# Geometry of lightlike submanifolds in Lorentzian space forms 

Angel Ferrández, Angel Giménez and Pascual Lucas<br>Departamento de Matemáticas, Universidad de Murcia<br>Campus de Espinardo, E-30100 Espinardo Murcia<br>e-mails: aferr@um.es, agpastor@um.es, plucas@um.es


#### Abstract

In a Lorentzian space (or more generally in a pseudo-Riemannian space) appears a class of submanifolds where the induced metric is degenerate; they are called lightlike submanifolds. This work tries to give relations between geometric objects of a lightlike submanifold and those of a (non-degenerate) Riemannian submanifold in a Lorentzian space. These relations allow us to obtain some characterization results for totally geodesic submanifolds in Lorentzian space forms.


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## 1 Introduction

It is well-know that in a Lorentzian manifold we can find three types of submanifolds: spacelike (or Riemannian), timelike (or Lorentzian) and lightlike (degenerate or null), depending on the induced metric in the tangent vector space. The growing importance of lightlike submanifolds in global Lorentzian geometry, and their use in general relativity, motivated the study of degenerate submanifolds in a semi-Riemannian manifold. Due to the degeneracy of the metric, basic differences occur between the study of lightlike submanifolds and the classical theory of Riemannian as well as semi-Riemannian submanifolds (see, for example, [1], [3], [5], [7], [11], and [14]).

Lightlike submanifolds (in particular, lightlike hypersurfaces) appear on innumerable papers of physics. For example, the lightlike submanifolds are very interesting for general relativity since they produce models of different types of horizons (event horizons, Cauchy's horizons, Kruskal's horizons). The idea that the Universe we live in can be represented as a 4-dimensional submanifold embedded in a $(4+d)$-dimensional space-time manifold has attracted the attention of many physicists. Higher dimensional semi-Euclidean spaces should provide theoretical framework in which the fundamental laws of physics
may appear to be unified, as in the Kaluza-Klein scheme. Lightlike hypersurfaces are also studied in the theory of electromagnetism (see, for example, [4], [15] and [16]).

In the following section we introduce the necessary geometric objects to study the geometry of lightlike submanifolds. The aim of this work is to relate the geometry of a lightlike submanifold in a Lorentzian space with the geometry of a Riemannian submanifold in the same Lorentzian space. Making use of the well known results in the non-degenerate geometry we obtain results for lightlike submanifolds. In the last section, and as application of the results obtained before, we completely describe the totally geodesic lightlike submanifolds in a Lorentzian space of constant curvature.

## 2 Preliminaries

Let us consider $\left(M_{1}^{n+1}, \nabla^{\circ}\right)$ an $(n+1)$-dimensional Lorentzian manifold with metric $g$ (which will also be denoted by $\langle$,$\rangle ). An m$-dimensional submanifold $P^{m}$ in $M_{1}^{n+1}$ is said to be a lightlike submanifold if the induced metric on $P^{m}$ is degenerate. In the sequel, and for simplicity of notation, we will write $M$ and $P$ instead of $M_{1}^{n+1}$ and $P^{m}$, respectively. Also, we will suppose that $m \geq 3$.

The basic facts on lightlike submanifolds can be found in [5]. In the notation of [5], the tangent vector bundle on $M$ restricted to $P$ can be decomposed (in a non unique way) as

$$
\begin{equation*}
\left.T M\right|_{P}=T P \oplus \operatorname{tr}(T P)=S(T P) \perp(K \oplus \widetilde{K}) \perp S\left(T P^{\perp}\right), \tag{1}
\end{equation*}
$$

where $K$ is the 1-dimensional radical distribution of $T P$, that is, $K=T P \cap$ $T P^{\perp} . S(T P)$ and $S\left(T P^{\perp}\right)$, called a screen distribution and a screen transversal vector bundle of $P$, verify

$$
T P=S(T P) \perp K \quad \text { and } \quad T P^{\perp}=S\left(T P^{\perp}\right) \perp K .
$$

$\tilde{K}$ is called a lightlike transversal vector bundle.
Bearing in mind the decomposition

$$
\left.T M\right|_{P}=T P \oplus \operatorname{tr}(T P),
$$

we obtain the Gauss formula for lightlike submanifolds,

$$
\nabla_{X}^{\circ} Y=\nabla_{X} Y+\theta(X, Y), \quad \text { for all } X, Y \in \Gamma T P,
$$

where $\nabla_{X} Y=\left(\nabla_{X}^{\circ} Y\right)^{\top}$ is the tangent part and $\theta(X, Y)=\left(\nabla_{X}^{\circ} Y\right)^{t}$ is the transversal part. $\nabla$ is called the induced connection on $P$ and $\theta$ is called the
lightlike second fundamental form of $P \subset M$. The lightlike second fundamental form is bilineal and symmetric. If $U$ and $X$ are normal and tangent sections, respectively, we can consider

$$
A_{U}(X)=-S\left(\nabla_{X}^{\circ} U\right)
$$

where $S: \Gamma T P \longrightarrow \Gamma S(T P)$ denotes the projection map on the screen distribution. The operator $A_{U}$ is called the lightlike shape operator with respect to the section $U$. The properties of these operators can be encountered in [5]. Now we will restrict our attention on a special type of lightlike submanifolds defined by Kupeli, [11].

Definition 1 A lightlike submanifold $P$ of $M$ is said to be irrotational if $\nabla_{X}^{\circ} \xi \in \Gamma T P$ for all tangent section $X$ of $P$, where $\xi$ is a section of the radical distribution $K$.

It is easy to check that this definition is independent of the chosen section $\xi$, and it is equivalent to the condition $\theta(X, \xi)=0$ for any decomposition.

A submanifold $P$ is called geodesic or totally geodesic if it contains the geodesics of $M$ which are somewhere tangent to it. In other words, if $q \in P$ and $v \in T_{q} P$, then the geodesic $\gamma$ in $M$ with initial conditions $\gamma(0)=q$ and $\gamma^{\prime}(0)=v$ lies in $P$. This is equivalent to saying that the vector fields on $P$ are invariant by covariant derivation (this equivalence is valid for all torsion free connections, [9]). Although there is no way to induce connections on arbitrary lightlike submanifolds, the totally geodesic ones have such a connection, which is compatible with the degenerate metric but not derived from it, as degenerate metrics do not have Levi-Civita connections.

Trivial examples of irrotational submanifolds are the totally geodesic lightlike submanifolds, since $\nabla_{X}^{\circ} Y=\nabla_{X} Y$, or equivalently, $\theta=0$. Another important example are the lightlike hypersurfaces, since $\left\langle\nabla_{X}^{\circ} \xi, \xi\right\rangle=0$ and so the transversal part vanishes.

Definition 2 Let $P$ be an $m$-dimensional lightlike submanifold of an $(n+1)$ dimensional Lorentzian manifold $M$. We say that $P$ is a totally umbilical lightlike submanifold if for all $U \in \Gamma T P^{\perp}$, there exist a differentiable function $\lambda_{U}$ verifying

$$
\begin{equation*}
A_{U}(X)=\lambda_{U} S X, \quad \text { for all } X \in \Gamma T P \tag{2}
\end{equation*}
$$

This definition is independent of the chosen screen distribution $S(T P)$.
If $P$ is an irrotational lightlike submanifold, for $\xi \in \Gamma K, U \in \Gamma T P^{\perp}$ and $W \in \Gamma S(T P)$, we have

$$
\left\langle\nabla_{\xi}^{\circ} U, W\right\rangle=-\left\langle\nabla_{\xi}^{\circ} W, U\right\rangle=0
$$

then $S\left(\nabla_{\xi}^{\circ} U\right)=0$, which implies

$$
\begin{equation*}
A_{U}(X)=A_{U}(S X), \quad \text { for all } U \in \Gamma T P^{\perp}, X \in \Gamma T P . \tag{3}
\end{equation*}
$$

From now on, $\left\{E_{1}, \ldots, E_{m-1}, \xi, \eta, N_{1}, \ldots, N_{n-m}\right\}$ will denote a pseudo-orthonormal basis of $\left.T M\right|_{P}$ adapted to the decomposition (1), where $E_{i} \in \Gamma S(T P)$, $N_{j} \in \Gamma S\left(T P^{\perp}\right), \xi \in \Gamma K, \eta \in \Gamma \widetilde{K}$, verifying

$$
\left\langle E_{i}, E_{i}\right\rangle=\left\langle N_{j}, N_{j}\right\rangle=\langle\xi, \eta\rangle=1, \quad\langle\xi, \xi\rangle=\langle\eta, \eta\rangle=0 .
$$

The following proposition can be found in [11], page 80; here we present a different point of view.

Proposition 1 Let $P$ be an m-dimensional irrotational lightlike submanifold of a Lorentzian manifold $M$. The following statements are equivalent:
(i) There exist a transversal section $H$ verifying that

$$
\theta(X, Y)=\langle X, Y\rangle H, \quad \text { for all } X, Y \in \Gamma T P .
$$

(ii) $P$ is totally umbilical.

Proof. Let $\left\{E_{1}, \ldots, E_{m-1}, \xi, \eta, N_{1}, \ldots, N_{n-m}\right\}$ be a pseudo-orthonormal basis adapted to $\left.T M\right|_{P}$. We claim that $\theta(X, Y)=\theta(S X, S Y)$ for $X, Y \in \Gamma T P$. The proof is a consequence of the following computation.

$$
\begin{align*}
\theta(S X, S Y) & =\left\langle\nabla_{S X}^{\circ} S Y, \xi\right\rangle \eta+\sum_{j=1}^{n-m}\left\langle\nabla_{S X}^{\circ} S Y, N_{j}\right\rangle N_{j} \\
& =-\left\langle\nabla_{S X}^{\circ} \xi, S Y\right\rangle \eta-\sum_{j=1}^{n-m}\left\langle\nabla_{S X}^{\circ} N_{j}, S Y\right\rangle N_{j}  \tag{4}\\
& =\left\langle A_{\xi}(S X), S Y\right\rangle \eta+\sum_{j=1}^{n-m}\left\langle A_{N_{j}}(S X), S Y\right\rangle N_{j} .
\end{align*}
$$

Let us assume (i), then $\theta(S X, S Y)=\langle S X, S Y\rangle H, H$ being

$$
\begin{equation*}
H=\lambda \eta+\sum_{j=1}^{n-m} \lambda_{j} N_{j} \tag{5}
\end{equation*}
$$

and so, combining this equality with (4), we obtain

$$
\left\langle A_{\xi}(S X), S Y\right\rangle=\lambda\langle S X, S Y\rangle, \quad\left\langle A_{N_{j}}(S X), S Y\right\rangle=\lambda_{j}\langle S X, S Y\rangle .
$$

These equations are equivalent to

$$
\left\langle A_{\xi}(S X)-\lambda S X, S Y\right\rangle=0, \quad\left\langle A_{N_{j}}(S X)-\lambda_{j} S X, S Y\right\rangle=0, \quad \forall X, Y \in \Gamma T P,
$$

which implies

$$
A_{\xi}(X)=A_{\xi}(S X)=\lambda S X, \quad A_{N_{j}}(X)=A_{N_{j}}(S X)=\lambda_{j} S X, \quad \forall X, Y \in \Gamma T P .
$$

Taking into account that $\left\{\xi, N_{1}, \ldots, N_{n-m}\right\}$ is a basis of $T P^{\perp}$, we obtain (ii).
Conversely, if $P$ is totally umbilical, then equation (2) implies that there exist differentiable functions $\lambda, \lambda_{j}$ verifying $A_{\xi}(X)=\lambda S X$ and $A_{N_{j}}(X)=$ $\lambda_{j} S X$ for all $X \in \Gamma T P$. Substituting these equalities into (4) we complete the proof, where $H$ is given by (5).

If $P$ is irrotational, we know that $S\left(\nabla_{\xi}^{\circ} \xi\right)=0$, and so there exist a differentiable function $\rho$ such that $\nabla_{\xi}^{\circ} \xi=-\rho \xi$. As a consequence of this computation we have the following result.

Proposition 2 Let $P$ be an irrotational lightlike submanifold of a Lorentzian manifold $M$. Then the integral curves of $\xi \in \Gamma K$ are null pregeodesics of $M$.
Proof. Let $h$ be a function verifying $\xi(\log h)=\rho$, and take $\tilde{\xi}=h \xi$. It is easy to check that $\nabla_{\tilde{\xi}}^{\circ} \tilde{\xi}=0$, hence the integral curves of $\tilde{\xi}$ are geodesics and, in consequence, the integral curves of $\xi$ are pregeodesics.

The above statement is true for all lightlike hypersurfaces. Furthermore, if $M$ have constant curvature, it is well know that the null geodesics are null lines, which implies that $P$ is ruled by null lines.

## 3 Lightlike submanifolds and submersions

Let $P$ be an $m$-dimensional lightlike submanifold immersed in a $(n+1)$ dimensional Lorentzian manifold $M$ by $\psi: P \longrightarrow M$. Let $\pi: M \longrightarrow B$ a Lorentzian submersion of codimension one (that is, $\operatorname{dim}(B)=1$ ), and consider $\tilde{\pi}=\pi \circ \psi$. Let us assume that $\tilde{\pi}: P \longrightarrow \tilde{B}=\tilde{\pi}(P)$ is a submersion, or equivalent, $P$ is not contained in any fiber of $\pi$.

Let us denote by $\mathcal{F}_{t}$ and $\Sigma_{t}$ the fibers of $\pi$ and $\tilde{\pi}$, respectively. It is clear that $\Sigma_{t}$ is a Riemannian submanifold immersed in $\mathcal{F}_{t}$. The following diagram illustrates the situation:


The Levi-Civita connections $\hat{\nabla}_{t}^{o}$ and $\hat{\nabla}_{t}$ on $\mathcal{F}_{t}$ and $\Sigma_{t}$, respectively, can be extended to connections $\hat{\nabla}^{\circ}$ and $\hat{\nabla}$ on the vertical distributions $\mathcal{V}(T M)$ and $\tilde{\mathcal{V}}(T P)$, respectively.

Our aim now is to define geometric objects with respect to these submersions.

Definition 3 Let $P$ be a lightlike submanifold of a Lorentzian manifold $M$ and $\pi: M \longrightarrow B$ a submersion as before. The screen vector bundle $S(T P)=$ $\tilde{\mathcal{V}}(T P)$ on $P$ is called the canonical screen distribution associated to the submersion $\pi$.

Bearing in mind the above diagram, definitions and notations, we can decompose the tangent vector bundle $\left.T M\right|_{P}$ in a different way that (1), as follows,

$$
\begin{align*}
\left.T M\right|_{P} & =\left.\left.\mathcal{V}(T M)\right|_{P} \perp \mathcal{H}(T M)\right|_{P} \\
& =\left.\left(\tilde{\mathcal{V}}(T P) \perp \tilde{\mathcal{V}}(T P)^{\perp}\right) \perp \mathcal{H}(T M)\right|_{P}  \tag{7}\\
& =S(T P) \oplus\left(\tilde{\mathcal{V}}(T P)^{\perp} \perp \tilde{\mathcal{H}}(T P)\right),
\end{align*}
$$

where $\tilde{\mathcal{V}}(T P)^{\perp}$ denotes the orthogonal of $\tilde{\mathcal{V}}(T P)$ in $\left.\mathcal{V}(T M)\right|_{P}$. Comparing the decompositions (1) and (7) we deduce

$$
(K \oplus \tilde{K}) \perp S\left(T P^{\perp}\right)=\tilde{\mathcal{V}}(T P)^{\perp} \perp \tilde{\mathcal{H}}(T P) .
$$

Let $\chi \in \mathcal{H}(T M)$ be a unit local basic vector field with respect to $\pi$ and write $\tilde{\chi}=\left.\chi\right|_{P} \in \tilde{\mathcal{H}}(T P)$. As $P$ is not contained in any fiber of $\pi$, then $\langle\xi, \chi\rangle \neq 0$ for $\xi \in \Gamma K$, so that $K \oplus \tilde{\mathcal{H}}(T P)$ is a hiperbolic plane. Choose $\tilde{K}$ such that $\Pi=K \oplus \tilde{K}=K \oplus \tilde{H}(T P)$. We can construct local references $\{\xi, \eta\}$ and $\{N, \tilde{\chi}\}$, with $N \in \tilde{\mathcal{V}}(T P)^{\perp}$ and $\eta \in \tilde{K}$, satisfying

$$
\begin{equation*}
\varepsilon=\langle N, N\rangle=-\langle\tilde{\chi}, \tilde{\chi}\rangle, \quad \xi=\frac{1}{\sqrt{2}}(N+\tilde{\chi}), \quad \eta=\frac{\varepsilon}{\sqrt{2}}(N-\tilde{\chi}), \tag{8}
\end{equation*}
$$

where $\varepsilon= \pm 1$. In this case $S\left(T P^{\perp}\right)$ is necessarily the orthonormal complementary of span $\{N\}$ in $\tilde{\mathcal{V}}(T P)^{\perp}$.

Definition 4 The section $\xi$ and the vector bundle $S\left(T P^{\perp}\right)$ defined above are called the canonical radical section and the canonical screen transversal vector bundle associated to the submersion $\pi$.

If $M$ is time-oriented, we can choose $\xi$ and $\eta$ pointing out to the future. In this case they are completely determined by the submersion $\pi$. In these conditions, if $\left\{N_{0}=N, N_{1}, \ldots, N_{n-m}\right\}$ is a basis of $\tilde{\mathcal{V}}(T P)^{\perp}$, where $\left\{N_{1}, \ldots, N_{n-m}\right\}$
expands the canonical screen transversal vector bundle, we consider the operators

$$
\left.\begin{array}{rlrl}
\tilde{A}_{N_{j}}: \tilde{\mathcal{V}}(T P) & \longrightarrow \tilde{\mathcal{V}}(T P) & \tilde{\sigma}: \tilde{\mathcal{V}}(T P) \times \tilde{\mathcal{V}}(T P) & \longrightarrow \tilde{\mathcal{V}}(T P)^{\perp} \\
W & \rightsquigarrow & -\tilde{\mathcal{V}}\left(\widehat{\nabla}_{W}^{\circ} N_{j}\right) & \left(W_{1}, W_{2}\right)
\end{array}>\left(\widehat{\nabla}_{W_{1}}^{\circ} W_{2}\right)^{\perp}\right) ~ ل
$$

where $0 \leq j \leq n-m$. Moreover, these operators restricted to each fiber are the shape operator respect to $N_{j}$ and the second fundamental form of the immersion $\psi_{t}: \Sigma_{t} \longrightarrow \mathcal{F}_{t}$, respectively.

On the other hand, bearing in mind the diagram (6), we can consider the operators defined by

$$
\begin{array}{cccc}
\hat{A}^{\circ}: \mathcal{V}(T M) \longrightarrow \mathcal{V}(T M) & \hat{\sigma}^{\circ}: \mathcal{V}(T M) \times \mathcal{V}(T M) \longrightarrow \mathcal{H}(T M) \\
V & \rightsquigarrow & -\nabla_{V}^{\circ} \chi & \left(V_{1}, V_{2}\right)
\end{array}>\mathcal{H}\left(\nabla_{V_{1}}^{\circ} V_{2}\right)
$$

These operators, restricted to each fiber $\mathcal{F}_{t}$, are the shape operator and the second fundamental form of the immersion $i_{t}: \mathcal{F}_{t} \rightarrow M$. We will consider both operators acting on $\left.\mathcal{V}(T M)\right|_{P}$.

We can write the following equations relating the above geometric objects,

$$
\begin{align*}
\nabla_{V_{1}}^{\circ} V_{2} & =\widehat{\nabla}_{V_{1}}^{\circ} V_{2}+\hat{\sigma}^{\circ}\left(V_{1}, V_{2}\right)=\widehat{\nabla}_{V_{1}}^{\circ} V_{2}-\varepsilon\left\langle\hat{A}^{\circ}\left(V_{1}\right), V_{2}\right\rangle \tilde{\chi} \\
\widehat{\nabla}_{W_{1}}^{\circ} W_{2} & =\widehat{\nabla}_{W_{1}} W_{2}+\tilde{\sigma}\left(W_{1}, W_{2}\right)  \tag{9}\\
& =\widehat{\nabla}_{W_{1}} W_{2}+\varepsilon\left\langle\tilde{A}_{N}\left(W_{1}\right), W_{2}\right\rangle N+\sum_{j=1}^{n-m}\left\langle\tilde{A}_{N_{j}}\left(W_{1}\right), W_{2}\right\rangle N_{j},
\end{align*}
$$

where $W_{1}, W_{2}$ are sections on $\tilde{\mathcal{V}}(T P)=S(T P)$ and $V_{1}, V_{2}$ are sections on $\left.\mathcal{V}(T M)\right|_{P}$. These equations restricted to each fiber represent the Gauss equations of both immersions $\mathcal{F}_{t} \subset M$ and $\Sigma_{t} \subset \mathcal{F}_{t}$, respectively.

We are going to state some results relating the different geometric objects defined above. From these relationships we will obtain interesting applications for particular cases.

Proposition 3 Let $P$ be an m-dimensional irrotational lightlike submanifold of a Lorentzian manifold $M$ of dimension $n+1$, and $\pi: M \longrightarrow B$ a totally umbilical semi-Riemannian (Riemannian or Lorentzian) submersion. Let $S(T P)$ be the canonical screen distribution, $\xi$ the canonical radical section $\xi$ associated to $\pi$ and $\left\{N, N_{1}, \ldots, N_{n-m}\right\}$ an orthonormal basis of $\tilde{\mathcal{V}}(T P)^{\perp}$. Then the following statements hold:
(i) $\nabla_{W}^{\circ} \xi=A_{\xi}(W)$, for all $W \in \Gamma S(T P)$.
(ii) $A_{N_{j}}=\tilde{A}_{N_{j}}$ for $1 \leq i \leq n-m$, and $A_{\xi}=\frac{1}{\sqrt{2}}\left(\tilde{A}_{N}+\mu \mathrm{Id}\right)$, where $\mu$ is the differentiable function verifying the equation $\hat{A}^{\circ}(V)=\mu V$.

Proof. "(i)" By hypothesis, $\nabla_{X}^{\circ} \xi$ is a section of $T P$. The proof is completed by showing that $\left\langle\nabla_{W}^{\circ} \xi, \eta\right\rangle=0$. Indeed,

$$
\begin{aligned}
\left\langle\nabla_{W}^{\circ} \xi, \eta\right\rangle & =\left\langle\frac{1}{\sqrt{2}} \nabla_{W}^{\circ}(N+\tilde{\chi}), \frac{\varepsilon}{\sqrt{2}}(N-\tilde{\chi})\right\rangle \\
& =\frac{\varepsilon}{2}\left(-\left\langle\nabla_{W}^{\circ} N, \tilde{\chi}\right\rangle+\left\langle\nabla_{W}^{\circ} \tilde{\chi}, N\right\rangle\right) \\
& =\varepsilon\left\langle\nabla_{W}^{\circ} \tilde{\chi}, N\right\rangle \\
& =\varepsilon\left\langle-\hat{A}^{\circ}(W), N\right\rangle=0 .
\end{aligned}
$$

"(ii)" Trivially, $A_{N_{j}}=\tilde{A}_{N_{j}}$ since $S(T P)=\tilde{\mathcal{V}}(T P)$. Bearing in mind the above statement and that the fibers of $\pi$ are totally umibilical, that is, $\hat{A}^{\circ}(W)=\mu W$, we obtain

$$
\begin{aligned}
A_{\xi}(W) & =-\frac{1}{\sqrt{2}} \nabla_{W}^{\circ}(N+\tilde{\chi}) \\
& =-\frac{1}{\sqrt{2}}\left(\nabla_{W}^{\circ} N+\nabla_{W}^{\circ} \tilde{\chi}\right) \\
& =-\frac{1}{\sqrt{2}}\left(\hat{\nabla}_{W}^{\circ} N-\hat{A}^{\circ}(W)\right) \\
& =\frac{1}{\sqrt{2}}\left(\tilde{A}_{N}(W)+\mu W\right) .
\end{aligned}
$$

Proposition 4 Let $P$ be an irrotational lightlike submanifold of a Lorentzian manifold $M$ and $\pi: M \longrightarrow B$ a totally umbilical submersion with semiRiemannian fibers $\mathcal{F}_{t}$. Let $\tilde{\pi}$ be the submersion induced by $\pi$ on $P$ with fibers $\Sigma_{t}$. Then $P$ is totally umbilical if and only if $\Sigma_{t}$ is totally umbilical in $\mathcal{F}_{t}$ for all $t \in \tilde{\pi}(P)$.

The proof is a direct consequence of Proposition 3. In particular, for totally geodesic lightlike submanifolds we have $\tilde{A}_{N_{j}}=0,1 \leq j \leq n-m$, and $\tilde{A}_{N}=-\mu \mathrm{Id}$. Then if the fibers $\mathcal{F}_{t}$ are totally geodesics $(\mu=0)$, then the immersions $\Sigma_{t} \subset \mathcal{F}_{t}$ are totally geodesics.

## 4 Applications to Lorentzian space forms

This section contains some applications to Lorentzian manifolds of constant curvature $M_{1}^{n+1}(c)$. In particular we describe the totally geodesic lightlike
submanifolds in $M_{1}^{n+1}(c)$. We study separately the ambient spaces $\mathbb{R}_{1}^{n+1}$, $\mathbb{S}_{1}^{n+1}$ and $\mathbb{H}_{1}^{n+1}$.

## Lightlike submanifolds in $\mathbb{R}_{1}^{n+1}$

As it is well known, a irrotational lightlike submanifold is ruled by null geodesics, and these geodesics can be naturally enlarged to complete geodesics. The problem is the appearance of singular points, but we can work locally.

Fix a vector $a \in \mathbb{R}_{1}^{n+1}$ such that $\langle a, a\rangle=-1$ and consider $\pi_{a}: \mathbb{R}_{1}^{n+1} \longrightarrow \mathbb{R}$ the map defined by $\pi_{a}(x)=\langle x, a\rangle$. It is easy to prove that $\pi$ is a totally geodesic submersion with Riemannian fibers, and $\tilde{\pi}$ is a submersion, since $P$ can not be contained in any fiber (they are Riemannian). In particular, if we choose $a=(1,0, \ldots, 0)$ we obtain a submersion where the fibers are $\{t\} \times \mathbb{R}^{n}$.

We have the following situation:


Remark 1 Whenever $P$ is a lightlike hypersurface of the $(n+1)$-dimensional Lorentz-Minkowski space and $\pi_{a}$ is as above, with $a=(1,0, \ldots, 0)$, the lightlike transversal vector bundle expanded by the vector field $\eta$, given by (8), agrees with the canonical lightlike transversal vector bundle introduced in [2] (up the orientation). In particular, if $n=3$, the Gauss map $N^{t}$ of the immersion $\Sigma_{t} \subset \mathcal{F}_{t}$ defined by

$$
\begin{aligned}
N^{t}: \Sigma_{t} & \longrightarrow \mathbb{S}^{2} \\
& \left.\rightsquigarrow N\right|_{\Sigma_{t}}(p)
\end{aligned}
$$

where the $N$ is given by (8), agrees with the Gauss map associated to a lightlike hypersurface $P$ with base $\Sigma_{t}$ introduced by Kossowski in [10].

It is well-known that the only totally geodesic submanifolds of $\mathbb{R}^{n}$ are pieces of $r$-planes, with $2 \leq r<n$. Moreover, the only non geodesic totally umbilical hypersurfaces of $\mathbb{R}^{n}$ are pieces of spheres. Then from this fact and by using Proposition 4 we deduce the following results already known.

Proposition 5 The only totally geodesic lightlike submanifods in the LorentzMinkowski space $\mathbb{R}_{1}^{n+1}$ are pieces of null m-planes.

Proof. We know that $P$ is ruled by null lines, and the sections $\Sigma_{t}$ provided by the submersion $\tilde{\pi}$ are $(m-1)$-planes. We only have to prove that $\xi$ is a parallel section with respect to $\tilde{\mathcal{V}}(T P)=S(T P)$. Using Proposition 3 we have

$$
\nabla_{W}^{\circ} \xi=A_{\xi}(W)=0, \quad \text { for all } W \in \Gamma S(T P)
$$

which is the desired conclusion.
The following result, already proved by Akivis and Goldberg in [1], can also be easily deduced.

Proposition 6 The only totally umbilical lightlike hypersurfaces in $\mathbb{R}_{1}^{n+1}$ are the lightlike cones.

Proof. Let $P$ be a totally umbilical lightlike hypersurface, then the fibers $\Sigma_{t}$ are spherical. It is easy to see that there are only two lightlike hypersurfaces that contain the fiber $\Sigma_{t}$, but we can construct two lightlike cones containing $\Sigma_{t}$. This concludes the proof.

## Lightlike submanifolds in the De-Sitter space $\mathbb{S}_{1}^{n+1}$

Fix a vector $a \in \mathbb{R}_{1}^{n+2}$ such that $\langle a, a\rangle=-1$ and consider as before $\bar{\pi}_{a}$ : $\mathbb{R}_{1}^{n+2} \longrightarrow \mathbb{R}$ the map defined by $\bar{\pi}_{a}(x)=\langle x, a\rangle$. It is not difficult to prove that $\pi_{a}=\left.\bar{\pi}_{a}\right|_{\mathbb{S}_{1}^{n+1}}$ is a totally umbilical submersion with Riemannian fibers. To make easier the computations, choose $a=(1,0, \ldots, 0)$. Then, for each $t \in \mathbb{R}$, the fiber $\mathcal{F}_{t}=\pi_{a}^{-1}(t)$ is a totally umbilical hypersurface with shape operator $\hat{A}^{\circ}=\mu I$, where $\mu=t / \sqrt{t^{2}+1}$. Therefore $\mathcal{F}_{t}$ have positive constant curvature $1 /\left(t^{2}+1\right)$, hence $\mathcal{F}_{t}$ is a sphere of radius $\sqrt{t^{2}+1}$. Note that the fiber $\mathcal{F}_{0}$ is totally geodesic $\left(\left.\mu\right|_{\mathcal{F}_{0}}=0\right)$. We have the following situation:


Proposition 7 The m-dimensional totally geodesic lightlike submanifolds $P$ in the De-Sitter space $\mathbb{S}_{1}^{n+1} \subset \mathbb{R}_{1}^{n+2}$ are exactly the intersections of null $(m+1)$ planes in $\mathbb{R}_{1}^{n+2}$ passing through the origin with $\mathbb{S}_{1}^{n+1}$.

Proof. Consider $\pi_{a}: \mathbb{S}_{1}^{n+1} \longrightarrow \mathbb{R}$ with $a=(1,0, \ldots, 0)$, defined as above. Then the fibers $\mathcal{F}_{t}$ are exactly $\mathbb{S}^{n}\left(1 /\left(1+t^{2}\right)\right.$. Since totally geodesic lightlike submanifolds are ruled by null lines, then it is very important to study the immersion $\Sigma_{0} \subset \mathcal{F}_{0}=\mathbb{S}^{n}(1)$. We work in $\mathcal{F}_{0}$ because it is the unique totally geodesic fiber in $\mathbb{S}_{1}^{n+1}$, and so the computations are easier.

By using Proposition 3, for a suitable basis $\left\{N, N_{1}, \ldots, N_{n-m}\right\}$ of $\tilde{\mathcal{V}}(T P)^{\perp}$, we have

$$
A_{\xi}=\frac{1}{\sqrt{2}}\left(\tilde{A}_{N}+\mu \mathrm{Id}\right), \quad A_{N_{j}}=\tilde{A}_{N_{j}}, \quad 1 \leq j \leq n-m
$$

Then $P$ is totally geodesic iff $\tilde{A}_{N_{j}}=0$ and $\tilde{A}_{N}=-\mu$ Id. In particular, if we restricted these operators to the fiber $\Sigma_{0}$, then $\tilde{A}_{N}=0$ and consequently the immersion $\Sigma_{0} \subset \mathbb{S}^{n}$ is totally geodesic. It is well-known that the totally geodesics submanifolds of $\mathbb{S}^{n}$ are $(m-1)$-dimensional spheres of maximum radius (that is, intersections of $m$-planes passing through the origin with $\mathbb{S}^{n}$ ). Until now, we have proved that $\Sigma_{0}=\Pi^{m} \cap \mathbb{S}^{n}$ where $\Pi^{m}$ is an $m$-plane contained in $\mathcal{F}_{0}=\mathbb{R}^{n+1}$. We are going to prove that $\xi$ is a parallel section on $\Sigma_{0}$ in $\mathbb{R}_{1}^{n+2}$.

We denote by $\nu$ the normal vector field of the immersion $\mathbb{S}_{1}^{n+1} \subset \mathbb{R}_{1}^{n+2}$, so that the shape operator of $\mathbb{S}_{1}^{n+1}$ is given by $A_{\nu}=-$ Id. If $W \in S(T P)=\tilde{\mathcal{V}}(T P)$ and $\xi$ is the canonical radical section, then

$$
\bar{\nabla}_{W}^{\circ} \xi=\nabla_{W}^{\circ} \xi+\left\langle A_{\nu}(W), \xi\right\rangle \nu=A_{\xi}(W)=0 .
$$

Let $\bar{\Pi}$ be the null $(m+1)$-plane given by $\bar{\Pi}=\Pi^{m} \perp$ span $\{v\}$, where $v$ have the same direction of $\xi \mid \Sigma_{0}$, then it is easy to check that $P=\bar{\Pi} \cap \mathbb{S}_{1}^{n+1}$.

## Lightlike submanifolds of the Anti-De Sitter space $\mathbb{H}_{1}^{n+1}$

We can obtain analogous results as in the De Sitter space. Fix a vector $a \in$ $\mathbb{R}_{2}^{n+2}$ such that $\langle a, a\rangle=1$ and let us consider $\bar{\pi}_{a}: \mathbb{R}_{2}^{n+2} \longrightarrow \mathbb{R}$ the map defined by $\bar{\pi}_{a}(x)=\langle x, a\rangle$. Consider $\pi_{a}=\left.\bar{\pi}_{a}\right|_{\mathbb{H}_{1}^{n+1}}$. It can be proved that $\pi_{a}$ is a totally umbilical submersion with Lorentzian fibers. The fibers $\mathcal{F}_{t}=\pi_{a}^{-1}(t)$ are totally umbilical hypersurfaces with shape operator $\hat{A}^{\circ}=\left(-t / \sqrt{t^{2}+1}\right) I$. Then they are of negative constant curvature $-1 /\left(1+t^{2}\right)$ and therefore they are pseudo-hyperbolic spaces $\mathbb{H}_{1}^{n}\left(-1 /\left(1+t^{2}\right)\right)$. Note that the fiber $\mathcal{F}_{0}$ is totally geodesic in $\mathbb{H}_{1}^{n+1}$. The induced map $\tilde{\pi}$ is not in general a submersion, but in this case, since the fibers are again anti De-Sitter spaces, we can suppose that $P$ is not contained in any fiber and then $\tilde{\pi}$ is a submersion. Choosing the
point $a=(0, \ldots, 0,1)$, we are in the following situation.


Denote by $\bar{\Pi}_{i, r}^{m+1}$ an $(m+1)$-plane passing through the origin of $\mathbb{R}_{2}^{n+2}$, where $i$ and $r$ denote the index and the dimension of the radical distribution, respectively. In the semi-Euclidean space $\mathbb{R}_{2}^{n+2}$ there exist six different types of $(m+1)$-planes, they are: $\bar{\Pi}_{0,0}^{m+1}, \bar{\Pi}_{1,0}^{m+1}, \bar{\Pi}_{2,0}^{m+1}, \bar{\Pi}_{1,1}^{m+1}, \bar{\Pi}_{0,1}^{m+1}$ and $\bar{\Pi}_{0,2}^{m+1}$.

The intersection of $(m+1)$-planes of index 0 with the Anti De-Sitter space is empty. On the other hand, $\bar{\Pi}_{1,0}^{m+1} \cap \mathbb{H}_{1}^{n+1}$ is an hiperbolic space $\mathbb{H}^{m}$ and $\bar{\Pi}_{2,0}^{m+1} \cap \mathbb{H}_{1}^{n+1}$ is an Anti De-Sitter space $\mathbb{H}_{1}^{m}$. The following proposition describes the intersection $\bar{\Pi}_{1,1}^{m+1} \cap \mathbb{H}_{1}^{n+1}$.

Proposition 8 The m-dimensional totally geodesic lightlike submanifolds $P$ of the Anti De-Sitter space $\mathbb{H}_{1}^{n+1} \subset \mathbb{R}_{2}^{n+2}$ are exactly the intersections of ( $m+$ 1)-planes $\bar{\Pi}_{1,1}^{m+1}$ in $\mathbb{R}_{2}^{n+2}$ passing through the origin with $\mathbb{H}_{1}^{n+1}$.

Proof. Consider the submersion $\pi_{a}: \mathbb{H}_{1}^{n+1} \longrightarrow \mathbb{R}$ defined above and let $\tilde{\pi}$ be the submersion induced by $\pi$. Similar considerations as in the De-Sitter space apply to this case, and prove that $P$ is of the form $\Sigma_{0} \times \ell$ where $\ell$ represents a constant null direction and $\Sigma_{0}$ is a $(m-1)$-dimensional totally geodesic Riemannian submanifold of $\mathcal{F}_{0}=\mathbb{H}_{1}^{n}$, that is, $\Sigma_{0}=\mathbb{H}^{m-1}$. Actually, we can write $\Sigma_{0}=\Pi^{m} \cap \mathbb{H}^{n}$, where $\Pi^{m}$ is a Lorentzian plane of dimension $m$. Take $\bar{\Pi}_{1,1}^{m+1}=\Pi^{m} \perp$ span $\{v\}$, where $v$ have the same direction of $\ell$, then it is easy to show that $P=\bar{\Pi}_{1,1}^{m+1} \cap \mathbb{H}_{1}^{n+1}$.

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