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NORMAL FORM OF THE NK-CURVATURE OPERATORS

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0. INTRODUCTION

Let M be a Riemannian manifold and M_m the tangent space at each point $m \in M$. The sectional curvature $r(P)$ of a plane $P \subset M_m$ is defined, [13], as the Gauss curvature at m of the surface $\mathrm{Exp}_m(P)$. So, r can be considered as a function on the Grassmann manifold of all planes of M_m, $G(2, M_m)$, and the Riemannian curvature tensor R can be defined in terms of the Plücker coordinates of $G(2, M_m)$ by polarization of r, [10]. The purpose of this paper is to determine R by analizing the critical point behaviour of the sectional curvature function r_R for a special class of curvature operators.

If $S : \mathbb{R}^n \to \mathbb{R}^n$ is a symmetric operator, one can define $\sigma_S : S^{n-1} \to \mathbb{R}$, by $\sigma_S(w) = \langle Sw, w \rangle$, which is projected at $\hat{\sigma}_S : \mathbb{R}P^{n-1} \to \mathbb{R}$. It is well known that w is a critical point of σ_S if, and only if, w is an eigenvector of S; and its critical value $\sigma_S(w)$ is exactly the correspondent eigenvalue. Algebraicly, if V is a real metric vector space, by a curvature operator on V we mean a symmetric operator R on $\Lambda^2(V)$ and the function attached according to the above way is the sectional curvature function r_R. Now, the points and critical values of r_R play the rôle of the eigenvectors and eigenvalues of S. The fundamental question is: "How and at what rate do the critical points and values of r_R determine R?".

On the whole, if R is a curvature operator verifying the first Bianchi identity, it is said that R has a normal form if the critical points and values of r_R determine R. In [11], it is shown that any Einstein curvature operator in dimension four has a normal form and this fact depends strongly of the existence of a determined number of critical points of the sectional curvature function. For Kaehler curvature operators, that is, curvature operators verifying the first curvature condition, it is possible to consider those for which the curvature function is a Morse function; this allows to establish lower bounds on the number of distinct critical points, [6], [7].

Since the abstract treatment of the curvature operators has take geometric sense when V is, at each point, the tangent space of a Riemannian manifold, the question naturally arises when V is identified with the tangent space at each point of a Nearly Kaehler manifold, [5], [7]. Since the Riemann curvature tensor of such manifolds verifies the second curvature condition, one can define the space of all curvature operators on V verifying the above condition of curvature. Using similar technics to [7], one determines normal forms in dimensions four and six. All geometric objects will be considered of class C^∞.

1. PRELIMINARIES

If $(V, J, \langle\,,\rangle)$ is an hermitian complex vector space of real dimension $2n$, by a curvature operator on V one refers to a symmetric operator R on $\Lambda^2(V)$. If, also, R verifies the first Bianchi identity, it is said that R *is a Riemann curvature operator* and it is denoted by $\mathscr{R}(2n)$ the metric vector space of all curvature operators on V. It is defined an element of $\mathscr{R}(2n)$, also represented by J, by $J(x \wedge y) = Jx \wedge Jy$, for all $x, y \in V$.

Identifying the Grassmann manifold $G(2, V)$ of the planes of V with the space of the unitary decomposable bivectors of $\Lambda^2(V)$, for each R of $\mathscr{R}(2n)$, one defines the curvature function associated to R, $r_R : G(2, V) \to \mathbb{R}$, by $r_R(P) = \langle R(P), P\rangle$. Since V is isomorphic to $\mathbb{C}^n$, by abuse of notation, when it is convenient, it will be written $\Lambda^2(\mathbb{C}^n)$ by $\Lambda^2(V)$ and $G(2, \mathbb{C}^n)$ or $G(2, 2n)$ by $G(2, V)$. Furthermore, by $G(2, 2n)^J$ we denote the holomorphic planes of $G(2, 2n)$. We shall frequently identify $\Lambda^2(V)$ and $o(2n)$. As $u(n) = \{M \in o(2n)/JM = M\}$. if $I \in u(n)$ is such that $JI = I$ and $\{v_i, v_{i*}\}$ is an unitary basis of $\mathbb{C}^n$, $I = \sum_{i=1}^{n} v_i \wedge v_{i*}$, where $v_{i*} = Jv_i$. If $P \in G(2, 2n)$, choosing an orthonormal base $\{v_\alpha\}$ of $\mathbb{C}^n$, P is holomorphic if, and only if $\langle P, I\rangle = \pm 1$.

Definition 1.1. $R \in \mathscr{R}(2n)$ is called a *Nearly Kaehler curvature operator* or NK-curvature operator if it satisfies the second curvature condition; that is,

$$R_{xyzw} = R_{JxJyzw} + R_{JxyJzw} + R_{xJyJzw}$$

for $x, y, z, w \in V$. It will denote by $\mathscr{N}\mathscr{K}(n)$ the set of such curvature operators, which with the restriction of the inner product is a metric vectorial subspace of $\mathscr{R}(2n)$.

Like [4], it will be useful to consider the tensor $\lambda^R = R - RJ$, such that if $\lambda^R = 0$, the space $\mathscr{N}\mathscr{K}(n)$ is reduced to the space of Kaehler curvature operators.

2. TYPES OF CRITICAL PLANES

Next, one can think about $G(2, V)$ as a complex hypersurface of $\mathbb{C}P^{n-1}$. This fact allows, using the Lagrange multipliers, to give an algebraic characterization of the critical points of the curvature function r_R for a curvature operator $R \in \mathscr{R}(2n)$. In this way we have the following

Proposition 2.1. *If* $R \in \mathscr{N}\mathscr{K}(n)$, *any critical plane of* $r_{R|G(2,2n)^J}$ *is a critical plane of* r_R.

Proof. Choosing an unitary basis $\{v_\alpha\}$, $\alpha = 1, 1^*, \ldots, n, n^*$, such that $P = v_1 \wedge v_{1^*}$ is a critical plane of $r_{R|G(2,2n)J}$, one gets

$$0 = \frac{d}{dt}\Big|_{t=0} r_R((\cos tv_1 + \sin tv_\alpha) \wedge (\cos tv_{1^*} + \sin tv_{\alpha^*}))$$

for $\alpha = 2, 2^*, \ldots, n, n^*$. It follows that $R_{11^*1\alpha^*} = 0$, $\alpha = 2, 2^*, \ldots, n, n^*$ and $R_{11^*1^*\alpha} = 0$. By Proposition 2.2 in [7], $P = v_1 \wedge v_{1^*}$ is a critical plane of r_R.

In [2] it is shown that if S is a Kaehler curvature operator, its curvature function r_S achieves the maximum value on the holomorphic planes. It seems reasonable to establish the following

Conjecture. *If the holomorphic sectional curvature of a Nearly-Kaehler manifold is non-negative, then at each point the sectional curvature achieves the maximum value on the holomorphic planes.*

First, one gives an example where this conjecture is verified. We denote by M the naturally reductive homogeneous space $U(3)/(U(1) \times U(1) \times U(1))$. It is known, [1], with the complex structure given by

$$x = \begin{pmatrix} 0 & a_{12} & a_{13} \\ -\bar{a}_{12} & 0 & a_{23} \\ -\bar{a}_{13} & -\bar{a}_{23} & 0 \end{pmatrix} \to Jx = \begin{pmatrix} 0 & ia_{12} & -ia_{13} \\ i\bar{a}_{12} & 0 & ia_{23} \\ i\bar{a}_{13} & i\bar{a}_{23} & 0 \end{pmatrix},$$

M is a Nearly Kaehler manifold non-Kaehler, [5]. From [8],

$$R_{xyxy} = \tfrac{1}{4}\langle [x, y]_m, [x, y]_m \rangle + \langle [x, y]_k, [x, y]_k \rangle .$$

An easy calculation shows that $[x, Jx]_m = 0$ and

$$[x, Jx]_k = \begin{pmatrix} 2i(A_{12} - A_{13}) & 0 & 0 \\ 0 & 2i(-A_{12} - A_{13}) & 0 \\ 0 & & 2i(A_{12} - A_{23}) \end{pmatrix}$$

where $A_{ij} = a_{ij}\bar{a}_{ij} \in \mathbb{R}^+$. Hence the function

$$H(x) = \frac{R_{xJxxJx}}{\|x \wedge Jx\|^2} = 2 - \frac{6(A_{12}A_{13} + A_{12}A_{23} + A_{13}A_{23})}{(A_{12} + A_{13} + A_{23})^2}$$

is bounded by $0 \leq H(x) \leq 2$.

Choosing the orthonormal base of M_m

$$v_1 = 1/\sqrt{2} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad v_2 = 1/\sqrt{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad v_3 = 1/\sqrt{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$v_{1^*} = 1/\sqrt{2} \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad v_{2^*} = 1/\sqrt{2} \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad v_{3^*} = 1/\sqrt{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}$$

it is easily proved that $R_{ijij} = \frac{1}{8}(j + i)$ and $R_{ii^*ii^*} = 2$.

It can be checked directly that, for any given vectors $x, y \in M_m$, $r_R(x \wedge y) < 2$.

Next, one analyzes the non-holomorphic critical planes, because they provide the greatest information about R. We shall denote by $G(2, \mathbb{C}^n) - G(2, \mathbb{C}^n)^J$ the manifold of such planes. For any non-holomorphic plane Q one can choose an unitary base $\{v_\alpha\}$ of V such that $Q = av_1 \wedge v_{1*} + bv_1 \wedge v_2\,(b \neq 0)$ and $U = Q + JQ/|Q + JQ| \in$ $\in u(n)$.

Proposition 2.2. *Given the set*

$$\Sigma = \{U \in u(n)/|U| = 1,\ \mathrm{rank}_R\, U = 4,\ \det_C U > 0\},$$

it is defined a map $F : G(2, \mathbb{C}^n) - G(2, \mathbb{C}^n)^J \to \Sigma$ by $F(Q) = (Q + JQ)/|Q + JQ|$. Then, (i) *F is a submersion;* (ii) *if $Q_1, Q_2 \in F^{-1}(U)$ and $R \in \mathcal{N}\mathcal{K}(n)$, $r_R(Q_1) =$ $= r_R(Q_2)$.*

Note that, from (ii), $r_{R|G(2,\mathbb{C}^n)-G(2,\mathbb{C}^n)^J}$ projects to $\sigma_R : \Sigma \to \mathbb{R}$. As r_R and F are both real differentiable, so is σ_R.

Using similar techniques to those in [7], the main result follows from Lemma (4.4).

Corollary 2.3. *Let $R \in \mathcal{N}\mathcal{K}(n)$ and $P \in G(2, \mathbb{C}^n)$. If P is a non-holomorphic critical plane of r_R, so is any $Q \in F^{-1}\, F(P)$. Moreover, P is a critical plane of r_R if, and only if, $F(P)$ is a critical plane of σ_R.*

In low dimensions a critical point of a Nearly-Kaehler curvature function has a behaviour very close to an eigenvector of R as it is shown at the following

Theorem 2.4. ([3]). *Let $R \in \mathcal{N}\mathcal{K}(n)$ be Riemann, let P be an holomorphic critical plane of r_R with critical value A and let Q be a non-holomorphic critical plane with critical value B. Then,*

(1) *If $n = 2$, $R(P) = AP + A' * P$;*

$$R(Q) = B(Q + JQ - \langle Q, I\rangle I) + \lambda^R_Q * (Q).$$

(2) *If $n = 3$, there are holomorphic planes P', P'' such that P, P', P'' are mutually orthogonal and $R(P) = AP + A'P' + A''P''$; and*

$$R(Q) = B(Q + JQ - \langle Q, I\rangle I_Q) + \lambda^R_Q *_Q (Q) + B'(I - I_Q),$$

where $*$ is the Hodge operator, $*_Q = *_{|\Lambda^2(Q \wedge JQ)}$ and

$$I_Q = I + * \frac{Q \wedge JQ}{|Q \wedge JQ|}.$$

Remark 2.5. One writes λ^R_Q to mean the above mentioned tensor $\lambda^R = R - RJ$ when we take $Q = av_1 \wedge v_{1*} + bv_1 \wedge v_2$, which will be very useful through this paper.

It is known that the normal forms of a curvature operator depend strongly on the number of critical points of the associated curvature function. Thus, we will establish lower bounds on the number of distinct critical points of the sectional curvature.

Definition 3.1. For $R \in \mathcal{NK}(n)$, the function r_R is said *non-degenerate* if all holomorphic critical points of r_R and all critical points of σ_R are non-degenerate.

Theorem 3.2. *Let $A = \{R \in \mathcal{NK}(n) / r_R$ is non-degenerate$\}$. Then there exists an open dense subset S of $\mathcal{NK}(n)$ such that $S \subset A$.*

The proof is similar to that of the Kaehler case [7]. See [3] for a detailed account.

In [3], it is shown that such critical planes exist satisfying Theorem 3.2. Using this theorem one can give lower bounds on the number of critical points of r_R. By similar calculations to [7] it is obtained the following

Theorem 3.3. *If $R \in \mathcal{NK}(n)$ is non-degenerate, then*

(1) *if $n = 2$, r_R has at least four distinct critical planes, at least two of which are holomorphic.*

(2) *if $n = 3$, r_R has at least nine distinct critical planes, at least three of which are holomorphic.*

Now, we shall try to give additional conditions to $R \in \mathcal{NK}(2)$ such that the minimum number of critical points can be fixed more exactly and, also, to locate such points. So, for each non-holomorphic plane Q, according to Theorem 2.4, one can consider the curvature operator $R^Q = R - \lambda_Q^R *$, where $*$ is the Hodge operator. It is easy to show that R^Q is a Kaehler curvature operator and $b(R^Q) \neq 0$. Considering the set

$$\Delta = \{Q \in G(2, 2n) - G(2, 2n)^J / \langle Q, R^Q(I) \rangle = 0\}$$

one gets the following

Theorem 3.4. *Let $R \in \mathcal{NK}(2)$, such that $b(R) = 0$ and $\langle R^Q(I), I \rangle \neq 0$, for each non-holomorphic plane Q. Then Q is a non-holomorphic critical plane of r_R if, and only if, $Q \in \Delta$ and Q is a critical point of $r_{R|\Delta}$.*

Proof. By Theorem 2.4, $\langle R^Q(I), I \rangle = 0$ if Q is critical; so, such critical planes are in Δ. Let $Q = av_1 \wedge v_{1*} + bv_1 \wedge v_2$. If Q is a critical point of $r_{R|\Delta}$ then $R^Q(Q) = AQ + B * Q + T$, for appropriate coefficients A and B, and T tangent to $G(2, 2n)$ at Q. But T can be written as

$$T = Cv_1 \wedge v_{2*} + Dv_2 \wedge v_{1*} + E(av_1 \wedge v_2 - bv_1 \wedge v_{1*}) + \\ + F(av_1 \wedge v_{1*} + bv_2 \wedge v_{2*}).$$

Since $R^Q(Q) \in u(2)$ and $\langle R^Q(Q), I \rangle = 0$, $E = F$ and $B = -A$. If Π is the projection on

$$T_Q(G(2, 4)) = \{P \in \Lambda^2(\mathbb{C}^2) / P \wedge Q = 0 = \langle P, Q \rangle\},$$

$\Pi(R^Q(I))$ spans the normal space to Δ at Q. As Q is a critical point of $r_{R|\Delta}$, $\Pi(R^Q(I)) = \lambda T$ and with the above expression for T, $T = 0$.

Theorem 3.5. *If $R \in \mathcal{N}\mathcal{K}(2)$ such that $b(R) = 0$ and $\langle R^Q(I), I\rangle \neq 0$, for all non-holomorphic plane Q, then r_R has at least two non-holomorphic critical planes.*

Proof. Since each $\sigma \in \Sigma$ has two eigenvalues ia, ib with $ab < 0$, one can choose a unitary base of C^n such that

$$\sigma = av_1 \wedge v_{1*} + bv_1 \wedge v_2$$

and

$$\Sigma = \{\sigma \in u(2)/|\sigma| = 1,\ |\langle\sigma, I\rangle| < 1\}.$$

For each non-holomorphic plane Q it is now defined the subspaces

$$I^\perp = \{\sigma \in u(2)/\langle\sigma, I\rangle = 0\}$$

and

$$R^Q(I)^\perp = \{\sigma \in u(2)/\langle\sigma, R^Q(I)\rangle = 0\}.$$

One can also define a map $f : R^Q(I)^\perp \to I^\perp$ by $f(\sigma) = \sigma - \frac{1}{2}\langle\sigma, I\rangle$. Thus if $Q \in \Delta$, Q is a critical point of r_R, with critical value B, if, and only if $f(Q + JQ)$ is an eigenvector of $R^Q \circ f^{-1}$. A direct computation yields likewise to show that there exist at least two eigenvectors v_1, v_2 of $R^Q \circ f^{-1}$ such that $f^{-1}(v_1)/|f^{-1}(v_i)| \in \Sigma$, $i = 1, 2$; this completes the proof.

Corollary 3.6. *Theorem 3.5 holds even though $\langle R^Q(I), I\rangle = 0$.*

Definition 3.7.

$$\mathcal{N}\mathcal{K}(2)^+ = \{R \in \mathcal{N}\mathcal{K}(2)/b(R) = 0;\ \langle R^Q(I),\ v_i \wedge v_{i*}\rangle > 0\}.$$

The next proposition extends to $\mathcal{N}\mathcal{K}(2)^+$ the above results improving the knowledge of the set where the lower bounds on the number of critical points are achieved.

Proposition 3.8. *If $R \in \mathcal{N}\mathcal{K}(2)^+$, r_R has at least three distinct non-holomorphic critical planes. If r_R is non-degenerate, $F(Q_i)$ are mutually orthogonal.*

For the proof suffice it to show that $R^Q(I)^\perp \subseteq \Sigma$.

In higher dimensions the situation is more complicated, however we can use the results above obtained in dimension two. For any non-holomorphic plane Q, it is considered the space $G(2, Q \wedge JQ)$ and, similar to the case $n = 2$, it is defined

$$\mathcal{N}\mathcal{K}(3)^+ = \{R \in \mathcal{N}\mathcal{K}(3)/b(R) = 0;$$
$$\varrho(R^Q|_{\Lambda^2(Q \wedge JQ)}) > 0,\quad \text{for all}\quad Q \in G(2, 6) - G(2, 6)^J\}.$$

Also, for each non-holomorphic plane Q and $R \in \mathcal{N}\mathcal{K}(3)^+$ let

$$\Delta_Q = \{P \in G(2, Q \wedge JQ)/\langle P, R^Q(I_Q)\rangle = 0\}$$

and

$$\Delta = \bigcup_{Q} \Delta_Q ,$$

that is,

$$\Delta = \{P \in G(2, 6)/P \neq JP, \langle P, R^P(I_P)\rangle = 0\} .$$

Proposition 3.9. *Δ is a compact, locally trivial fibration over $\mathbb{C}P^2$, such that the projection $\Pi : \Delta \to \mathbb{C}P^2$ is given by*

$$\Pi(Q) = - * \frac{Q \wedge JQ}{|Q \wedge JQ|} .$$

Proof. If one considers the map $f: G(2, 6) - G(2, 6)^J \to \mathbb{R}$ given by $f(Q) = \langle Q, R^Q(I_Q)\rangle$, then $\Delta = f^{-1}(0)$. The rest of the proof is straightforward, [3].

Proposition 3.10. *If $R \in \mathcal{N}\mathcal{K}(3)^+$, any critical point of $r_{R|\Delta}$ is a critical plane of r_R and any non-holomorphic critical plane of r_R is on Δ.*

Suffice it to say that $T_Q(G(2, \mathbb{C}^3)) = T_Q(G(2, Q \wedge JQ)) + T_Q\Delta$.

The following main fact shows that the above obtained lower bounds are achieved on $\mathcal{N}\mathcal{K}(3)^+$. From [7], through the $\mathbb{Z}_2$-cohomology of the space $\hat{\Delta} = F(\Delta)/\sigma = -\sigma$, one gets

Proposition 3.11. *If $R \in \mathcal{N}\mathcal{K}(3)^+$ and r_R, $r_{R|\Delta}$ are non-degenerate, r_R has at least three distinct holomorphic critical planes and nine distinct non-holomorphic critical planes.*

4. NORMAL FORMS OF THE NK-CURVATURE OPERATORS

If $R \in \mathcal{N}\mathcal{K}(n)$ using the above mentioned tensor $\lambda^R = R - RJ$, we can define, [9], a new tensor given by

$$\Gamma^R_{xyzw} = \tfrac{1}{2}\lambda^R_{xyzw} + \tfrac{1}{2}\lambda^R_{xzyw} - \tfrac{1}{4}\lambda^R_{xwyz} .$$

Let $R, S \in \mathcal{N}\mathcal{K}(n)$, with $b(R) = b(S) = 0$, such that $R_{xJxxJx} = S_{xJxxJx}$. Then

$$R_{xyzw} - \Gamma^R_{xyzw} = S_{xyzw} - \Gamma^S_{xyzw} .$$

This fact shows that two Nearly-Kaehler curvature operators having the same holomorphic sectional curvature do not coincide everywhere, against the well-known property for Kaehler curvature operators. That justifies the following

Definition 4.1. Let $\mathcal{S} \subset \mathcal{N}\mathcal{K}(n)$ a subspace. Let $R \in \mathcal{S}$ with $b(R) = 0$ and let $\{(P_i, A_i)\}$ be a set of critical points P_i of the sectional curvature r_R with critical values A_i. It is said that $\{(P_i, A_i)\}$ is a *normal form of R relative to $\mathcal{S}$* if for each $S \in \mathcal{S}$, with $b(S) = 0$, such that $r_{S-\Gamma^S+\Gamma^R}$ has critical points $\{(P_i, A_i)\}$ with $r_{S-\Gamma^S+\Gamma^R}(P_i) = A_i = r_R(P_i)$, then $R = S - \Gamma^S + \Gamma^R$.

This definition is not manageable to checking normal forms for a given curvature operator R and for this reason one will establish an algebraic condition such that when this condition is verified one gets the existence of a normal form of R.

Definition 4.2. For any plane $P \in G(2, 2n)$, $P = a_\alpha v_\alpha \wedge b_\beta v_\beta = a_{\alpha\beta} v_\alpha \wedge v_\beta$, it is defined a map

$$\Omega(P): \mathcal{NK}(n) \to \mathbb{C}^{2n}$$

by

$$\Omega(P)\, R = \left(a_\alpha a_{\varkappa\delta}(R - \Gamma^R)_{\varkappa t\gamma\delta} - \mathrm{i} b_\beta a_{\gamma\delta}(R - \Gamma^R)_{t\beta\gamma\delta}\right)$$

where $t = 1, 1^*, \ldots, n, n^*$.

In general, if $P_1, \ldots, P_k \in G(2, 2n)$

$$\Omega(P_1, \ldots, P_k)\, R = \left(\Omega(P_1)\, R, \ldots, \Omega(P_k)\, R\right) \in \mathbb{C}^{2nk}\,.$$

Let $R, R' \in \mathcal{NK}(n)$, such that r_R and $r_{R'-\Gamma^{R'}+\Gamma^R}$ have the same critical points P_i, $1 \leqq i \leqq k$, and the same critical values A_i. Then, by (7), $\Omega(P_i)\, R = \Omega(P_i)\, R'$, $1 \leqq i \leqq k$; that is, $R - R' \in \operatorname{Ker} \Omega(P_1, \ldots, P_k)$.

Conversely, given $R \in \mathcal{NK}(n)$, let $K \in \mathcal{NK}(n)$, such that $K \in \operatorname{Ker} \Omega(P_1, \ldots, P_k)$. It is considered $R' = R + K$. A direct computation shows that r_R and $r_{R'-\Gamma^{R'}+\Gamma^R}$ have the same critical points $P_1, \ldots, P_k$ with the same critical values $A_1, \ldots, A_k$.

So, to show that $\{(P_1, A_1), \ldots, (P_k, A_k)\}$ is a normal form of R it will be suffice to prove that

$$\operatorname{Ker} \Omega(P_1, \ldots, P_k)\big|_{\operatorname{Ker}(b)} = \{S \in \mathcal{NK}(n) / S = \Gamma^S\}\,.$$

Thus, to determine normal forms of curvature operators in $\mathcal{NK}(n)$ suffice it to look over the kernel of $\Omega(P_i)$.

Theorem 4.3. *Any $R \in \mathcal{NK}(2)$, $b(R) = 0$, has a normal form relative to $\mathcal{NK}(2)$.*

Proof. We shall give a sketch of the proof, which can be found in [7], since it is not substantially different from Kaehler case.

From the above paragraph one can suppose that r_R has two distinct holomorphic critical planes P_1 and P_2 and two distinct nonholomorphic Q_1 and Q_2. One takes $P_1 = v_1 \wedge v_{1^*}$ and $K \in \operatorname{Ker} \Omega(P_1, P_2, Q_1, Q_2)$, with $b(K) = 0$. Putting $K' = K - \Gamma^K$, it will be sufficient to prove that $K' = 0$. As P_1 is critical of r_R, $K_{11^*1\alpha} = 0$, $\alpha = 1, 1^*, 2, 2^*$.

Also, one can choose v_2 such that $K_{1212^*} = 0$. Next, one considers the possible elections of P_2, Q_1, Q_2, which form the matrix of $\Omega(P_1, P_2, Q_1, Q_2)$. Looking over the kernel of this matrix one can easily compute the others components of K'.

One could hope for direct generalization of the preceding theorem, however the functions and spaces involved, as we can deduce from § 3, are very complicated. Since the normal forms of a curvature operator depend strongly on the number of critical points of the curvature function, we shall restrict to $\mathcal{NK}(3)^+$, where one can use the result of § 3.

Lemma 4.4. ([3]). *There exists an $R \in \mathcal{NK}(3)^+$ such that r_R has three distinct* *holomorphic critical planes and nine distinct non-holomorphic. Also, r_R is non-degenerate and R has two distinct types of normal form.*

Proof. From Theorem 4.3 in [4], there exists an $A \in \mathcal{T}$ such that $S = \sigma(A) \in \mathcal{N}\mathcal{K}(3)^+$ and for each non-holomorphic plane Q, $\Delta(S^Q) = I^\perp \cap G(2, 6)$. Indeed, if $Q = av_1 \wedge v_{1*} + bv_1 \wedge v_2$ one considers $Q \wedge JQ$ as a real subspace of dimension four of $\mathbb{C}^3$. Then, there is an $A \in \mathcal{T}$ such that $\sigma(A) = S = aR_{FS} + \lambda_Q^S *_Q$, where R_{FS} is the curvature tensor of $\mathbb{C}P^n$ with the Fubini-Study metric; it is sufficient to observe that

$$S_{ii^*ii^*} = 2\langle Av_i, v_i\rangle + \frac{a}{4},$$

$$S_{ii^*jj^*} = \tfrac{1}{2}\{\langle Av_i, v_i\rangle + \langle Av_j, v_j\rangle\} - \frac{a}{24},$$

$$S_{ijij} = S_{ij^*ij^*} = \tfrac{1}{4}\{\langle Av_i, v_i\rangle + \langle Av_j, v_j\rangle\} + \frac{5a}{16}$$

and

$$\lambda_Q^S = \lambda_{12}^S = S_{1212} - S_{121^*2^*} = \frac{a}{3}.$$

Then, $S^Q = aR_{FS}$; and

$$\begin{aligned}\Delta(S) &= \{Q \in G(2, 6) - G(2, 6)^J / \langle Q, S^Q(I_Q)\rangle = 0\} = \\ &= \{Q \in G(2, 6) - G(2, 6)^J / \langle Q, aR_{FS}(I_Q)\rangle = \} = \\ &= I^\perp \cap G(2, 6).\end{aligned}$$

Hence, for this S, $v \wedge w \in \Delta$ if, and only if, $\langle v \wedge w, I\rangle = 0$; equivalently, $v \wedge w$ is an antiholomorphic plane; that is, $\langle v, Jw\rangle = 0$. So, one can choose the eigenvectors of A such that $|\langle v, Jw\rangle| < \frac{1}{3}$.

If $v \wedge w$ is a critical point of r_R with critical value C, then v and w can be choosen eigenvectors of A. Thus

$$Cw = S(v \wedge w)\, v = \frac{1}{10}\{\langle \hat{A}v, w\rangle\, w + \hat{A}w - 3\langle v, Jw\rangle\, \hat{A}Jv - 3\langle \hat{A}v, Jw\rangle\, Jv\} + \\ + \frac{33a}{160}\, w + \frac{61a}{160}\langle v, Jw\rangle\, Jv\,;$$

So,

$$\hat{A}w = 10Cw - \langle \hat{A}v, v\rangle\, w + 3\langle v, Jw\rangle\, \hat{A}Jv + 3\langle \hat{A}v, Jw\rangle\, Jv - \frac{33a}{160}\, w - \frac{61a}{160}\langle v, Jw\rangle Jv,$$

$$\hat{A}v = 100v - \langle \hat{A}w, w\rangle\, v - 3\langle v, Jw\rangle\, \hat{A}Jw - 3\langle \hat{A}v, Jw\rangle\, Jw - \frac{33a}{160}\, v + \frac{61a}{160}\langle v, Jw\rangle\, Jw.$$

Solving for $\hat{A}v$:

$$(1 - 9\langle v, Jw\rangle^2)\,\hat{A}v = \Big\{\langle \hat{A}v, v\rangle + 15\langle v, Jw\rangle\,\langle \hat{A}v, Jw\rangle +$$

$$+ \frac{33a}{160} + \frac{61a}{160}\langle v, Jw\rangle^2 - \frac{183}{160}\langle v, Jw\rangle\Big\} +$$

$$+ \{-30C\langle v, Jw\rangle + 3\langle v, Jw\rangle\,\langle \hat{A}v, v\rangle + \langle v, Jw\rangle - J\langle \hat{A}v, Jw\rangle\}\, Jw\,.$$

Thus, $\hat{A}v \in \{v, Jv, w, Jw\}$.

Similary, $\hat{A}w \in \{v, Jv, w, Jw\}$.

If $\{v_\alpha\}$ is a base of V consisting of eigenvectors of A, also are eigenvectors of $\hat{A}$, and by diagonalization of $A_{|v\wedge w}$, if $v \wedge w$ is a critical plane of r_S, we can choose v, w such that $\langle Av, w\rangle = 0$. Then $v \wedge w \in \Lambda^2(v_i, v_{i*}, v_j, v_{j*})$ and $v \wedge w$ is a critical point or r_S restricted to $G(2, 6) \cap \Lambda^2(v_i, v_{i*}, v_j, v_{j*})$.

If $v \wedge w$ is holomorphic, that is, $w = Jv$, then v is also an eigenvector of A, therefore the unique holomorphic critical planes are those of the form $v_i \wedge v_{i*}$.

If $w \neq Jv$, from Proposition 2.2 and Corollary 2.3

$$v \wedge w \in \{v_i \wedge v_j,\ v_i \wedge v_{j*},\ v_{i*} \wedge v_j,\ v_{i*} \wedge v_{j*}\}$$

or

$$F(v \wedge w) = a_{ij}(v_i \wedge v_{i*}) + b_{ij}(v_j \wedge v_{j*})\,,$$

where a_{ij}, b_{ij} are determined up to sign by the equation

$$a_{ij}(S_{ii*ii*} + S_{ii*jj*}) + b_{ij}(S_{ii*jj*} + S_{jj*jj*}) = 0\,.$$

Similar arguments to those of [7] yield to conclude this critical planes achieve the bounds claimed.

Next it will be proved that R has two distinct types of normal form: one correspondent to the critical planes $v_i \wedge v_{i*}$, $v_i \wedge v_j$, $v_i \wedge v_{j*}$; another one using only the nine distinct non-holomorphic critical planes.

First, let $R \in \mathcal{NK}(3)$ such that $K \in \operatorname{Ker} \Omega(v_i \wedge v_{i*},\ v_i \wedge v_j,\ v_i \wedge v_{j*})$, with $b(K) = 0$. Then, putting $K' = K - \Gamma^K$, $K'_{ii*i\alpha} = K'_{iji\alpha} = K'_{ij*i\alpha} = 0$, for all α. For the other terms, by the Kaehler identities and the first Bianchi identity

$$\begin{aligned} K'_{ii*jj*} &= K'_{ijij} = K'_{ij*ij*} = 0 \\ K'_{ii*jk} &= -K'_{ijik*} + K'_{ikij*} = 0 \\ K'_{ij*jk*} &= K'_{ijik} + K'_{ij*ik*} = 0\,. \end{aligned}$$

Then $K' = 0$.

Secondly, by the above argument $K'_{iji\alpha} = K'_{ij*i\alpha} = 0$ and $K'_{ii*jj*} = K'_{ii*jk} = K'_{ii*jk*} = 0$. The other terms are of the form K'_{ii*ii*}. Let Q_{ij} be the non-holomorphic critical planes such that $F(Q_{ij}) = a_{ij}v_i \wedge v_{i*} + b_{ij}v_j \wedge v_{j*}$. But

$$a_{ij}(K'_{ii*ii*} + K'_{ii*jj*}) + b_{ij}(K'_{ii*jj*} + K'_{jj*jj*}) = 0$$

is reduced to $a_{ij}K'_{ii^*ii^*} + b_{ij}K'_{jj^*jj^*} = 0$. Therefore, $K'_{ii^*ii^*} = 0$ unless

$$\det\begin{pmatrix} a_{12} & b_{22} & 0 \\ a_{13} & 0 & b_{13} \\ 0 & a_{23} & b_{23} \end{pmatrix} = 0\,.$$

But one can make a generic choice of the eigenvalues such that this determinant becomes non-zero. This completes the proof.

Now, assume that $R \in \mathcal{N}\mathcal{K}(3)^+$ but does not have a normal form. By Theorem 3.2 one can suppose that r_R, $r_{R|G(2,6)J}$ and $\sigma_{R_{F(\Delta)}}$ are non-degenerate.

Note that

$$\{R \in \mathcal{N}\mathcal{K}(3)/r_R \text{ degenerate}\}$$

contains

$$\{R \in \mathcal{N}\mathcal{K}(3)/r_R \text{ has only a degenerate critical point with Null}\,(r_{R^{**}}) = 1\}$$

as an open dense subset.

Let aR_{FS} be a multiple of the operator R_{FS}, where a is large enough so that $r_{R_{FS}} > > r_R$, which is possible as $r_{R_{FS}} > 0$. By perturbing R if need be, suppose that the path $\mathbb{R} \to \mathcal{N}\mathcal{K}(3)$, given by $t \mapsto R_t = (1-t)\,aR_{FS} + tR$ meets the set $A = = \{R \in \mathcal{N}\mathcal{K}(3)/r_R \text{ degenerated}\}$ in a finite number of points $t_j, j = 1, \ldots, l$, for which r_{R_t} has only one degenerate critical point with Null $(r_{R^{**}}) = 1$. In fact, the condition to be r_{R_t} degenerate is determinated by a set of polinomial relations. If r_R has two distinct critical points of nullity 1, then R satisfies two distinct polinomial relations. By Theorem 3.2 R can be perturbed to satisfy only one of them. Similary, if r_R has one degenerate critical point with nullity more than 1, R satisfies several polinomial relations. R can be again perturbed does not satisfy one of them.

The proof is similar to that of the Kaehler case [7]. See [3] for a detailed account.

Theorem 4.5.

$$\{R \in \mathcal{N}\mathcal{K}(3)^+/R \text{ has a normal form relative to } \mathcal{N}\mathcal{K}(3)\}$$

contains an open dense subset of $\mathcal{N}\mathcal{K}(3)^+$.

As one pointed out, this result can be extended to $\mathcal{N}\mathcal{K}(n)^+$, $n > 3$, by suitable choice of the spaces and functions involved in the proof.

5. EXAMPLES

Theorem 5.1. *If* $R \in B \oplus D \subseteq \mathcal{N}\mathcal{K}(n)$, [3], *for some unitary base* $\{v_\alpha\}$ *of* $\mathbb{C}^n$ *and for all* $i, j \leqq n$, r_R *has critical points the planes* $v_i \wedge v_{i^*}$, $v_i \wedge v_j$, $v_i \wedge v_{j^*}$. *If* $n = 2, 3$, *these critical points and their correspondent critical values are a normal form of R relative to* $\mathcal{N}\mathcal{K}(n)$.

Proof. Given $T \in \mathcal{T}$, by [4], $R = \sigma(T) \in B \oplus D$. Applying the theorem of characterization of the critical points of r_R to the plane $P_1 = v_1 \wedge v_{1*}$ one gets:

$$A = R_{11*11*}\,; \quad R_{11*12} = R_{12*21*} = 0\,; \quad R_{11*12*} = R_{11*2*1*} = 0\,.$$

But

$$A = R_{11*11*} = \langle \hat{T}v_1, v_1 \rangle - \frac{11t}{48} = \frac{49}{24}\langle Tv_1, v_1 \rangle + \frac{25}{24}\langle Tv_2, v_2 \rangle\,.$$

$$R_{11*12} = \frac{1}{2}\langle \hat{T}v_{1*}, v_2 \rangle = 0\,.$$

Likewise for the other components of R. Thus, P_1 is a critical point of r_R. The same argument can be applied to the other planes. From Theorem 4.3 and Lemma 4.4 it is straightforward to see that these points constitute a normal form of R.

Theorem 5.2. *Let $R \in \mathcal{NK}(2)$, with $b(R) = 0$, such that R has a normal form of the type*

$$\{(v_1 \wedge v_{1*}, A_{11*}),\ (v_2 \wedge v_{2*}, A_{22*})\}\,.$$

Then, $A_{11*} = A_{22*}$ if, and only if, $R \in B \oplus D$.

Proof. As the plane $v_1 \wedge v_{1*}$ is critical for r_R, it is obtained the similar relations of the preceding theorem. Furthermore,

$$\langle a(R)\, v_1, v_1 \rangle = \left(A_{11*} + R_{11*22*} - \frac{\lambda^R_{12}}{2}\right) \mathrm{id}$$

if, and only if, $R \in B \oplus D$.

Let M^n be a complex submanifold of a generalized complex space form $\mathbb{P}^N$. The curvature tensor of $\mathbb{P}^N$ it is given by, [12],

$$R'_{xy} = \frac{\mu + 3\alpha}{4}\, x \wedge y + \frac{\mu - \alpha}{4}\,(Jx \wedge Jy + 2\langle x, Jy \rangle\, J)$$

for all $x, y \in \mathscr{X}(\mathbb{P}^N)$, where μ and α are the holomorphic sectional curvature and the type of $\mathbb{P}^N$, respectively.

Let s be the second fundamental form of the imbedding of M^n in $\mathbb{P}^N$ and $\langle h^\beta x, y \rangle = \langle s(x, y), \xi_\beta \rangle$, for a given unitary base $\{\xi_\beta\}$ of $M^\perp_m$. If R is the curvature of M, the Gauss-Codazzi equations are written, [8],

$$R = \Pi R' + \sum_\beta (h^\beta)_2$$

where Π is the projection on the tangent space of M and $h_2(x \wedge y) = (h^\beta x) \wedge (h^\beta y)$.

Proposition 5.3. *Let s be the second fundamental form of a complex submanifold M of a NK-manifold $\mathbb{P}^N$. Then,*

$$s(x, Jy) = s(Jx, y) = Js(x, y)\,, \quad \text{for all} \quad x, y \in \mathscr{X}(M)\,.$$

Corollary 5.4.

(i) $h^\beta J = -Jh^\beta$,

(ii) $h^{i^*} = Jh^i$,

(iii) *If x is a eigenvector of h^β with eigenvalue λ (necessarily real, since h^β is symmetric), also Jx is an eigenvector of h^β with eigenvalue $-\lambda$; that is, if $h^\beta x = \lambda x$, $h^\beta(Jx) = -\lambda Jx$.*

Proposition 5.5. *If $R = \Pi R' + h_2^1 + h_2^{1^*}$ is the curvature of a complex hypersurface M^n of a generalized complex space form $\mathbb{P}^{n+1}$, for an unitary base $\{v_\alpha\}$ of M^n formed by eigenvectors of h^1, the planes $v_i \wedge v_{i*}$, $v_i \wedge v_j$, $v_i \wedge v_{j*}$ are critical of r_R with critical values*

$$\mu - 2\lambda_i^2\,, \quad \frac{\mu + 3\alpha}{4} + \lambda_i\lambda_j\,, \frac{\mu + 3\alpha}{4} - \lambda_i\lambda_j,$$

respectively.

Proof. Developing the expression of R one gets

$$R_{ii^*ii^*} = \mu - 2\lambda_i^2\,,$$

$$R_{ijij} = \frac{\mu + 3\alpha}{4} + \lambda_i\lambda_j\,,$$

$$R_{ij^*ij^*} = \frac{\mu + 3\alpha}{4} - \lambda_i\lambda_j$$

where λ_i, λ_j are the eigenvalues correspondent to the elements of the base, respect to h^1. A direct calculation shows that the given planes are critical for r_R.

Corollary 5.6. *Let R be the curvature tensor at a point m of a complex hypersurface M of a NK-manifold $\mathbb{P}^{n+1}$ of constant holomorphic sectional curvature. Then, if $n = 3$ R has a normal form relative to $\mathcal{N}\mathcal{K}(n)$ correspondent to the critical points described in Proposition 5.5.*

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