} \rangle$. After a direct computation, where the curvature functions are renamed ($k_1 = \bar{k}_4$, $k_2 = \bar{k}_5$ and so on), we can show the following theorem.

Theorem 1 ([15]) *Let $\gamma : I \to M_1^n$, $n = m + 2$, be a null curve parametrized by the pseudo-arc such that $\{\gamma'(t), \gamma''(t), \ldots, \gamma^{(n)}(t)\}$ is a basis of $T_{\gamma(t)} M_1^n$ for all t. Then there exists only one Frenet frame satisfying the equations*

$$L' = W_1,$$
$$W_1' = \varepsilon k_1 L - \varepsilon N,$$
$$N' = -k_1 W_1 + k_2 W_2,$$
$$W_2' = -\varepsilon k_2 L + k_3 W_3, \tag{2}$$
$$W_i' = -k_i W_{i-1} + k_{i+1} W_{i+1} \quad i = 3, \ldots, m-1,$$
$$W_m' = -k_m W_{m-1},$$

and verifying

i) *For $1 \leqslant i \leqslant m-1$, $\{\gamma', \gamma'', \ldots, \gamma^{(i)}\}$ and $\{L, W_1, N, \ldots, W_{i-2}\}$ have the same orientation.*

ii) *$\{L, W_1, N, \ldots, W_m\}$ is positively oriented.*

Observe that when $m > 1$ then $\varepsilon = -1$ and $k_i \geq 0$ for $i \geq 2$; however, when $m = 1$ then $\varepsilon = -1$ or $\varepsilon = 1$ according to $\{\gamma', \gamma'', \gamma'''\}$ is positively or negatively oriented, respectively.

Definition 1 A null curve in M_1^n satisfying the conditions of the above theorem is called a *Cartan curve*. The above frame $\{L, W_1, N, \ldots, W_m\}$ and curvatures $\{k_1, k_2, \ldots, k_m\}$ are called the *Cartan frame* and the *Cartan curvatures*, respectively, of the curve γ.

It is not difficult to show that the Cartan curvatures of a curve C in M_1^n are invariant under Lorentzian transformations.

Definition 2 A curve is said to be a *helix* if it has constant Cartan curvatures.

A long and messy computation shows that if γ is a helix then it satisfies the following differential equation

$$\gamma^{(n+1)} = a_1\gamma' + a_2\gamma^{(3)} + \cdots + a_s\gamma^{(n-1)}, \quad \text{if } n \text{ is even, } n = 2s,$$

and

$$\gamma^{(n+1)} = a_1\gamma'' + a_2\gamma^{(4)} + \cdots + a_s\gamma^{(n-1)}, \quad \text{if } n \text{ is odd, } n = 2s + 1,$$

where the coefficients can be easily computed (see Ref. 15).

3   Null curves in $\mathbb{M}_1^n(c)$

In this section $\mathbb{M}_1^n(c)$ stands for $\mathbb{R}_1^n$, $\mathbb{S}_1^n(c)$ or $\mathbb{H}_1^n(c)$, according to $c = 0$, $c > 0$ or $c < 0$, respectively. Our first goal is to prove the following theorem.

Theorem 2 *Let $k_1, k_2, \ldots, k_m : [-\delta, \delta] \to \mathbb{R}$ be differentiable functions with $k_2 < 0$ and $k_i > 0$ for $i = 3, \ldots, m-1$. Let p be a point in $\mathbb{M}_1^n$, $n = m + 2$, and consider $\{L^0, W_1^0, N^0, \ldots, W_m^0\}$ a positively oriented pseudo-orthonormal basis of $T_p\mathbb{M}_1^n$. Then there exists a unique Cartan curve γ in $\mathbb{M}_1^n$, with $\gamma(0) = p$, whose Cartan frame $\{L, W_1, N, \ldots, W_m\}$ satisfies*

$$L(0) = L^0, N(0) = N^0, W_i(0) = W_i^0, \quad i = 1, \ldots, m.$$

Proof. Let us suppose $\mathbb{M}_1^n = \mathbb{R}_1^n$ (the remaining cases are similar). According to the general theory of differential equations, there exists a unique solution $\{L, W_1, N, \ldots, W_m\}$ of (2), defined on an interval $[-\delta, \delta]$, and satisfying the initial conditions of the theorem. A straightforward computation, bearing in mind (2), leads to

$$\frac{d}{dt}\left(\varepsilon L_i N_j + \varepsilon L_j N_i + \sum_{\alpha=1}^m W_{\alpha i} W_{\alpha j}\right) = 0, \quad i, j \in \{1, \ldots, n\}.$$

Since $\{L(0), W_1(0), N(0), \ldots, W_m(0)\}$ is a pseudo-orthonormal basis, then the above equation jointly with Lemma 6 of Ref. 15 implies

$$\varepsilon L_i(t) N_j(t) + \varepsilon L_j(t) N_i(t) + \sum_{\alpha=1}^{m} W_{\alpha i}(t) W_{\alpha j}(t) = \eta_{ij}, \quad t \in [-\delta, \delta].$$

Then by using again the same lemma we deduce that $\{L, W_1, N, \ldots, W_m\}$ is a pseudo-orthonormal basis for all t, and this concludes the proof.

The following result shows that the Cartan curvatures determine curves satisfying the nondegeneracy conditions stated in Theorem 1.

Theorem 3 *If two Cartan curves C and $\bar{C}$ in $\mathbb{M}_1^n$ have Cartan curvatures $\{k_1, \ldots, k_n\}$, where $k_i : [-\delta, \delta] \to \mathbb{R}$ are differentiable functions, then there exists a Lorentzian transformation of $\mathbb{M}_1^n$ which maps C into $\bar{C}$.*

3.1 Null helices in $\mathbb{R}_1^5$

The goal of this section is to classify the family of null helices in $\mathbb{R}_1^5$. Before to do that, we present some examples. From the general equation, we know that a null helix in $\mathbb{R}_1^5$ satisfies the following differential equation

$$\gamma^{(6)} + (2k_1 + k_3^2)\gamma^{(4)} - (k_2^2 - 2k_1 k_3^2)\gamma'' = 0,$$

which will help us to find the examples.

Example 1 Let ω, σ and h be three non-zero constants such that $\frac{1}{\sigma^2} < h^2 < \frac{1}{\omega^2}$ and let $\gamma : \mathbb{R} \to \mathbb{R}_1^5$ be the curve defined by

$$\gamma(t) = \left(ht, \frac{1}{\omega} a \sin \omega t, \frac{1}{\omega} a \cos \omega t, \frac{1}{\sigma} b \sin \sigma t, \frac{1}{\sigma} b \cos \sigma t \right),$$

where $a = \sqrt{h^2 \sigma^2 - 1}/\sqrt{\sigma^2 - \omega^2}$ and $b = \sqrt{1 - h^2 \omega^2}/\sqrt{\sigma^2 - \omega^2}$. Then it is easy to see that γ is a helix with curvatures $k_1 = \frac{1}{2}(\sigma^2 + \omega^2(1 - \sigma^2 h^2))$, $k_2^2 = -\omega^2 \sigma^2 (\omega^2 h^2 - 1)(\sigma^2 h^2 - 1)$ and $k_3^2 = \omega^2 \sigma^2 h^2$.

Example 2 Let ω, σ and h be three non-zero constants such that $0 < h^2 \omega^2 < 1$ and let $\gamma : \mathbb{R} \to \mathbb{R}_1^5$ be the curve defined by

$$\gamma(t) = \left(\frac{1}{\omega} a \sinh \omega t, \frac{1}{\omega} a \cosh \omega t, \frac{1}{\sigma} b \sin \sigma t, \frac{1}{\sigma} b \cos \sigma t, ht \right),$$

where $a = \sqrt{1 + h^2 \sigma^2}/\sqrt{\omega^2 + \sigma^2}$ and $b = \sqrt{1 - h^2 \omega^2}/\sqrt{\omega^2 + \sigma^2}$. Then γ is a helix with curvatures $k_1 = \frac{1}{2}(\sigma^2 - \omega^2(1 + \sigma^2 h^2))$, $k_2^2 = -\omega^2 \sigma^2(\omega^2 h^2 - 1)(\sigma^2 h^2 + 1)$ and $k_3^2 = \omega^2 \sigma^2 h^2$.

Example 3 Let σ and h be two non-zero constants such that $0 < 2h^2 < 1$ and let $\gamma : \mathbb{R} \to \mathbb{R}_1^5$ be the curve defined by

$$\gamma(t) = \left(\frac{3}{2}at + \frac{1}{6}h^2 t + t^3, \frac{h}{\sqrt{2}}t^2, \frac{3}{2}at - \frac{1}{6}h^2 t + t^3, b\sin\sigma t, b\cos\sigma t \right),$$

where $a = (1 - 2h^2)/(\sigma^2 h^2)$ and $b = \sqrt{1 - 2h^2}/\sigma^2$. Then γ is a helix with curvatures $k_1 = \frac{1}{2}\sigma^2(1 - 2h^2)$, $k_2^2 = 2\sigma^4 h^2(1 - 2h^2)$ and $k_3^2 = 2\sigma^2 h^2$.

Theorem 4 (^{15}) *A null curve fully immersed in $\mathbb{R}_1^5$ is a helix if and only if it is congruent to a helix of the families described in Examples 1-3.*

3.2 Null helices in $\mathbb{S}_1^4$

In this section we are going to classify the null helices in the 4-dimensional De Sitter space. A null curve γ in $\mathbb{S}_1^4 \subset \mathbb{R}_1^5$ is a helix if and only if it satisfies the differential equation $\gamma^{(5)} + 2k_1 \gamma^{(3)} - (1 + k_2^2)\gamma' = 0$, whose general solution is $\gamma(t) = A_1 \sinh \omega t + A_2 \cosh \omega t + A_3 \sin \sigma t + A_4 \cos \sigma t + A_5$, where A_1, A_2, A_3, A_4 and A_5 are constant vectors in $\mathbb{R}_1^5$.

Example 4 Let $\gamma : \mathbb{R} \to \mathbb{S}_1^4 \subset \mathbb{R}_1^5$ be the null curve defined by

$$\gamma(t) = \frac{1}{\sqrt{\omega^2 + \sigma^2}} \left(\frac{1}{\omega}\sinh\omega t, \frac{1}{\omega}\cosh\omega t, \frac{1}{\sigma}\sin\sigma t, \frac{1}{\sigma}\cos\sigma t, \sqrt{\frac{\omega^4 - 1}{\omega^2} + \frac{\sigma^4 - 1}{\sigma^2}} \right)$$

where $0 < \omega^2 \sigma^2 > 1$. A direct computation shows that γ is a null helix with curvatures $k_1 = \frac{1}{2}(\sigma^2 - \omega^2)$ and $k_2^2 = \omega^2 \sigma^2 - 1$.

Theorem 5 (^{15}) *A null curve fully immersed in $\mathbb{S}_1^4$ is a helix if and only if it is congruent to one of the family described in Example 4.*

3.3 Null helices in $\mathbb{H}_1^4$

Let γ be a null curve in $\mathbb{H}_1^4$, then it is a helix if and only if it verifies the ordinary differential equation $\gamma^{(5)} + 2k_1 \gamma^{(3)} + (1 - k_2^2)\gamma' = 0$. Before we state the main result of this section we present some examples of helices in the 4-dimensional anti De Sitter space.

Example 5 Let $0 < \omega^2 < 1$ and let γ be the curve in $\mathbb{H}_1^4$ defined by

$$\gamma(t) = \left(\frac{t}{2\omega} \cosh \omega t, \frac{1}{\omega^2} \left(\cosh \omega t - \frac{1}{2} \omega t \sinh \omega t \right), \frac{1}{\omega^2} \left(\sinh \omega t - \frac{1}{2} \omega t \cosh \omega t \right), \right.$$
$$\left. \frac{t}{2\omega} \sinh \omega t, \frac{\sqrt{1 - \omega^4}}{\omega^2} \right).$$

Then γ is a helix with curvatures $k_1 = -\omega^2$ and $k_2^2 = 1 - \omega^4$.

Example 6 Let $0 < \sigma^2 < 1$ and let γ be the curve in $\mathbb{H}_1^4$ defined by

$$\gamma(t) = \left(\frac{1}{\sigma^2} \left(\sin \sigma t - \frac{1}{2} \sigma t \cos \sigma t \right), \frac{1}{\sigma^2} \left(\cos \sigma t + \frac{1}{2} \sigma t \sin \sigma t \right), -\frac{t}{2\sigma} \cos \sigma t, \right.$$
$$\left. \frac{t}{2\sigma} \sin \sigma t, \frac{\sqrt{1 - \sigma^4}}{\sigma^2} \right).$$

Then γ is a helix with curvatures $k_1 = \sigma^2$ and $k_2^2 = 1 - \sigma^4$.

Example 7 Let $\omega^2 = 1$ and let γ be the null curve in $\mathbb{H}_1^4$ defined by

$$\gamma(t) = \left(1 - \frac{t^4}{24}, \frac{\omega(t^3 + t)}{2\sqrt{3}}, \frac{t^4}{24}, \frac{\omega(t^3 - t)}{2\sqrt{3}}, \frac{t^2}{2} \right).$$

Then γ is a helix with curvatures $k_1 = 0$ and $k_2^2 = 1$. γ will be called the *null quartic* in $\mathbb{H}_1^4$.

Example 8 Let $0 < \omega^2 < \sigma^2$ and $\omega^2 \sigma^2 < 1$, and let γ be the curve in $\mathbb{H}_1^4$ defined by

$$\gamma(t) = \frac{1}{\sqrt{\sigma^2 - \omega^2}} \left(\frac{1}{\omega} \sin \omega t, \frac{1}{\omega} \cos \omega t, \frac{1}{\sigma} \sin \sigma t, \frac{1}{\sigma} \cos \sigma t, \sqrt{\frac{1 + \omega^4}{\omega^2} - \frac{1 + \sigma^4}{\sigma^2}} \right).$$

Then γ is a helix with curvatures $k_1 = \frac{1}{2}(\omega^2 + \sigma^2)$ and $k_2^2 = 1 - \omega^2 \sigma^2$.

Example 9 Let $0 < \sigma^2 < \omega^2$ and $\omega^2 \sigma^2 < 1$, and let γ be the curve in $\mathbb{H}_1^4$ defined by

$$\gamma(t) = \frac{1}{\sqrt{\omega^2 - \sigma^2}} \left(\frac{1}{\omega} \sinh \omega t, \frac{1}{\sigma} \cosh \sigma t, \frac{1}{\sigma} \sinh \sigma t, \frac{1}{\omega} \cosh \omega t, \sqrt{\frac{1 + \sigma^4}{\sigma^2} - \frac{1 + \omega^4}{\omega^2}} \right).$$

Then γ is a helix with curvatures $k_1 = -\frac{1}{2}(\omega^2 + \sigma^2)$ and $k_2^2 = 1 - \omega^2 \sigma^2$.

Example 10 Let $\sigma \neq 0$ and let γ be the curve in $\mathbb{H}^4_1$ defined by

$$\gamma(t) = \left(\frac{2 + 2\sigma^4 - \sigma^2 t^2}{2\sigma^2 \sqrt{1 + \sigma^4}}, \frac{t}{\sigma}, \frac{t^2}{2\sqrt{1 + \sigma^4}}, \frac{1}{\sigma^2} \sin \sigma t, \frac{1}{\sigma^2} \cos \sigma t \right).$$

Then γ is a helix with curvatures $k_1 = \sigma^2/2$ and $k_2^2 = 1$.

Example 11 Let $\omega \neq 0$ and let γ be the curve in $\mathbb{H}^4_1$ defined by

$$\gamma(t) = \left(\frac{2 + 2\omega^4 + \omega^2 t^2}{2\omega^2 \sqrt{1 + \omega^4}}, \frac{1}{\omega^2} \sinh \omega t, \frac{1}{\omega^2} \cosh \omega t, \frac{t^2}{2\sqrt{1 + \omega^4}}, \frac{t}{\omega} \right).$$

Then γ is a helix with curvatures $k_1 = -\omega^2/2$ and $k_2^2 = 1$.

Example 12 Let $\omega \sigma \neq 0$ and let γ be the curve in $\mathbb{H}^4_1$ defined by

$$\gamma(t) = \frac{1}{\sqrt{\omega^2 + \sigma^2}} \left(\sqrt{\frac{1 + \omega^4}{\omega^2} + \frac{1 + \sigma^4}{\sigma^2}}, \frac{1}{\omega} \sinh \omega t, \frac{1}{\omega} \cosh \omega t, \frac{1}{\sigma} \sin \sigma t, \frac{1}{\sigma} \cos \sigma t \right)$$

Then γ is a helix with curvatures $k_1 = \frac{1}{2}(\sigma^2 - \omega^2)$ and $k_2^2 = 1 + \omega^2 \sigma^2$.

Example 13 Let $\omega^2 + \sigma^2 < 1$ and let γ be the curve in $\mathbb{H}^4_1$ defined by

$$\gamma(t) = \frac{1}{2\omega\sigma \left(\omega^2 + \sigma^2 \right)} \left(2\omega\sigma \cosh \omega t \sin \sigma t + \left(\omega^2 - \sigma^2 \right) \sinh \omega t \cos \sigma t, \right.$$
$$-2\omega\sigma \cosh \omega t \cos \sigma t + \left(\omega^2 - \sigma^2 \right) \sinh \omega t \sin \sigma t, \left(\omega^2 + \sigma^2 \right) \sinh \omega t \cos \sigma t,$$
$$\left. \left(\omega^2 + \sigma^2 \right) \sinh \omega t \sin \sigma t, 2\omega\sigma \sqrt{1 - (\omega^2 + \sigma^2)^2} \right).$$

Then γ is a helix with curvatures $k_1 = -\omega^2 + \sigma^2$ and $k_2^2 = 1 - (\omega^2 + \sigma^2)^2$.

Theorem 6 (^{15}) *A null curve fully immersed in $\mathbb{H}^4_1$ is a helix if and only if it is congruent to one helix of the families described in Examples 5–13.*

4   The Cartan frame for s-degenerate curves

Let γ be a differentiable curve in M^n_1, write $E_i(t) = \text{span} \left\{ \gamma'(t), \gamma''(t), \ldots, \gamma^{(i)}(t) \right\}$ and let d be the number defined by $d = \max \left\{ i : \dim E_i(t) = i \text{ for all } t \right\}$. The curve γ is said to be an *s-degenerate (or s-lightlike) curve* if for all $1 \leq i \leq d$ $\dim \text{Rad}(E_i(t))$ is constant for all t, and there exists s, $0 < s \leq d$, such that $\text{Rad}(E_s) \neq \{0\}$ and $\text{Rad}(E_j) = \{0\}$ for all $j < s$.

Note that 1-degenerate curves are precisely the null curves studied in preceding sections. In this section we will focus on s-degenerate curves $(s > 1)$ in Lorentzian spaces. Notice that they must be spacelike curves.

To find the Frenet frames, we will distinguish four cases:

Case 1: $d = n$ **and** $s \leq d$. One can show that there exist a set $\mathcal{F} = \{W_1, \ldots, W_{s-1}, L, W_s, N, W_{s+1}, \ldots, W_m\}$ such that the following equations hold

$$
\begin{aligned}
\gamma' &= \bar{k}_1 W_1, \\
W_1' &= \bar{k}_2 W_2, \\
W_i' &= -\bar{k}_i W_{i-1} + \bar{k}_{i+1} W_{i+1}, \quad 2 \leq i \leq s-2, \\
W_{s-1}' &= -\bar{k}_{s-1} W_{s-2} + \varepsilon \bar{k}_s L, \\
L' &= \varepsilon \bar{k}_{s+1} L + \bar{k}_{s+2} W_s, \\
W_s' &= \varepsilon \bar{k}_{s+3} L - \varepsilon \bar{k}_{s+2} N, \\
N' &= -\bar{k}_s W_{s-1} - \varepsilon \bar{k}_{s+1} N - \bar{k}_{s+3} W_s + \bar{k}_{s+4} W_{s+1}, \\
W_{s+1}' &= -\varepsilon \bar{k}_{s+4} L + \bar{k}_{s+5} W_{s+2}, \\
W_j' &= -\bar{k}_{j+3} W_{j-1} + \bar{k}_{j+4} W_{j+1}, \quad s+2 \leq j \leq m-1, \\
W_m' &= -\bar{k}_{m+3} W_{m-1},
\end{aligned}
$$

for certain functions $\{\bar{k}_1, \ldots, \bar{k}_{m+3}\}$ called the *curvature functions* of γ with respect to $\mathcal{F}$.

Case 2: $d < n$ **and** $s = d$. If M_1^n is a Lorentzian space form, then γ lies in a d-dimensional totally geodesic lightlike submanifold. This can be proved by adapting the proofs of Theorems 5 and 9 of Chapter 7 in Ref. 17. This case will be treated in a forthcoming paper[16].

Case 3. $d < n$ **and** $s = d - 1$. This case can not occur.

Case 4: $d < n$ **and** $s < d - 1$. Working as in the non-degenerate case (see, for example, the book of Spivak[17]) this case reduces to Case 1.

Note that the type s does not depend on the parameter of the curve. Also, this kind of curves are invariant under Lorentzian transformations, in the sense that the type s does not change under a Lorentzian transformation.

Now we are going to find a Frenet frame with the minimal number of curvatures and such that they are invariant under Lorentzian transformations. We will restrict to Case 1. Without loss of generality, let us assume that γ is arc-length parametrized, so that $W_1 = \gamma'$ and $\bar{k}_1 = 1$. By taking $\bar{k}_s = 1$, a

straightforward computation shows that

$$
\begin{aligned}
\gamma' &= W_1, \\
W_1' &= k_1 W_2, \\
W_i' &= -k_{i-1} W_{i-1} + k_i W_{i+1}, \quad 2 \le i \le s-2, \\
W_{s-1}' &= -k_{s-2} W_{s-2} + L, \\
L' &= k_{s-1} W_s, \\
W_s' &= \varepsilon k_s L - \varepsilon k_{s-1} N, \\
N' &= -\varepsilon W_{s-1} - k_s W_s + k_{s+1} W_{s+1}, \\
W_{s+1}' &= -\varepsilon k_{s+1} L + k_{s+2} W_{s+2}, \\
W_j' &= -k_j W_{j-1} + k_{j+1} W_{j+1}, \quad s+2 \le j \le m-1, \\
W_m' &= -k_m W_{m-1},
\end{aligned}
\tag{3}
$$

for certain functions $\{k_1, \dots, k_m\}$. We can easily deduce the following result.

Observe that when $m > s$ then $\varepsilon = -1$ and $k_i \ge 0$ for $i \ne s$; however, when $m = s$ then $\varepsilon = -1$ or $\varepsilon = 1$ according to $\{\gamma', \gamma'', \dots, \gamma^{(n)}\}$ is positively or negatively oriented, respectively.

Theorem 7 ([18]) *Let $\gamma : I \to M_1^n$, $n = m+2$, be an s-degenerate ($s > 1$) unit spacelike curve and suppose that $\{\gamma'(t), \gamma''(t), \dots, \gamma^{(n)}(t)\}$ spans $T_{\gamma(t)} M_1^n$ for all t. Then there exists a unique Frenet frame satisfying the equations (3).*
In this case γ is said to be an s-degenerate Cartan curve, the reference and curvature functions given by (3) are called the Cartan frame and Cartan curvatures of γ, respectively.

5 s-degenerate curves in Lorentzian space forms

Let $\gamma : I \to \mathbb{M}_1^n(c)$ be an s-degenerate ($s > 1$) Cartan curve, where $\mathbb{M}_1^n(c)$ stands for $\mathbb{R}_1^n$, $\mathbb{S}_1^n$ o $\mathbb{H}_1^n$, according to $c = 0$, $c = 1$ or $c = -1$, respectively. If $\{W_1, \dots, W_{s-1}, L, W_s, N, W_{s+1}, \dots, W_m\}$ is the Cartan frame then in equations (3) we must write $W_1' = k_1 W_2 - c\gamma$.

The following results can be obtained in a similar way as in the null case (for the proofs see Ref. 18).

Theorem 8 *Let $k_1, \dots, k_m : [-\delta, \delta] \to \mathbb{R}$ be differentiable functions with $k_i > 0$ for $i \ne s, m$. Let p be a point in $\mathbb{M}_1^n$, $n = m + 2$, and let $\{W_1^0, \dots, W_{s-1}^0, L^0, W_s^0, N^0, W_{s+1}^0, \dots, W_m^0\}$ be a positively oriented pseudo-orthonormal basis of $T_p \mathbb{M}_1^n(c)$. Then there exists a unique s-degenerate, $s > 1$,*

Cartan curve γ in $\mathbb{M}_1^n(c)$, with $\gamma(0) = p$, whose Cartan frame satisfies:

$$L(0) = L^0, N(0) = N^0, W_i(0) = W_i^0, \quad i \in \{1, \dots, m\}.$$

Theorem 9 (Congruence Theorem) *If two s-degenerate Cartan curves C and $\bar{C}$ in $\mathbb{M}_1^n(c)$ have Cartan curvatures $\{k_1, \dots, k_m\}$, where $k_i : [-\delta, \delta] \to \mathbb{R}$ are differentiable functions, then there exists a Lorentzian transformation of $\mathbb{M}_1^n(c)$ which maps bijectively C into $\bar{C}$.*

6   s-degenerate helices in $\mathbb{M}_1^4(c)$

This section is devoted to classify s-degenerate Cartan helices in Lorentzian space forms $\mathbb{M}_1^4(c)$, $c = -1, 0, 1$. If we assume that k_1 and k_2 are constant, then γ satisfies the differential equation $\gamma^{(5)} - (2\varepsilon k_1 k_2 - c) \gamma^{(3)} - (k_1^2 + 2\varepsilon c k_1 k_2) \gamma' = 0$. Without loss of generality, we can assume that γ is positively oriented, that is, $\varepsilon = -1$.

In what follows, we will present examples of 2-degenerate Cartan helices in $\mathbb{M}_1^4(c)$ and show the corresponding characterization theorems.

6.1   2-degenerate helices in $\mathbb{R}_1^4$

Example 14 Let γ be the curve in $\mathbb{R}_1^4$ defined by

$$\gamma(t) = \frac{1}{\sqrt{\omega^2 + \sigma^2}} \left(\frac{\sigma}{\omega} \cosh \omega t, \frac{\sigma}{\omega} \sinh \omega t, \frac{\omega}{\sigma} \sin \sigma t, \frac{\omega}{\sigma} \cos \sigma t \right), \quad \omega \sigma > 0,$$

Then γ is a helix with curvatures $k_1 = \omega \sigma$ and $k_2 = (\sigma^2 - \omega^2)/(2\omega\sigma)$.

Theorem 10 ([18]) *An s-degenerate spacelike Cartan curve $\mathbb{R}_1^4$ is a helix if and only if it is congruent to a helix of Example 14.*

6.2   2-degenerate helices in $\mathbb{S}_1^4$

Example 15 Let $0 < \sigma^2 < 1 < \omega^2$ and let γ be the curve in $\mathbb{S}_1^4$ defined by

$$\gamma(t) = \left(\sqrt{\frac{(\omega^2 - 1)(1 - \sigma^2)}{\omega^2 \sigma^2}}, \frac{1}{\omega} a \sin \omega t, \frac{1}{\omega} a \cos \omega t, \frac{1}{\sigma} b \sin \sigma t, \frac{1}{\sigma} b \cos \sigma t \right),$$

where $a = \sqrt{1 - \sigma^2}/\sqrt{\omega^2 - \sigma^2}$ and $b = \sqrt{\omega^2 - 1}/\sqrt{\omega^2 - \sigma^2}$. Then γ is a helix with curvatures $k_1 = \sqrt{(\omega^2 - 1)(1 - \sigma^2)}$ and $k_2 = \frac{1}{2}(\omega^2 + \sigma^2 - 1)/\sqrt{(\omega^2 - 1)(1 - \sigma^2)}$.

Example 16 Let $\sigma^2 > 1$ and let γ be the curve in $\mathbb{S}_1^4$ defined by

$$\gamma(t) = \left(\frac{1}{\omega} a \cosh \omega t, \frac{1}{\omega} a \sinh \omega t, \frac{1}{\sigma} b \sin \sigma t, \frac{1}{\sigma} b \cos \sigma t, \frac{1}{\omega\sigma} \sqrt{(\omega^2+1)(\sigma^2-1)} \right),$$

where $\omega \neq 0$, $a = \sqrt{\sigma^2-1}/\sqrt{\omega^2+\sigma^2}$ and $b = \sqrt{\omega^2+1}/\sqrt{\omega^2+\sigma^2}$. Then γ is a helix with curvatures $k_1 = \sqrt{(\sigma^2-1)(\omega^2+1)}$ and $k_2 = (\sigma^2 - \omega^2 - 1)/\sqrt{4(\sigma^2-1)(\omega^2+1)}$.

Example 17 Let $\sigma^2 > 1$ and let γ be the curve in $\mathbb{S}_1^4$ defined by

$$\gamma(t) = \left(\frac{1}{2} \frac{\sqrt{\sigma^4-1}}{\sigma^2-1} t^2, \sqrt{\frac{\sigma^2-1}{\sigma^2}} t, \frac{\sqrt{\sigma^4-1}}{\sigma^2} - \frac{\sqrt{\sigma^4-1}}{2(\sigma^2+1)} t^2, \frac{1}{\sigma^2} \sin \sigma t, \frac{1}{\sigma^2} \cos \sigma t \right)$$

Then γ is a helix with curvatures $k_1 = \sqrt{\sigma^2-1}$ and $k_2 = \frac{1}{2}\sqrt{\sigma^2-1}$.

Theorem 11 (^{18}) *An s-degenerate spacelike Cartan curve in $\mathbb{S}_1^4$ is a helix if and only if it is congruent to one of the families described in Examples 15-17.*

6.3   Helices in $\mathbb{H}_1^4$

Example 18 Let $0 < \sigma^2 < 1 < \omega^2$ and let γ be the curve in $\mathbb{H}_1^4$ defined by

$$\gamma(t) = \left(\frac{1}{\omega} a \cosh \omega t, \frac{1}{\sigma} b \cosh \sigma t, \frac{1}{\omega} a \sinh \omega t, \frac{1}{\sigma} b \sinh \sigma t, -\frac{1}{\omega\sigma} \sqrt{(\omega^2-1)(1-\sigma^2)} \right)$$

where $a = \sqrt{1-\sigma^2}/\sqrt{\omega^2-\sigma^2}$ and $b = \sqrt{\omega^2-1}/\sqrt{\omega^2-\sigma^2}$. Then γ is a helix with curvatures $k_1 = \sqrt{(\omega^2-1)(1-\sigma^2)}$ and $k_2 = -(\omega^2 + \sigma^2 - 1)/\sqrt{4(\omega^2-1)(1-\sigma^2)}$.

Example 19 Let $\omega^2 > 1$ and let γ be the curve in $\mathbb{H}_1^4$ defined by

$$\gamma(t) = \left(\sqrt{\frac{(\omega^2-1)(\sigma^2+1)}{\omega^2\sigma^2}}, \frac{1}{\omega} a \cosh \omega t, \frac{1}{\omega} a \sinh \omega t, \frac{1}{\sigma} b \sin \sigma t, \frac{1}{\sigma} b \cos \sigma t \right),$$

where $\sigma \neq 0$, $a = \sqrt{\sigma^2+1}/\sqrt{\omega^2+\sigma^2}$ and $b = \sqrt{\omega^2-1}/\sqrt{\omega^2+\sigma^2}$. Then γ is a helix with curvatures $k_1 = \sqrt{(\omega^2-1)(\sigma^2+1)}$ and $k_2 = (\sigma^2 - \omega^2 + 1)/\sqrt{4(\omega^2-1)(\sigma^2+1)}$.

Example 20 Let $\omega^2 > 1$ and let γ be the curve in $\mathbb{H}_1^4$ defined by

$$\gamma(t) = \left(\frac{\sqrt{\omega^4-1}}{\omega^2} + \frac{1}{2} \frac{\sqrt{\omega^4-1}}{\omega^2+1} t^2, \frac{1}{\omega^2} \cosh \omega t, \frac{1}{\omega^2} \sinh \omega t, \sqrt{\frac{\omega^2-1}{\omega^2}} t, \frac{1-\omega^4}{2(1+\omega^2)} t^2 \right)$$

Then γ is a helix with curvatures $k_1 = \sqrt{\omega^2-1}$ and $k_2 = -\frac{1}{2}\sqrt{\omega^2-1}$.

Theorem 12 (^{18}) *An s-degenerate spacelike Cartan curve in $\mathbb{H}_1^4$ is a helix if and only if it is congruent to one of the families described in Examples 18-20.*

References

1. B. O'Neill. *Semi-Riemannian Geometry*. Academic Press, New York - London, 1983.
2. E. Vessiot. Sur les curbes minima. *Comptes Rendus*, 140:1381–1384, 1905.
3. W.B. Bonnor. Null curves in a Minkowski space-time. *Tensor, N.S.*, 20:229–242, 1969.
4. M. Castagnino. Sulle formule di Frenet-Serre per le curve nulle di una V_4 riemanniana a metrica iperbolica normale. *Rendiconti di matematica, Roma, Ser. 5*, 23:438–461, 1964.
5. L.P. Hughston and W.T. Shaw. Classical strings in ten dimensions. *Proc. Roy. Soc. London Ser. A*, 414:423–431, 1987.
6. L.P. Hughston and W.T. Shaw. Constraint-free analysis of relativistic strings. *Classical Quatum Gravity*, 5:69–72, 1988.
7. L.P. Hughston and W.T. Shaw. Spinor parametrizations of minimal surfaces. In *The Mathematics of Surfaces, III (Oxford, 1989)*, pages 359–372. Oxford Univ. Press, New York, 1989.
8. W.T. Shaw. Twistors and strings. In *Mathematics and General Relativity (Santa Cruz, CA, 1986)*, pages 337–363. Amer. Math. Soc., RI, 1988.
9. H. Urbantke. On Pinl's representation of null curves in n dimensions. In *Relativity Today (Budapest, 1987)*, pages 34–36. World Sci. Publ., Teaneck, New York, 1988.
10. A. Nersessian and E. Ramos. Massive spinning particles and the geometry of null curves. *Phys. Lett. B*, 445:123–128, 1998.
11. A. Nersessian and E. Ramos. A geometrical particle model for anyons. *Modern Phys. Lett. A*, 14(29):2033–2037, 1999.
12. A. Bejancu. Lightlike curves in Lorentz manifolds. *Publ. Math. Debrecen*, 44:145–155, 1994.
13. K. L. Duggal and A. Bejancu. *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, volume 364 of *Mathematics and its Aplications*. Kluwer Academic Publishers Group, Dordrecht, The Netherlands, 1996.
14. A. Ferrández, A. Giménez, and P. Lucas. Degenerate curves in pseudo-Euclidean spaces of index two. To appear in Proceedings of Geometry, Integrability and Quantization, Varna. Ed. I. Mladenov, 2002.
15. A. Ferrández, A. Giménez, and P. Lucas. Null helices in Lorentzian space forms. International Journal of Modern Physics A 16 (2001), 4845–4863.
16. A. Ferrández, A. Giménez, and P. Lucas. Spacelike curves in lightlike submanifolds of a Lorentzian space form. In preparation, 2001.
17. M. Spivak. *A Comprehensive Introduction to Differential Geometry*, 3rd Ed, Vol. IV. Publish or Perish, Houston, Texas, 1999.

18. A. Ferrández, A. Giménez, and P. Lucas. s-degenerate curves in Lorentzian space forms. Preprint, 2001.