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Null generalized helices and the Betchov-Da Rios equation in Lorentz-Minkowski spaces

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Abstract

We study null Cartan generalized helices in the four dimensional Lorentz-Minkowski $\mathbb{L}^4$. We find a Lancret type theorem and the solutions to the natural equations problem. We also give some solutions of the null Betchov-Da Rios equation.

Key words: null helix, Betchov-Da Rios equation, localized induction equation
MSC 2000: 53B30, 53B50, 53C80, 53Z05

1 Introduction

A Lancret curve (or generalized helix) in $\mathbb{R}^3$ is a curve whose tangent makes a constant angle with a fixed direction (called the axis). The study of these curves in $\mathbb{R}^3$ dates from 1802 when M.A. Lancret stated that "a curve is a generalized helix if and only if the ratio of curvature to torsion is constant" (see [9] for details). The Lancret theorem was revisited and solved by M. Barros (see [1]) in 3-dimensional real space forms by using Killing vector fields along curves. Recently, new improvements have been achieved in Lorentzian space forms (see [2]).

The interest of non null generalized helices is well known (see [7] and [8]). We have found in [3] parametrized solutions of the localized induction equation (LIE) $\partial\gamma/\partial s = \partial\gamma/\partial t \times \partial^2\gamma/\partial t^2$ in the 3-dimensional Lorentzian space forms, so that the soliton solutions are the null geodesics of the Lorentzian cylinders or B-scrolls. In [6] we propose the equation $\partial\gamma/\partial s = \partial^2\gamma/\partial t^2 \times \partial^3\gamma/\partial t^3 \times \cdots \times \partial^n\gamma/\partial t^n$ as the corresponding LIE for null curves $\gamma(t) = \gamma(t, 0)$ in the n-dimensional Lorentz-Minkowski space. Then we find that null generalized helices in $\mathbb{L}^n$ evolving in the axis direction are solutions of the null LIE.

Throughout this paper we will deal with fully immersed curves, which means that all of its Cartan curvatures do not vanish anywhere.

2 Null generalized helices in 4-dimensional Lorentz-Minkowski spaces

Let $\gamma : I \longrightarrow \mathbb{L}^4$ be a null Cartan curve (see [4]) with Cartan frame $\{L = \gamma'(t), W_1, N, W_2\}$ satisfying the equations

$$L' = W_1; \quad W_1' = -kL + N; \quad N' = -kW_1 + \tau W_2; \quad W_2' = \tau L. \tag{1}$$

Definition 2.1. *A null Cartan curve $\gamma : I \longrightarrow \mathbb{L}^4$ is said to be a* generalized helix *if there exists a constant vector $\upsilon \neq 0$ such that the product $\lambda = \langle L(t), \upsilon\rangle \neq 0$ is constant.*

The straight line generated by υ, which can be spacelike, timelike or lightlike, is uniquely determined and will be called the *axis* of γ. When υ is spacelike or timelike, we can suppose that υ is of unit length.

The general characterization of null generalized helices in $\mathbb{L}^4$ states as follows.

Theorem 2.2. *Let $\gamma : I \longrightarrow \mathbb{L}^4$ be a null Cartan curve. Then γ is a generalized helix if and only if its Cartan curvatures satisfy the following differential equation*

$$(k')^2 = \tau^2 (2k + C), \quad k' \neq 0, \tag{2}$$

where C is a constant.

Proof. Let γ be a Cartan generalized helix with axis $\upsilon = \mu(t)L + \lambda_1(t)W_1 - \lambda N + \lambda_2(t)W_2$, where λ is a constant and $\mu, \lambda_i : I \longrightarrow \mathbb{R}$ are differentiable functions. Then

$$\frac{d\upsilon}{ds} = \left(\mu' - \lambda_1 k + \lambda_2\tau\right) L + \left(\mu + \lambda_1' + \lambda k\right) W_1 + \lambda_1 N + \left(\lambda_2' - \lambda\tau\right) W_2.$$

As υ is a constant vector we get

$$\lambda_1 = 0; \quad \mu + \lambda k = 0; \quad \mu' + \lambda_2\tau = 0; \quad \lambda_2' - \lambda\tau = 0. \tag{3}$$

As $\tau \neq 0$, we deduce that $\mu = -\lambda k$ and $\lambda_2 = \lambda k'/\tau$, so that the axis of the helix is

$$\upsilon = -\lambda k L - \lambda N + \lambda\frac{k'}{\tau}W_2. \tag{4}$$

Finally, as $\langle\upsilon, \upsilon\rangle = \varepsilon = \pm 1$ we obtain the equation (2) with $C = \varepsilon/\lambda^2$. Furthermore, $k' = 0$ yields $\lambda_2 = 0$ and so $\lambda\tau = 0$, which can not be hold by definition of generalized helix.

Conversely, let us suppose that the curvatures of γ satisfiy the equation (2), and consider the vector field along γ defined by (4), where $\lambda = \sqrt{|C|}$. Then is easy to prove that υ is a constant vector and $\langle L, \upsilon\rangle = \lambda$. $\square$

3 The solving natural equations problem

This classical problem concerns with finding out a fully immersed parametrized curve which has as Cartan curvatures a pair of given functions. As we are now dealing with helices, we have to distinguish according to whether the axis is non-null or null.

3.1 Non-null axis

Let $\gamma : I \longrightarrow \mathbb{L}^4$ be a null Cartan curve and let $\upsilon \neq 0$ be a unit constant vector (spacelike or timelike), that is, $\langle \upsilon, \upsilon \rangle = \varepsilon = \pm 1$. Let Σ denote the hyperplane orthogonal to υ, P the projection map onto the hyperplane Σ and $\bar{\beta} = P(\gamma)$. Following the ideas of [5], we deduce that $\bar{\beta}$ is a spacelike (resp. timelike) curve provided that υ is timelike (resp. spacelike).

Let $\beta : J \longrightarrow \Sigma$ be the arc-length parametrization of $\bar{\beta}$ with curvature functions $\tilde{k}$ and $\tilde{\tau}$.

Theorem 3.1. *Let γ be a null Cartan curve in $\mathbb{L}^4$, $\upsilon \neq 0$ a constant unit vector, Σ the hyperplane orthogonal to υ in $\mathbb{L}^4$ and $\bar{\beta}$ the projection of γ onto Σ. Then γ is a generalized helix with axis υ if and only if $\bar{\beta}$ is a curve of Σ with constant curvature and non-constant torsion.*

As a consequence we have

Theorem 3.2. *Let γ be a null Cartan curve in $\mathbb{L}^4$. Then γ is a generalized helix with non-null axis if and only if it is a null geodesic of a Lorentzian cylinder constructed on a spacelike curve in $\mathbb{R}^3$ or a timelike curve in $\mathbb{L}^3$ with constant curvature and non-constant torsion.*

3.2 Null axis

There exists a close relation between timelike generalized helices and null Cartan generalized helices having both the same null axis. Let us begin doing a short study of non-degenerate generalized helices.

Let $\beta : J \longrightarrow \mathbb{L}^4$ be a non-degenerate curve with Frenet frame $\{\ell = \beta'(s), n_1, n_2, n_3\}$, where $\langle \ell, \ell \rangle = \varepsilon_0$ and $\langle n_i, n_i \rangle = \varepsilon_i$. We will say that β is a *generalized helix* if there exist a constant vector $\upsilon \neq 0$ such that the product $\langle \ell(s), \upsilon \rangle \neq 0$ is constant.

Theorem 3.3. *Let β be a non-degenerate curve in $\mathbb{L}^4$. Then β is a generalized helix if and only if its curvatures functions satisfy the following differential equation*

$$(\phi')^2 = \varepsilon_0 \varepsilon_1 \tilde{k}_3^2 \left(\phi^2 + C \right), \qquad \phi' \neq 0, \tag{5}$$

where $\phi = \tilde{k}_1 / \tilde{k}_2$ and C is a constant.

The proof of this theorem is quite similar to that of Theorem 2.2.

On the other hand, let $\beta : I \longrightarrow \mathbb{L}^4$ be a timelike generalized helix with null axis υ. We choose υ as

$$2\upsilon = \ell + \frac{\tilde{k}_1}{\tilde{k}_2} n_2 + \frac{1}{\tilde{k}_3} \frac{d}{ds} \left(\frac{\tilde{k}_1}{\tilde{k}_2} \right) n_3,$$

satisfying that $\langle \ell, \upsilon \rangle = -\frac{1}{2}$. Then, the surface S locally parametrized by $X(s, \omega) = \beta(s) + \omega \upsilon$ is a Lorentzian surface in $\mathbb{L}^4$. The null geodesics of S can be parametrized by $\bar{\gamma}(s) = \beta(s) - (s + \sigma)\upsilon$, where σ is a constant. Let t be the pseudo-arc parameter of $\bar{\gamma}$ (as a curve in $\mathbb{L}^4$) and write $\gamma(t(s)) = \beta(s) - (s + \sigma)\upsilon$. An easy computation yields $\sqrt{\tilde{k}_1(s)}\, \langle L, \upsilon \rangle = -1/2$. Therefore, γ is a Cartan generalized helix if and only if $\tilde{k}_1$ is constant. Furthermore, the Cartan curvatures of γ and those of the timelike generalized helix β are related by

$$k(t(s)) = \frac{\tilde{k}_2(s)^2 - \tilde{k}_1^2}{2\tilde{k}_1}, \qquad \tau(t(s))^2 = \frac{\tilde{k}_2(s)^2 \tilde{k}_3(s)^2 + \tilde{k}_2'(s)^2}{\tilde{k}_1^2}.$$

Then we have proved the following

Theorem 3.4. *Let γ be a null Cartan curve in $\mathbb{L}^4$. Then γ is a generalized helix with null axis if and only if it is a geodesic of a Lorentzian ruled surface whose directrix is a timelike generalized helix in $\mathbb{L}^4$ (with null axis and constant first curvature) and whose rulings have the axis direction.*

4 The null Betchov-Da Rios equation

In this section we present a similar soliton equation to the *Betchov-Da Rios or localized induction equation (LIE)*, such that the evolving curves are null ones. We are interested in finding out evolution equations where the null helices provide soliton solutions.

Definition 4.1. *Let $\gamma(t, s)$ be the evolution equation by null Cartan curves, in a n-dimensional Lorentzian space $\mathbb{L}^n$, of $\gamma(t) = \gamma(t, 0)$, where t is the pseudo-arc parameter. The equation*

$$\frac{\partial \gamma}{\partial s} = \frac{\partial^2 \gamma}{\partial t^2} \wedge \cdots \wedge \frac{\partial^n \gamma}{\partial t^n} \tag{6}$$

will be called the null Betchov-Da Rios equation or null localized induction equation (NLIE).

Observe that, from a geometric point of view, $\gamma(t,s)$ is a parametrized surface, then $V(t,s) = \frac{\partial\gamma}{\partial s}(t,s)$ is the *variational vector field* on the surface. The classical Betchov-Da Rios equation is usually written as $\frac{\partial\gamma}{\partial s} = kB$ (then known as filament equation), that is, in terms of the Frenet frames and curvature functions of the state curves. We can do the same for NLIE in the three, four and five dimensional cases, respectively, obtaining

$$\frac{\partial\gamma}{\partial s} = -(kL+N), \qquad \frac{\partial\gamma}{\partial s} = k_2(k_1L+N) - k_1'W_2,$$
$$\frac{\partial\gamma}{\partial s} = -k_2^2k_3(k_1L+N) - k_1'k_2k_3W_2 - (k_2^3 - k_2k_1'' + k_2'k_1')W_3.$$

As a consequence it is easy to see the following

Proposition 4.2. *Let $\gamma(t,s)$ be the evolution equation by null Cartan curves in $\mathbb{L}^n$ and let $\{L, W_1, N, W_2, \ldots, W_{n-2}\}$ be the Cartan frame associated to any state curve of the evolution. Then the variational vector field V of $\gamma(t,s)$ satisfies the following equations:*

$$\text{(A1)}\ \langle\nabla_L V, L\rangle = 0, \qquad \text{(A2)}\ \left\langle\nabla_L^2 V, W_1\right\rangle = 0.$$

Then the solutions of NLIE satisfy the following conditions

	3-dimensional case	4-dimensional case	5-dimensional case
(A1)	always	$\frac{\partial\tau}{\partial t} = 0$	$\frac{\partial}{\partial t}(k_2^2k_3) = 0$
(A2)	$\frac{\partial k}{\partial t} = 0$	always	$\frac{\partial k_1}{\partial t} = 0$

Finally, we present some interesting examples in low dimensions.

Example 4.1. Let $\gamma : I \longrightarrow \mathbb{L}^3$ a null Cartan helix with constant curvature k and Cartan frame $\{L, W, N\}$, and consider

$$\gamma(t,s) = \gamma(t) - s(kL(t) + N(t)),$$

which means that $\gamma(t)$ evolves by translation along its axis. It is easy to see that $\gamma(t,s)$ is a solution of the NLIE in $\mathbb{L}^3$.

Example 4.2. Let γ be a null Cartan generalized helix in $\mathbb{L}^4$ with Cartan frame $\{L, W_1, N, W_2\}$ and constant second curvature τ. Its axis was given in (4), which now is written as

$$\upsilon = -\frac{\lambda}{\tau}\left(\tau(k(t)L(t) + N(t)) - k'(t)W_2(t)\right),$$

where $\lambda = \langle L, \upsilon \rangle$ and $k(t)$ satisfies (2). Then we obtain that

$$k(t) = \frac{1}{2}\left(\tau^2(t+\sigma)^2 - \frac{\varepsilon}{\lambda^2}\right),$$

σ being a constant, and $\gamma(t,s) = \gamma(t) - (\tau/\lambda)s\upsilon$ is a solution of the NLIE in $\mathbb{L}^4$.

Example 4.3. Let γ be a null Cartan helix in $\mathbb{L}^5$ with Cartan frame $\{L, W_1, N, W_2, W_3\}$. Its axis is given by $\upsilon = k_3k_2(k_1L + N) + W_3$, where k_1, k_2 and k_3 are constant. Then $\gamma(t,s) = \gamma(t) - sk_2^3\upsilon$ is a simple solution of the NLIE.

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