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Geometry of Relativistic Particles with Torsion in 4-dimensional Spaces

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Abstract

We study particle paths determined by an action which is linear in the torsion. The Euler-Lagrange equations associated are solved obtaining the curvatures of the paths. We also integrate the corresponding Frenet equations and find out explicitly the trajectories.

Key words: torsion, spinning massless and massive particles, moduli spaces of solutions.
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1 Introduction

The model of a particle with torsion in (2+1)-Minkowski space has been deeply investigated. For example, it was shown that, at classical level, the squared mass of the system is restricted from above and that, besides the massive solutions of the equations of motion, the model must also have massless and tachyonic solutions, [1]. The same author obtains in [2] the classical equations of motion of the model whose Lagrangian is $f = -m + \alpha\tau$. This model of relativistic particle with torsion (whose action appears in the Bose-Fermi transmutation mechanism) is also studied in [3], where it is canonically quantized in the (2+1)-Minkowski and 3-Euclidean spaces.

In d=(3+1) there are also some geometrical models of relativistic particles that involve the curvatures of the particle path. In these cases, it seems interesting to investigate the models and establish which of them have a maximal symmetry, [4].

Recently, in [5] the authors consider a relativistic particle whose dynamics is determined by an action depending on the torsion k_2. The Euler-Lagrange equations are obtained but unfortunately, as the authors pointed out, these higher order differential equations do not appear to be tractable in general.

The main goal of this work is to solve the Euler-Lagrange equations when the action depends linearly on the torsion and the background is 4-dimensional and flat. Furthermore, we are able to integrate the Frenet-Serret equations of the critical curves obtaining explicit expressions for its coordinates.

This paper advances some results contained in [6].

2 The model and the equations of motion

Let $\mathbb{R}^4_\nu$ be the 4-dimensional pseudo-Euclidean space with background gravitational field $\mathrm{d}s^2 = \langle , \rangle$ given by $\mathrm{d}s^2 = -\sum_{i=1}^{\nu} \mathrm{d}x_i^2 + \sum_{i=\nu+1}^{4} \mathrm{d}x_i^2$, where (x_1, x_2, x_3, x_4) denote the usual rectangular coordinates. As usual, let ∇ denote the Levi-Civita connection on $\mathbb{R}^4_\nu$.

A non-null differentiable curve $\gamma : [0, L] \to \mathbb{R}^4_\nu$ is said to be a Frenet curve if there exist functions $\{k_1, k_2, k_3\}$ and vector fields $\{T = \gamma', N_1, N_2, N_3\}$ along γ such that the following equations (called the Frenet-Serret equations) hold:

$$\begin{aligned}
\nabla_T T &= \varepsilon_1 k_1 N_1, \\
\nabla_T N_1 &= -\varepsilon_0 k_1 T + \varepsilon_2 k_2 N_2, \\
\nabla_T N_2 &= -\varepsilon_1 k_2 N_1 + \varepsilon_3 k_3 N_3, \\
\nabla_T N_3 &= -\varepsilon_2 k_3 N_2.
\end{aligned}$$

Here $\varepsilon_0 = \langle T, T \rangle$ and $\varepsilon_i = \langle N_i, N_i \rangle$ for $i = 1, 2, 3$. It should be noted that curves in Euclidean space $\mathbb{R}^4$ and time-like curves in Lorentzian space $\mathbb{L}^4$ are always Frenet curves.

We are now to investigate Lagrangians which are linear functions on the torsion of the relativistic particle. The space Λ of elementary fields in this model is that of Frenet curves fulfilling given first order boundary data to drop out the boundary terms which appear in the first variation formula of the action. We consider the action $\mathcal{L} : \Lambda \to \mathbb{R}$ given by

$$\mathcal{L}(\gamma) = \int_\gamma (pk_2 + q) ds,$$

p and q being constant.

By using a standard argument involving some integrations by parts we find, after a long and messy computation using the Frenet equations, the first-order variation of this action, which is given by

$$\mathcal{L}'(0) = [\mathcal{B}(\gamma, W)]_0^L - \int_\gamma \langle \nabla_T P, W \rangle \, ds, \tag{1}$$

where the vector field P reads

$$P = \varepsilon_0 q T + \varepsilon_0 p (k_1 - \varepsilon k_3 \varphi) N_2 + \varphi' N_3$$

and the boundary term is given by

$$\mathcal{B}(\gamma, W) \;=\; \left\langle \nabla_T^2 W, \varepsilon_1 \frac{p}{k_1} N_2 \right\rangle + \langle \nabla_T W, -\varepsilon_1 \varepsilon_3 p \varphi N_3 \rangle + \langle W, P \rangle \,,$$

W standing for a generic variational vector field along γ, $\varepsilon = \varepsilon_0\varepsilon_1\varepsilon_2\varepsilon_3$ and $\varphi = k_3/k_1$. To drop out the term $[\mathcal{B}(\gamma, W)]_0^L$, we must consider curves with the same endpoints and having the same Frenet frame on them. Then we have obtained the following result.

The trajectory $\gamma \in \Lambda$ is the worldline of a relativistic particle in our model if and only if the vector field P is constant along γ.

A straightforward computation from the above equations shows that P is constant if and only if the following equations of motion hold:

$$p k_2 (1 - \varepsilon \varphi^2) - q \;=\; 0, \tag{2}$$

$$k_1' (1 - \varepsilon \varphi^2) - 3 \varepsilon k_1 \varphi \varphi' \;=\; 0, \tag{3}$$

$$-\varepsilon_2 \varepsilon_3 \varphi'' - \varepsilon k_1^2 \varphi (1 - \varepsilon \varphi^2) \;=\; 0. \tag{4}$$

Before integrating these equations, we are going to present some easy consequences. First, let us consider the function $\Psi = k_1^2 (1 - \varepsilon \varphi^2)^3$. Then from Eq. (3) we find $\Psi' = 0$, so that $\Psi = A$ is constant. Second, assume without loss of generality that $k_1 \neq 0$ (otherwise, γ would be a geodesic). If $A = 0$ then $\varphi^2 = 1$, so that $k_3 = \pm k_1$, and from equation (2) we deduce that $q = 0$. Note that this situation can not appears in the Lorentzian background. The most interesting case happens when $A \neq 0$, which will be solved in the next section.

3 The solutions of the equations of motion

The main goal of this section is to integrate the motion equations of Lagrangians giving models for relativistic particles that involve linearly the torsion of the worldline.

Let Z_1 and Z_2 be constant vector fields and consider the vector field $W = \gamma \wedge Z_1 \wedge Z_2$, then the boundary term reads

$$\mathcal{B}(\gamma, W) = \langle (p\, N_1 \wedge N_2 - \varepsilon_1\varepsilon_3 p\varphi\, T \wedge N_3 + \gamma \wedge P) \wedge Z_1, Z_2 \rangle\, .$$

As P is a constant vector field, we obtain two constant vectors

$$\begin{aligned} Q &= p\, N_1 \wedge N_2 \wedge P - \varepsilon_1\varepsilon_3 p\varphi\, T \wedge N_3 \wedge P, \\ V &= p\, N_1 \wedge N_2 \wedge Q - \varepsilon_1\varepsilon_3 p\varphi\, T \wedge N_3 \wedge Q + \gamma \wedge P \wedge Q. \end{aligned}$$

Then $J = -\gamma \wedge P \wedge Q + V$ is a Killing vector field along γ. Furthermore, P, Q and J read

$$P = \varepsilon_0 q\, T + \varepsilon_0 p k_1 (1 - \varepsilon\varphi^2)\, N_2 + \varepsilon_1\varepsilon_3 p\varphi'\, N_3, \tag{5}$$

$$Q = -\varepsilon_0\varepsilon_1\varepsilon_3 p^2 \varphi'\, T + \varepsilon_0\varepsilon_3 p^2 \varphi k_1 (1 - \varepsilon\varphi^2)\, N_1 + \varepsilon_0\varepsilon_3 pq\, N_3, \tag{6}$$

$$J = p^2(-\varepsilon_3 q\, T - \varepsilon_0\varepsilon_1\varepsilon_2 p\varphi^2 k_1 (1 - \varepsilon\varphi^2)\, N_2 - \varepsilon_0\varepsilon_1 p\varphi' N_3), \tag{7}$$

and they can be interpreted as generators of the particle mass M and spin S, with the mass-shell condition and the Majorana-like relation between M and S given by $\langle P, P \rangle = M^2$ and $\langle P, J \rangle = MS$.

By using that P and Q are constant vector fields along γ, so that $\langle P, P \rangle = \varepsilon_P u^2$ and $\langle Q, Q \rangle = \varepsilon_Q v^2$ also are, we obtain the following two first integrals for φ:

$$\varepsilon_3 p^2 (\varphi')^2 + \varepsilon_0 q^2 + \varepsilon_2 p^2 k_1^2 (1 - \varepsilon\varphi^2)^2 = \varepsilon_P u^2, \tag{8}$$

$$p^2[\varepsilon_0 p^2 (\varphi')^2 + \varepsilon_1 \varphi^2 p^2 k_1^2 (1 - \varepsilon\varphi^2)^2 + \varepsilon_3 q^2] = \varepsilon_Q v^2. \tag{9}$$

Then, as A is not zero, we get

$$(\varphi')^2 = \frac{(\varepsilon_P u^2 - \varepsilon_0 q^2)(1 - \varepsilon\varphi^2) - \varepsilon_2 p^2 A}{\varepsilon_3 p^2 (1 - \varepsilon\varphi^2)}.$$

This ODE can be integrate and its solution reads

$$a\mathrm{E}\left(\arcsin\left(b\varphi\right), \frac{\varepsilon}{b^2}\right) = t + C_1,$$

where C_1 is an arbitrary constant, $a = \sqrt{\frac{\varepsilon_3 p^2}{\varepsilon(\varepsilon_P u^2 - \varepsilon_0 q^2)}}$, $b = \sqrt{\frac{\varepsilon(\varepsilon_P u^2 - \varepsilon_0 q^2)}{-\varepsilon_2 p^2 A + \varepsilon_P u^2 - \varepsilon_0 q^2}}$ and E stands for the elliptic function of second kind. From here and Eq. (2) we obtain the curvatures in terms of the function φ:

$$k_1^2 = \frac{A}{(1 - \varepsilon\varphi^2)^3}, \qquad k_2 = \frac{q}{p\,(1 - \varepsilon\varphi^2)},$$

$$k_3^2 = \frac{A\varphi^2}{(1 - \varepsilon\varphi^2)^3}.$$

4 Integration of the Frenet equations

As P and Q determine privileged directions, it is natural to introduce cylindrical coordinates in $\mathbb{R}^4_\nu$ such that P and Q are axes. Without loss of generality we can assume that P and Q are linearly independent and one of them is non-null; otherwise, we have $\varphi^2 = 1$, which can not be integrated. Then let $\Pi = P \wedge Q$ be the plane determined by P and Q. We can introduce orthonormal (or pseudo-orthonormal) coordinates (z_1, z_2) in Π such that P and Q are collinear with ∂_{z_1} and ∂_{z_2}, respectively. On the other hand, let Π^* a complementary plane, i.e. $\mathbb{R}^4_\nu = \Pi \oplus \Pi^*$ getting coordinates (z_3, z_4) on Π^* such that $\{\partial_{z_1}, \partial_{z_2}, \partial_{z_3}, \partial_{z_4}\}$ is an orthonormal (or pseudo-orthonormal) frame satisfying that $\partial_{z_4} = \partial_{z_1} \wedge \partial_{z_2} \wedge \partial_{z_3}$.

Let $\{R_\theta\}$ be the uniparametric group of rotations of $\mathbb{R}^4_\nu$ leaving invariant the plane Π (the expression of this group depends on the causal character of P and Q). Consider the parametrization Ψ of $\mathbb{R}^4_\nu$ given by

$$\Psi(z_1, z_2, r, \theta) = R_\theta(z_1\partial_{z_1} + z_2\partial_{z_2} + r\partial_{z_3})$$

that provides us a new coordinate system (z_1, z_2, r, θ) in $\mathbb{R}^4_\nu$. Note that the parameter θ can be chosen in such a way that $\gamma \wedge \partial_{z_1} \wedge \partial_{z_2} = \partial_\theta$.

By using the invariance under translations it is not difficult to see that the vector field J can be written as

$$J = c\partial_\theta - \varepsilon_Q v^2 H,$$

where $H = -\frac{\varepsilon_P}{u}\partial_{z_1}$ or $H = \partial_{z_3}$ according to P is non-null or null respectively, and c is a constant. Here we have used that $\langle P, J\rangle = -\langle Q, Q\rangle$.

The unit tangent vector reads $T = (z_1)_s\partial_{z_1} + (z_2)_s\partial_{z_2} + r_s\partial_r + \theta_s\partial_\theta$, and taking into account Eqs. (5)–(7) we can obtain three ordinary differential equations from the products $\langle T, P\rangle$, $\langle T, Q\rangle$ and $\langle T, J\rangle$. These equations jointly with the equation obtained from $\langle J, J\rangle$ allow us to determine γ.

We have to consider three cases: (i) P and Q are non-null; (ii) P is null and Q is non-null; (iii) P is non-null and Q is null. All of them can be solved following the method described above. For example, in case (i) we can take Π^* as the orthogonal complement $\Pi^\perp$ to Π and (r, θ) the polar coordinates in $\Pi^\perp$. Then (z_1, z_2, r, θ) are the cylindrical coordinates in the background space.

As $T = (z_1)_s\partial_{z_1} + (z_2)_s\partial_{z_2} + r_s\partial_r + \theta_s\partial_\theta$ we have

$$(z_1)_s = \varepsilon_P \frac{q}{u}, \qquad (z_2)_s = -\varepsilon_1\varepsilon_3\varepsilon_Q \frac{p^2\varphi'}{v}.$$

On the other hand, from $\langle J, J\rangle = \varepsilon_\theta r^2u^2v^2 + \varepsilon_P v^4/u^2$ and (7) we are able to find the radial function r. Finally, as we can compute $\langle J, T\rangle$ in two different ways, we deduce that $-\varepsilon_\theta uvr^2\theta_s - \varepsilon_P\varepsilon_Q(v^2/u)(z_1)_s = -\varepsilon_0\varepsilon_3 p^2 q$. Then we can easily integrate to get θ.

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