Classification of certain semi-Riemannian constant mean curvature surfaces

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1. Introduction. Let $x : M^n \longrightarrow \mathbb{R}^m$ be a Riemannian submanifold. The products of the coordinate functions can be organized to define a smooth map φ from M^n into the set $SM(m)$ of $(m \times m)$-real symmetric matrices defined by $\varphi = xx^t$. This map will be called the quadric representation of M^n. I. Dimitric, [4], made a nice study of φ pointing out that, in general, it is not an isometric immersion. This map was first considered by A. Ros, [5], M. Barros and B.Y. Chen, [2], and M. Barros and F. Urbano, [1], to distinguish minimal submanifolds in the sphere. Actually, they asked for the eigenvalue behaviour of the products of the Laplacian eigenfunctions. In a very recent paper, [3], M. Barros and O.J. Garay have shown that the Clifford torus and the totally geodesic 2-sphere are the only compact minimal surfaces in $\mathbb{S}^3$ whose quadric representations live minimally in a certain hyperquadric of $SM(4)$.

As for semi-Riemannian surfaces, we are interested in a more general problem: *classify CMC semi-Riemannian surfaces in the non-flat 3-dimensional semi-Riemannian space forms whose quadric representations into the set $SA(4,\nu)$ of selfadjoint matrices satisfies a certain Laplacian differential equation*. Notice that according to [4, Theorem 1] the flat case should be avoided. We have observed that, under Barros-Garay conditions, the quadric representation φ of a CMC semi-Riemannian surface in $\bar{M}_1^3(k)$, $k \neq 0$, satisfies the matricial Laplacian equation $\Delta\varphi = A * \varphi + B$ (see section 3 for the definition of the star product $*$). Then an interesting problem arises as follows: *could you characterize CMC semi-Riemannian surfaces into $\bar{M}_1^3(k)$ whose quadric representation satisfies that equation?*

It should be pointed out that, on one hand, we do not need the surface to be either compact or minimal. On the other hand, since the surface is now endowed with a semi-Riemannian metric, our problem naturally generalizes that of Barros-Garay. Therefore, it seems reasonable to hope for finding a richer class of CMC semi-Riemannian surfaces than in the Riemannian situation.

2. Set up. Let $\mathbb{R}^4_\nu$ be the pseudo-Euclidean 4-dimensional space endowed with the standard inner product of index ν given by $\langle a, b\rangle = a^tGb$, where $G = \mathrm{diag}[\delta_1, \delta_2, \delta_3, \delta_4]$, $\delta_i = \pm 1$, stands for the matrix of the metric with respect to the usual rectangular coordinates. Throughout this paper, vectors in $\mathbb{R}^4_\nu$ will be regarded as column matrices and $(\cdot)^t$ will denote the transpose matrix. As usual, let $\mathbb{S}^3_1 = \{x \in \mathbb{R}^4_1 : \langle x, x\rangle = 1\}$ and $\mathbb{H}^3_1 = \{x \in \mathbb{R}^4_2 : \langle x, x\rangle = -1\}$ be the unit pseudosphere and the unit pseudohyperbolic space, respectively, viewed as hypersurfaces of index one with constant sectional curvature $k = +1$ and $k = -1$, respectively. From now on, $\bar{M}_1^3(k)$ will denote $\mathbb{S}^3_1$ or $\mathbb{H}^3_1$ according to $k = 1$ or $k = -1$, and $\mathbb{R}^4_\nu$ the pseudo-Euclidean space where $\bar{M}_1^3(k)$ is lying.

Let $SA(4,\nu) = \{B \in \mathfrak{g}l(4,\mathbb{R}) : B^tG = GB\}$ be the set of selfadjoint endomorphisms of $\mathbb{R}^4_\nu$ equipped with the metric $g(B,C) = \frac{k}{2}\mathrm{trace}(BC)$. Let $f : \bar{M}_1^3(k) \longrightarrow SA(4,\nu)$ be the map defined by $f(x) = xx^tG$. It is easy to see that f is an isometric immersion, that is

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called the second standard immersion of $\bar{M}^3_1(k)$, and its second fundamental form $\bar{\sigma}$ is given by $\bar{\sigma}(X,Y) = (XY^t + YX^t)G - 2k\,\langle X,Y\rangle\, f(x)$, for any $x \in \bar{M}^3_1(k)$ and $X, Y \in T_x\bar{M}^3_1(k)$.

Given an isometric immersion $x : M^2_s \longrightarrow \bar{M}^3_1(k)$ of a semi-Riemannian surface M^2_s into $\bar{M}^3_1(k)$, the map $\varphi : M^2_s \longrightarrow SA(4,\nu)$ defined by $\varphi = f\circ x$ is also an isometric immersion that will be called the quadric representation of M^2_s. Then the mean curvature vector fields H_1 and H associate to the immersions x and φ, respectively, are related by the formula $H = (H_1x^t + xH_1^t)G + \sum_{i=1}^2 \varepsilon_i E_i E_i^t G - 2k\varphi$, where $\{E_1, E_2\}$ is an orthonormal frame field tangent to M^2_s and $\varepsilon_i = \langle E_i, E_i\rangle$, $i = 1, 2$.

3. First results. Let us suppose that the quadric representation $\varphi = (\varphi_{ij})$ of the isometric immersion $x : M^2_s \longrightarrow \bar{M}^3_1(k)$ satisfies the system of differential equations $\Delta\varphi_{ij} = a_{ij}\varphi_{ij} + b_{ij},$ for all i, j, for some real numbers a_{ij} and b_{ij}. From the definition of φ, it is easy to see that $a_{ij} = a_{ji}$ and $b_{ij} = \delta_i\delta_j b_{ji}$. Therefore the above conditions can be globally written as

$$\Delta\varphi = A * \varphi + B,$$

where $A = (a_{ij})$ is a symmetric matrix, $B = (b_{ij})$ is a selfadjoint endomorphism, and the star product $*$ associates to each pair of matrices $C = (c_{ij})$ and $D = (d_{ij})$ the matrix $C*D = (c_{ij}d_{ij})$.

Lemma 11 *Let $x : M^2_s \longrightarrow \bar{M}^3_1(k)$ be an isometric immersion whose quadric representation satisfies $\Delta\varphi = A*\varphi + B$. Then M^2_s is contained in the hyperquadric defined by $\langle Bx, x\rangle = c$, for some real constant c.*

In what follows, let N be a unit vector field normal to M^2_s in $\bar{M}^3_1(k)$, with $\varepsilon = \langle N, N\rangle$, and let S be the shape operator associated to N. The following lemma gives an accurate description of the endomorphism B.

Lemma 12 *Let $x : M^2_s \longrightarrow \bar{M}^3_1(k)$ be a surface contained in the hyperquadric defined by $\langle Bx, x\rangle = c$, where $B \in SA(4,\nu)$ and $c \in \mathbb{R}$. Then there exists a smooth function β on M^2_s such that*

$$\begin{aligned} Bx &= \beta N + kcx, & (2)\\ BN &= \varepsilon\,\mathrm{grad}(\beta) + (\mathrm{trace}(B) - 3kc + \beta\mathrm{trace}(S))N + k\varepsilon\beta x, & (3)\\ BX &= -\beta SX + kcX + X(\beta)N, & (4) \end{aligned}$$

where $\mathrm{grad}(\beta)$ stands for the gradient of β and X is a tangent vector field.

It is worth pointing out that the function β contains a nice geometric information about the surface M^2_s. In the following proposition we go further into the shape of the surface provided that β is a constant.

Proposition 13 *Let $x : M^2_s \longrightarrow \bar{M}^3_1(k)$ be a constant mean curvature surface contained in the hyperquadric defined by $\langle Bx, x\rangle = c$, where $B \in SA(4,\nu)$ and $c \in \mathbb{R}$. Assume that the function β given in Lemma 12 is constant. Then:*
1) *If $\beta \neq 0$, M^2_s is a flat isoparametric surface.*
2) *If $\beta = 0$ and $\mathrm{trace}(B) \neq 4kc$, M^2_s is a totally geodesic surface.*

The adjoint table explicitly exhibits certain families of surfaces in $\mathbb{S}^3_1$ and $\mathbb{H}^3_1$ satisfying equation (1) and in his turn they will support the classification we are looking for.

Equation	Surface	A	B	a, b, c
$x_1 = \rho$	$\mathbb{S}^2(r) \subset \mathbb{S}^3_1$ $r = \sqrt{1+\rho^2}$	$\begin{pmatrix} 0 & a & a & a \\ a & b & b & b \\ a & b & b & b \\ a & b & b & b \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$	
$x_4 = \rho$	$\mathbb{H}^2(-r) \subset \mathbb{S}^3_1$ $r = \sqrt{\rho^2-1}$	$\begin{pmatrix} -b & -b & -b & -a \\ -b & -b & -b & -a \\ -b & -b & -b & -a \\ -a & -a & -a & 0 \end{pmatrix}$	$\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$a = 2/r^2$ $b = 6/r^2$
$x_4 = \rho$	$\mathbb{S}^2_1(r) \subset \mathbb{S}^3_1$ $r = \sqrt{1-\rho^2}$	$\begin{pmatrix} b & b & b & a \\ b & b & b & a \\ b & b & b & a \\ a & a & a & 0 \end{pmatrix}$	$\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	
$x_1 = \rho$	$\mathbb{S}^2_1(r) \subset \mathbb{H}^3_1$ $r = \sqrt{\rho^2-1}$	$\begin{pmatrix} 0 & a & a & a \\ a & b & b & b \\ a & b & b & b \\ a & b & b & b \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$	
$x_1 = \rho$	$\mathbb{H}^2(-r) \subset \mathbb{H}^3_1$ $r = \sqrt{1-\rho^2}$	$\begin{pmatrix} 0 & -a & -a & -a \\ -a & -b & -b & -b \\ -a & -b & -b & -b \\ -a & -b & -b & -b \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$	$a = 2/r^2$ $b = 6/r^2$
$x_4 = \rho$	$\mathbb{H}^2_1(-r) \subset \mathbb{H}^3_1$ $r = \sqrt{1+\rho^2}$	$\begin{pmatrix} -b & -b & -b & -a \\ -b & -b & -b & -a \\ -b & -b & -b & -a \\ -a & -a & -a & 0 \end{pmatrix}$	$\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	
$-x_1^2 + x_2^2 = r^2$	$\mathbb{S}^1_1(r) \times \mathbb{S}^1(\sqrt{1-r^2}) \subset \mathbb{S}^3_1$	$\begin{pmatrix} a & a & b & b \\ a & a & b & b \\ b & b & c & c \\ b & b & c & c \end{pmatrix}$	$\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$	$a = 4/r^2$ $c = 4/(1-r^2)$
$-x_1^2 + x_2^2 = -r^2$	$\mathbb{H}^1(-r) \times \mathbb{S}^1(\sqrt{1+r^2}) \subset \mathbb{S}^3_1$	$\begin{pmatrix} a & a & b & b \\ a & a & b & b \\ b & b & c & c \\ b & b & c & c \end{pmatrix}$	$\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$	$a = -4/r^2$ $c = 4/(1+r^2)$
$-x_1^2 - x_2^2 = -r^2$	$\mathbb{H}^1_1(-r) \times \mathbb{S}^1(\sqrt{r^2-1}) \subset \mathbb{H}^3_1$	$\begin{pmatrix} a & a & b & b \\ a & a & b & b \\ b & b & c & c \\ b & b & c & c \end{pmatrix}$	$\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$	$a = -4/r^2$ $c = 4/(r^2-1)$
$-x_1^2 + x_3^2 = r^2$	$\mathbb{S}^1_1(r) \times \mathbb{H}^1(-\sqrt{1+r^2}) \subset \mathbb{H}^3_1$	$\begin{pmatrix} a & b & a & b \\ b & c & b & c \\ a & b & a & b \\ b & c & b & c \end{pmatrix}$	$\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$	$a = 4/r^2$ $c = -4/(1+r^2)$
$-x_1^2 + x_3^2 = -r^2$	$\mathbb{H}^1(-r) \times \mathbb{H}^1(-\sqrt{1-r^2}) \subset \mathbb{H}^3_1$	$\begin{pmatrix} a & b & a & b \\ b & c & b & c \\ a & b & a & b \\ b & c & b & c \end{pmatrix}$	$\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$	$a = -4/r^2$ $c = -4/(1-r^2)$

In the last five examples, the constant b is given by $b = (a+c)/4$.

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4. Main result. Before to get down to the general situation, it is worthwhile to pay attention to the following interesting case.

Proposition 14 *Let M_s^2 be a surface in $\bar{M}_1^3(k)$ whose quadric representation satisfies $\Delta\varphi = \lambda\varphi + B$. Then M_s^2 is totally geodesic.*

Now we are ready to show our first main theorem.

Theorem 15 *Let $x : M_s^2 \longrightarrow \bar{M}_1^3(k)$ be a constant mean curvature isometric immersion whose quadric representation satisfies $\Delta\varphi = A * \varphi + B$. Then M_s^2 is an isoparametric surface.*
Let M_s^2 be a semi-Riemannian constant mean curvature surface in $\bar{M}_1^3(k)$ whose quadric representation satisfies $\Delta\varphi = A * \varphi + B$. Then M_s^2 is an open piece of one of the following surfaces:
1) $\mathbb{S}^2(r)$, $\mathbb{S}_1^2(r)$, $\mathbb{H}^2(-r)$, $\mathbb{H}_1^2(-r)$.
2) $\mathbb{H}^1(-r)\times\mathbb{S}^1(\sqrt{1+r^2})$, $\mathbb{S}_1^1(r)\times\mathbb{S}^1(\sqrt{1-r^2})$, $\mathbb{H}_1^1(-r)\times\mathbb{S}^1(\sqrt{r^2-1})$, $\mathbb{S}_1^1(r)\times\mathbb{H}^1(-\sqrt{1+r^2})$, $\mathbb{H}^1(-r)\times\mathbb{H}^1(-\sqrt{1-r^2})$.

Remark 16 Obviously, the same computations work when the ambient space is a non-flat Riemannian space form $\mathbb{S}^3$ or $\mathbb{H}^3$. In both cases, the equation (1) characterizes the totally umbilical surfaces and Riemannian standard products.

This theorem allows us to distinguish minimal surfaces in $\bar{M}_1^3(k)$ via its quadric representation. More precisely, we have the following consequences.

Corollary 17 *Let M_s^2 be a minimal surface in $\bar{M}_1^3(k)$ whose quadric representation satisfies $\Delta\varphi = A * \varphi + B$. Then M_s^2 is totally geodesic or an open piece of one of the following products:* $\mathbb{S}_1^1(\sqrt{2}/2)\times\mathbb{S}^1(\sqrt{2}/2)$, $\mathbb{H}^1(-\sqrt{2}/2)\times\mathbb{H}^1(-\sqrt{2}/2)$.

The Clifford torus characterization found by Barros-Garay in [3] can be directly obtained from Theorem 15 as follows.

Corollary 18 *Let M^2 be a compact, minimal surface in $\mathbb{S}^3$ whose quadric representation is minimal in some hyperquadric of $SA(4,0)$. Then M^2 is totally geodesic or the Clifford torus.*

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