Some variational problems involving Lancret curves

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This is a survey on a series of joint papers with my colleagues
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1. Introduction

Helical configurations are structures commonly found in Nature. They appear in microscopic systems (biomolecules, bacterial fibers, nanosprings, protein chains in particular DNA,...) as well as in macroscopic phenomena (strings, ropes, climbing plants,...) (see for example [1, 9, 11, 12, 13, 14, 21, 27] and references therein). In particular, they are very important and ubiquitous in biology as a consequence of the following known, in the biological community since the work of Pauling, theorem: *Identical objects, regularly assembled, form a helix* (see [10] and references therein).

As far as we know, helical structures are usually identified with the simplest idea of circular helices. However, nothing could be further from the truth. Nobody can believe that squirrels chasing one another up and around tree trunks follow a circular helix path. First, because the cross section of a tree trunk is not circular, but also because its axis is not exactly a straight line. On the other hand, many types of bacteria, such as certain strains of *Escherichia coli* or *Salmonella typhimorium* swim by rotating flagellar filaments. These are polymers which are flexible enough to switch among different helical forms quite different from circular helices. Then we will deal with generalized helices to get answers to those questions.

2. Lancret curves in 3-dimensional space forms ([2] and [5])

The seminal paper was that of M. Barros [2].

Let $M(0)$ be $\mathbb{R}^3$ or $\mathbb{L}^3$. A Lancret curve (or general helix) in $M(0)$ is a Frenet curve whose tangent indicatrix is contained in some plane $\Pi \subset M(0)$. It will be called degenerate or nondegenerate according to the causal character of such a plane. As in the Euclidean setting, the Lancret curves in $\mathbb{L}^3$ are those for which the ratio of curvature to torsion is constant.

In [2, 5] the notion of Lancret curve was extended to real space forms and spacetimes $M(C)$, with $C \neq 0$, respectively, where the notion of Killing vector field along a curve played an important role. We will consider the class of Lancret curves including not only those curves with torsion vanishing identically, but also the ordinary helices (or simply helices), whose curvature and torsion are both nonzero constants. We will refer to these two cases as trivial Lancret curves. As a resume from [2] and [5] we have:

Lancret curves in $\mathbb{R}^3$. The only ones are geodesics of a right cylinders.

Lancret curves in $\mathbb{L}^3$. The only ones are geodesics in either right cylinders (nondegenerate case) or flat B-scrolls (degenerate case) [see Graves [20] for details on scrolls].

Lancret curves in the hyperbolic space $\mathbb{H}^3(C)$, $C < 0$. A curve is Lancret if and only if either its torsion vanishes identically and it is lying in some hyperbolic plane $\mathbb{H}^2(C)$ or it is an ordinary helix.

Lancret curves in the 3-sphere $\mathbb{S}^3(C)$, $C > 0$. A curve is Lancret if and only if either its torsion vanishes identically and it is lying in some 2-sphere $\mathbb{S}^2(C)$ or there exists a constant b such that curvature κ and the torsion τ are related by $\tau = q\kappa \pm \sqrt{C}$, where q will be viewed as a sort of slope.

Lancret curves in the de Sitter space $\mathbf{dS}_3(C)$, $C > 0$. A curve is Lancret if and only if either its torsion vanishes identically or it is an ordinary helix.

Lancret curves in the anti de Sitter space $\mathbf{AdS}_3(C)$, $C < 0$. A curve γ is Lancret if and only if either its torsion vanishes identically or the curvature κ and the torsion τ are related by $\tau = q\kappa \pm \sqrt{-C}$, where q will be viewed as a sort of slope.
(i) **Nondegenerate case:** γ is Lancret if and only if it is a geodesic of either a Hopf tube or a hyperbolic Hopf tube.
(ii) **Degenerate case:** γ is Lancret if and only if it is a geodesic of a flat scroll over a null curve.

3. Variational problem & Euler-Lagrange equations in 3-dimensional Lorentzian space forms([7])

Let $M(C)$ be a 3-dimensional Lorentzian space with constant curvature C. In a suitable space Λ of Frenet curves in $M(C)$ (for example, the space of closed curves or curves satisfying certain second order boundary data, such as clamped curves), we have a three-parameter family $\{\mathcal{F}_{mnp} : \Lambda \to \mathbb{R} \,|\, m, n, p \in \mathbb{R}\}$ of lagrangians defined by

$$\mathcal{F}_{mnp}(\gamma) = \int_\gamma (m + n\kappa + p\tau) ds, \tag{1}$$

where s, κ and τ stand for the arclength parameter, curvature and torsion of γ, respectively, and the parameters m, n and p are not allowed to be zero simultaneously.

We have found out the moduli space of trajectories regarding the model $[M(C), \mathcal{F}_{mnp}]$, as well as the corresponding algorithms to obtain the trajectories of a given model.

The closed trajectories, when there exist, are also obtained from an interesting quantization principle.

We have used standard arguments, involving integrations by parts, to get the variation of $\mathcal{F}_{mnp}$ along γ in the direction of W

$$\delta\mathcal{F}_{mnp}(\gamma)[W] = \int_\gamma \langle \Omega(\gamma), W \rangle \, ds + [\mathcal{B}(\gamma, W)]_0^L \,, \tag{2}$$

where $\Omega(\gamma)$ and $\mathcal{B}(\gamma, W)$ stand for the Euler-Lagrange and boundary operators, respectively, which are given by

$$\Omega(\gamma) = (-\varepsilon_1\varepsilon_2 m\kappa + \varepsilon_1\varepsilon_2 p\,\kappa\tau - \varepsilon_2\varepsilon_3 n\tau^2 + \varepsilon_1 n\,C)N + (-\varepsilon_1 p\,\kappa_s + \varepsilon_3 n\tau_s)B,$$

$$\begin{aligned}\mathcal{B}(\gamma, W) &= \varepsilon_2 \frac{p}{\kappa}\left\langle \nabla_T^2 W, B\right\rangle + n\left\langle \nabla_T W, N\right\rangle \\ &+ \varepsilon_1 m\left\langle W, T\right\rangle + \left(-\varepsilon_3 n\tau + \varepsilon_1\varepsilon_2 \frac{p\,C}{\kappa} + \varepsilon_1 p\,\kappa\right)\left\langle W, B\right\rangle.\end{aligned}$$

Second order boundary conditions *Given $q_1, q_2 \in M$ and $\{x_1, y_1\}$, $\{x_2, y_2\}$ orthonormal vectors in $T_{q_1}M$ and $T_{q_2}M$, respectively, define the space of curves*

$$\Lambda = \{\gamma : [t_1, t_2] \to M \,|\, \gamma(t_i) = q_i, T(t_i) = x_i, N(t_i) = y_i, 1 \le i \le 2\}.$$

Then the critical points of the variational problem $\mathcal{F}_{mnp} : \Lambda \to \mathbb{R}$ are characterized by the following Euler-Lagrange equations

$$\varepsilon_3 m\kappa - \varepsilon_3 p\,\kappa\tau + \varepsilon_1 n\tau^2 + \varepsilon_1 n\,C = 0, \tag{3}$$

$$-\varepsilon_1 p\,\kappa_s + \varepsilon_3 n\tau_s = 0. \tag{4}$$

4. The moduli spaces of trajectories ([7])

The field equations (3) and (4) can be nicely integrated. First, notice that they can be written as

$$\begin{aligned} \varepsilon_1 p\kappa - \varepsilon_3 n\tau &= a, && (5)\\ -\varepsilon_1 m\kappa + a\tau &= \varepsilon_3 nC, && (6) \end{aligned}$$

where a denotes an undetermined integration constant. Then we have

- If $\varepsilon_1 pa + \varepsilon_2 mn \neq 0$, then the solutions are helices (trivial Lancret curves) with curvature and torsion given by

$$\kappa = \frac{a^2 + n^2 C}{\varepsilon_1 pa + \varepsilon_2 mn}, \qquad \tau = \frac{ma + \varepsilon_3 npC}{pa - \varepsilon_3 nm}.$$

- Otherwise, the existence of solutions is equivalent to $n^2 C + a^2 = ma + \varepsilon_3 pnC = 0$. A first consequence is that $C \leq 0$. Therefore, the trajectories in the relativistic particle model $[\mathbf{dS}_3, \mathcal{F}_{mnp}]$ are helices in the de Sitter space with curvature and torsion given as above. This seems reasonable since the de Sitter space is free of non-trivial Lancret curves.

- Now, in the above setting, we may assume that $n \neq 0$, otherwise we have the free fall particle model. Thus, the field equations reduces to equation (5). If $m = p = 0$,

then $\tau^2 = -C$, so that γ is a plane curve (when $C = 0$) or the horizontal lift, via the Hopf map π_- or λ, of a curve in either $\mathbb{H}^2(4C)$ or $\mathbf{AdS}_2(4C)$ (when $C < 0$). Otherwise, the trajectories are curves whose curvatures satisfy

$$\tau = -\varepsilon_2 q\kappa \pm \frac{m}{p}, \qquad \text{with} \qquad q = \frac{p}{n} \quad \text{and} \quad -C = \frac{m^2}{p^2}. \tag{7}$$

Then they are Lancret curves in either $\mathbb{L}^3$ or in $\mathbf{AdS}_3$. The slope in both cases is p/n.

The moduli space of trajectories is summarized in the following tables which correspond with Lorentz-Minkowski, de Sitter and anti de Sitter spaces, respectively. All solutions are Lancret curves. Similarly to the classical Euclidean case, helices are considered as special cases of Lancret curves (trivial Lancret curves). For simplicity of interpretation, we have represented different cases according to the values of the parameters defining the action.

m	n	p	**Solutions in $\mathbb{L}^3$, $C=0$**
$\neq 0$	$=0$	$=0$	Geodesics ($\kappa=0$)
$=0$	$=0$	$\neq 0$	Circles (κ constant and $\tau=0$)
$=0$	$\neq 0$	$=0$	Plane curves ($\tau=0$)
$\neq 0$	$\neq 0$	$=0$	Helices with arbitrary τ and $\kappa=\varepsilon_2\frac{n\tau^2}{m}$
$\neq 0$	$=0$	$\neq 0$	Helices with arbitrary κ and $\tau=\frac{m}{p}$
$=0$	$\neq 0$	$\neq 0$	Circles and **Lancret curves** with $\tau=-\varepsilon_2\frac{p}{n}\kappa$
$\neq 0$	$\neq 0$	$\neq 0$	Helices with $\kappa=\frac{\varepsilon_1 a^2}{ap-\varepsilon_3 nm}$ and $\tau=\frac{ma}{ap-\varepsilon_3 nm}$, $a\in\mathbb{R}-\{\frac{\varepsilon_3 nm}{p}\}$

m	n	p	**Solutions in $\mathbf{dS}_3$, $C = c^2$**
$\neq 0$	$= 0$	$= 0$	Geodesics ($\kappa = 0$)
$= 0$	$= 0$	$\neq 0$	Circles (κ constant and $\tau = 0$)
$= 0$	$\neq 0$	$= 0$	Do not exist
$\neq 0$	$\neq 0$	$= 0$	Helices with arbitrary τ and $\kappa = \varepsilon_2 \frac{n(c^2+\tau^2)}{m}$
$\neq 0$	$= 0$	$\neq 0$	Helices with arbitrary κ and $\tau = \frac{m}{p}$
$= 0$	$\neq 0$	$\neq 0$	Helices with $\kappa = \varepsilon_1 \frac{n^2c^2+a^2}{ap}$ and $\tau = \varepsilon_3 \frac{n\,c^2}{a}$, $a \in \mathbb{R} - \{0\}$
$\neq 0$	$\neq 0$	$\neq 0$	Helices with $\kappa = \frac{n^2c^2+a^2}{\varepsilon_1 p\,a+\varepsilon_2 mn}$ and $\tau = \frac{ma+\varepsilon_3 np\,c^2}{p\,a-\varepsilon_3 mn}$, $a \in \mathbb{R} - \{\varepsilon_3 \frac{mn}{p}\}$

m	n	p	**Solutions in $\mathbf{AdS}_3$, $C=-c^2$**
$\neq 0$	$=0$	$=0$	Geodesics ($\kappa=0$)
$=0$	$=0$	$\neq 0$	Circles (κ constant and $\tau=0$)
$=0$	$\neq 0$	$=0$	Horizontal lifts, via a Hopf map π_- or λ, of curves in either $\mathbb{H}^2(-4c^2)$ or $\mathbf{AdS}_2(-4c^2)$
$\neq 0$	$\neq 0$	$=0$	Helices with arbitrary τ and $\kappa=\varepsilon_2\frac{n(\tau^2-c^2)}{m}$
$\neq 0$	$=0$	$\neq 0$	Helices with arbitrary κ and $\tau=\frac{m}{p}$
$=0$	$\neq 0$	$\neq 0$	Helices with $\kappa=\varepsilon_1\frac{a^2-n^2c^2}{ap}$ and $\tau=-\varepsilon_3\frac{n\,c^2}{a}$, $a\in\mathbb{R}-\{0\}$
$\neq 0$	$\neq 0$	$\neq 0$	Helices with $\kappa=\frac{a^2-n^2c^2}{\varepsilon_1 p\,a+\varepsilon_2 mn}$ and $\tau=\frac{ma-\varepsilon_3 np\,c^2}{p\,a-\varepsilon_3 mn}$, $a\in\mathbb{R}-\{\varepsilon_3\frac{mn}{p}\}$
$\neq 0$	$\neq 0$	$\neq 0$	**Lancret curves** with $\tau=-\varepsilon_2\frac{p}{n}\kappa\pm\frac{m}{p}$ and $c=\pm\frac{m}{p}$

5. Going deeply into non flat backgrounds: creating algorithms ([7])

The uninteresting case corresponds to relativistic particles evolving in the de Sitter background. This is due in part to the absence of non trivial Lancret curves in $\mathbf{dS}_3$. Therefore, most of the models $[\mathbf{dS}_3(C), \mathcal{F}_{mnp}]$ admit a one-parameter family of trajectories which are trivial Lancret curves. The exception to this rule is the model $[\mathbf{dS}_3(C), \mathcal{F}_{0n0}]$, which is associated with the action measuring the total curvature of trajectories (known as the Plyushchay model for a massless relativistic particle, [25, 26]) and does not provide any consistent dynamics (see [4] for more details).

Particles evolving in anti de Sitter backgrounds

The most interesting models in the anti de Sitter space $\mathbf{AdS}_3$ are $[\mathbf{AdS}_3(C), \mathcal{F}_{0n0}]$ and $[\mathbf{AdS}_3(C), \mathcal{F}_{mnp}]$ with $mnp \neq 0$. The former corresponds again with the action giving the total curvature, that we have called the Plyushchay model describing a massless relativistic particle. In [4] it is shown that the three-dimensional anti de Sitter space is the only spacetime (no matter the dimension) with constant curvature providing a consistent dynamics for this action. More precisely, the trajectories of this model are nothing but the horizontal lifts, via either the usual Hopf map π_- or the Lorentzian Hopf map λ, of arbitrary curves in either the hyperbolic plane or the anti de Sitter plane, respectively

(see Appendix B). It should be noticed that those horizontal curves are Lancret ones, where the curvature is an arbitrary function and the torsion is determined by the radius of the anti de Sitter space (for instance, $\tau = \pm 1$ if $C = -1$).

However, the latter provides a model very rich in solutions. We are going to describe explicitly the trajectories for a better understanding of their nice dynamics. First, the model admits a one-parameter class $\mathcal{T}$ of trajectories which are ordinary helices (see Table 3). They can be geometrically obtained as geodesics of either a Hopf tube over a curve with constant curvature in the corresponding hyperbolic plane or a hyperbolic Hopf tube over a curve with constant curvature in the anti de Sitter plane (see [3] and Appendix B for more details). The dynamics are completed with classes of non trivial Lancret paths whose existence is related to the values of the parameters defining the action. First of all, notice that the ratio $\frac{m}{p}$ and the curvature C of $\mathbf{AdS}_3(C)$ should satisfy $\frac{m}{p} = \pm\sqrt{-C}$. Therefore, without loss of generality, we may assume that $C = -1$ and $m = \pm p$, so we will put $m = p$ in the discussion. On the other hand, the non trivial Lancret curves in the anti de Sitter space are characterized by the following constraint between curvature and torsion

$$\tau = q\kappa \pm 1, \quad \text{for a certain constant } b \in \mathbb{R}.$$

Furthermore, as in the flat case, degenerate Lancret curves correspond with $q = \pm 1$ and spacelike acceleration, $\varepsilon_2 = 1$, [5]. Consequently, we have to distinguish two cases.

5.1. The dynamics in $[\mathbf{AdS}_3, \mathcal{F}_{mnp}]$ with $n^2 \neq p^2$

Besides the above mentioned class $\mathcal{T}$ of ordinary helices, this model has a second class, $\mathcal{T}_{n^2 \neq p^2}$, of trajectories which, according to Table 3, are nondegenerate Lancret curves (because $n^2 \neq p^2$) satisfying

$$\tau = -\varepsilon_2 \frac{p}{n} \kappa \pm 1.$$

This class of solutions is made up of curves that are geodesics in Hopf tubes over curves either in the hyperbolic plane or in the anti de Sitter plane. In both cases the slope is determined by n and p. The steps will be sketched as follows.

Trajectories being geodesics of Hopf tubes

The algorithm to get the solutions of this subfamily runs as follows.

1. Take a unit speed curve $\gamma(s)$ in the hyperbolic plane $\mathbb{H}^2(-4)$ and consider its Hopf tube $\pi_-^{-1}(\gamma)$ in $\mathbf{AdS}_3$ (see Appendix B).

2. This is a Lorentzian flat surface that can be parametrized with coordinate curves being, respectively, the fibers and the horizontal lifts of γ, in the following way

$$\Phi(s,t) = \cos(t)\, \bar{\gamma}(s) + \sin(t)\, i\bar{\gamma}(s).$$

3. Choose now the arclength parametrized geodesic of $\pi_-^{-1}(\gamma)$ defined by

$$\gamma_{np}(u) = \Phi(au, bu), \qquad a^2 - b^2 = \varepsilon_1, \qquad \frac{b^2}{a^2} = \frac{p^2}{n^2}.$$

4. Let ρ be the curvature function of γ into $\mathbb{H}^2(-4)$. Then a direct computation gives the curvature κ and the torsion τ of γ_{np} in $\mathbf{AdS}_3$

$$\begin{aligned} \kappa &= a^2\rho + 2ab, \\ \tau^2 &= \kappa^2 - \varepsilon_1\kappa\rho + 1. \end{aligned}$$

¿From these equations, we obtain that $\tau = -\varepsilon_2 \frac{p}{n}\kappa \pm 1$. Therefore, γ_{np} is a path in $[\mathbf{AdS}_3, \mathcal{F}_{mnp}]$ with $n^2 \neq p^2$.

5. Finally, notice that all of solutions γ_{np} of this kind are either spacelike or timelike, according to $n^2 > p^2$ or $n^2 < p^2$, respectively.

Trajectories being geodesics of hyperbolic Hopf tubes

In this case, the algorithm is as follows.

1. Choose a unit speed curve $\sigma(s)$ in the anti de Sitter plane $\mathbf{AdS}_2(-4)$ with curvature function ρ and causal character δ_1. Now, we consider its hyperbolic Hopf tube $\lambda^{-1}(\sigma)$ in $\mathbf{AdS}_3$ (see Appendix B).

2. This is a flat surface which is either Riemannian or Lorentzian, according to σ is spacelike or timelike, respectively. It can be parametrized with coordinate curves being, respectively, the fibers and the horizontal lifts of σ, in the following way

$$\Psi(s,t) = \cosh(t)\,\bar{\sigma}(s) + \sinh(t)\,i\bar{\sigma}(s).$$

3. Choose now the arclength parametrized geodesic of $\lambda^{-1}(\sigma)$ defined by

$$\sigma_{np}(u) = \Psi(au, bu), \qquad \delta_1\, a^2 + b^2 = \varepsilon_1, \qquad \frac{b^2}{a^2} = \frac{p^2}{n^2}.$$

4. Let ρ be the curvature function of σ into $\mathbf{AdS}_2(-4)$. A direct computation gives the curvature κ and the torsion τ of σ_{np} in $\mathbf{AdS}_3$

$$\begin{aligned} \kappa &= a^2\rho + 2ab, \\ \tau^2 &= \kappa^2 - \varepsilon_1\delta_1\kappa\rho + 1. \end{aligned}$$

¿From here we get $\tau = -\varepsilon_2 \frac{p}{n}\kappa \pm 1$. Therefore, σ_{np} is a trajectory of $[\mathbf{AdS}_3, \mathcal{F}_{mnp}]$ with $n^2 \neq p^2$.

5. When σ is chosen to be timelike in $\mathbf{AdS}_2(-4)$, then the solutions σ_{np} are either spacelike or timelike according to $n^2 < p^2$ or $n^2 > p^2$, respectively.

Furthermore, the converse of these algorithms also hold. That is, all trajectories of these relativistic particle models are obtained according to them.

5.2. The dynamics in $[\mathbf{AdS}_3, \mathcal{F}_{mnp}]$ with $n^2 = p^2$

Notice that, in this case, no solutions are obtained as geodesics in Hopf tubes. In addition, no solutions are obtained in Lorentzian hyperbolic Hopf tubes. Therefore, besides the one-parameter class of ordinary helices, the model presents the following families.

Trajectories being geodesics of Riemannian hyperbolic Hopf tubes

These solutions are obtained by means of the following algorithm.

1. Choose a spacelike unit speed curve $\sigma(s)$ in the anti de Sitter plane $\mathbf{AdS}_2(-4)$ with curvature function ρ. Consider its hyperbolic Hopf tube $\lambda^{-1}(\sigma)$, which is a Riemannian flat surface in $\mathbf{AdS}_3$ (see Appendix B). As above, it can be parametrized with coordinate curves being, respectively, the fibers and the horizontal lifts of σ, in the following way

$$\Psi(s,t) = \cosh(t)\,\bar{\sigma}(s) + \sinh(t)\,i\bar{\sigma}(s).$$

2. Take now the arclength parametrized geodesics of $\lambda^{-1}(\sigma)$ defined by

$$\sigma_{\pm pp}(u) = \Psi(\pm pu, pu).$$

3. Let ρ be the curvature function of σ into $\mathbf{AdS}_2(-4)$. The curvature κ and the torsion τ of $\sigma_{\pm pp}$ in $\mathbf{AdS}_3$ are

$$\begin{aligned} \kappa &= -\frac{1}{2}\rho + 1, \\ \tau^2 &= -\kappa^2 - \kappa\rho + 1. \end{aligned}$$

Then $\tau = \kappa - 1$, so that $\sigma_{\pm pp}$ are trajectories of $[\mathbf{AdS}_3, \mathcal{F}_{mnp}]$ with $n^2 = p^2$.

As the converse in this algorithm also holds, all nondegenerate Lancret of the model are obtained following this method.

Trajectories being geodesics of scrolls over null curves

These s are degenerate Lancret curves obtained from the following algorithm.

1. Take a null curve $\alpha(s)$, $s \in I \subset \mathbb{R}$, in $\mathbf{AdS}_3$. Given a Cartan frame $\{A(s), D(s), F(s)\}$ along α, consider the flat scroll $\mathbf{S}_{\alpha D}$ (notice that $\mu = \pm 1$, see Appendix A), which can be parametrized by

$$\Phi(s,t) = \alpha(s) + t\, D(s), \qquad (s,t) \in I \times \mathbb{R}.$$

2. For an arclength parametrized geodesic β of $\mathbf{S}_{\alpha D}$, one can see that its acceleration is spacelike. Furthermore, its curvature and torsion functions are computed to satisfy $\tau = \pm\varepsilon_1\kappa \pm 1$ and so it is a trajectory of this model.

3. The converse also holds. Indeed, for a degenerate Lancret path β of $[\mathbf{AdS}_3, \mathcal{F}_{mnp}]$ with $n^2 = p^2$, its acceleration is spacelike and we may assume that $\tau = \kappa + \varepsilon_1$. The remaining cases can be handled likewise.

4. Define a null curve α in $\mathbf{AdS}_3$ by

$$\alpha(s) = \beta(s) - \frac{1}{2}s(T(s) - B(s)).$$

5. Take the following vector fields along α

$$\begin{aligned} A(s) &= -\frac{\varepsilon_1}{2}s\,\beta(s) + \frac{1}{2}(T(s) + B(s)) + \frac{\varepsilon_1}{2}s\,N(s), \\ D(s) &= -\varepsilon_1(T(s) - B(s)), \\ F(s) &= -\frac{1}{2}s\,(T(s) - B(s)) + N(s). \end{aligned}$$

It is not difficult to see that $\{A(s), D(s), F(s)\}$ is a Cartan frame along α with $\mu = 1$ and $\rho = \tau$.

6. Consider the scroll $\mathbf{S}_{\alpha D}$, which is a flat surface in $\mathbf{AdS}_3$ and can be parametrized by $\Phi(s,t) = \alpha(s) + t\,D(s)$.

7. Finally, notice that β can be viewed as a geodesic in this scroll, because $\beta(s) = \Phi(s, \frac{\varepsilon_1}{2}s)$.

6. A summary of trajectories ([7])

We can describe the dynamics of the relativistic particle models $[\mathbf{AdS}_3, \mathcal{F}_{mnp}]$ by applying the above algorithms. We then summarize the corresponding moduli space of trajectories as follows.

(A) The model $[\mathbf{AdS}_3, \mathcal{F}_{mnp}]$ with $n^2 > p^2$

The moduli space of s is made up of the following classes of trajectories:

- A one-parameter class of ordinary helices.
- A class of spacelike nondegenerate Lancret curves obtained as geodesics of Hopf tubes
$$\Gamma_{(n^2>p^2)} = \{\gamma_{np} \,|\, \gamma \text{ is a curve in } \mathbb{H}^2(-4)\}.$$
- A class of spacelike nondegenerate Lancret curves obtained as geodesics of Riemannian hyperbolic Hopf tubes
$$\Sigma^+_{(n^2>p^2)} = \{\sigma_{np} \,|\, \sigma \text{ is a spacelike curve in } \mathbf{AdS}_2(-4)\}.$$

- A class of timelike nondegenerate Lancret curves obtained as geodesics of Lorentzian hyperbolic Hopf tubes

$$\Sigma^-_{(n^2>p^2)} = \{\sigma_{np} \,|\, \sigma \text{ is a timelike curve in } \mathbf{AdS}_2(-4)\}.$$

(B) The model $[\mathbf{AdS}_3, \mathcal{F}_{mnp}]$ with $n^2 < p^2$

The moduli space of s is made up of the following classes of trajectories.

- A one-parameter class of ordinary helices.
- A class of timelike nondegenerate Lancret curves obtained as geodesics of Hopf tubes

$$\Gamma_{(n^2<p^2)} = \{\gamma_{np} \,|\, \gamma \text{ is a curve in } \mathbb{H}^2(-4)\}.$$

- A class of spacelike nondegenerate Lancret curves obtained as geodesics of Riemannian hyperbolic Hopf tubes

$$\Sigma^+_{(n^2<p^2)} = \{\sigma_{np} \,|\, \sigma \text{ is a spacelike curve in } \mathbf{AdS}_2(-4)\}.$$

- A class of spacelike nondegenerate Lancret curves obtained as geodesics of Lorentzian hyperbolic Hopf tubes

$$\Sigma^-_{(n^2<p^2)} = \{\sigma_{np} \,|\, \sigma \text{ is a timelike curve in } \mathbf{AdS}_2(-4)\}.$$

(C) The model $[\mathbf{AdS}_3, \mathcal{F}_{mnp}]$ with $n^2 = p^2$

The moduli space of s is made up of the following classes of trajectories.

- A one-parameter class of ordinary helices.
- A class of spacelike nondegenerate Lancet curves obtained as geodesics of Riemannian hyperbolic Hopf tubes

$$\Sigma^{+}_{(n^2=p^2)} = \{\sigma_{np} \,|\, \sigma \text{ is a spacelike curve in } \mathbf{AdS}_2(-4)\}.$$

- A class of degenerate Lancret curves obtained as geodesics of scrolls over null curves

$$\Upsilon = \{\beta_{\alpha D}(s) = \alpha(s) \pm \frac{s}{2} D(s) \,|\, \alpha \text{ is a null curve in } \mathbf{AdS}_3\}.$$

Remark. It should be noticed that a Lancret curve in $\mathbf{AdS}_3$ with Lorentzian rectifying plane at any point is simultaneously degenerate and nondegenerate, because it admits both null and non null axes. Consequently, it can be viewed as a geodesic of a Hopf tube but also as one in a flat scroll over a null curve. Therefore, a Lancret curve can be regarded as a trajectory in different models.

7. A general Lagrangian density in 3-dimensional space forms ([19])

Let $M^3_\nu(C)$ be a 3-dimensional pseudo-Riemannian space form of curvature C and index ν. Let $\gamma : I \to M^3_\nu(C)$ be an immersed curve with speed $v(t) = |\gamma'(t)|$, curvature k, torsion τ and Frenet frame $\{T, N, B\}$. The Frenet equations write down as follows

$$\begin{cases} \nabla_T T = \varepsilon_2 k N, \\ \nabla_T N = -\varepsilon_1 k T + \varepsilon_3 \tau B, \\ \nabla_T B = -\varepsilon_2 \tau N, \end{cases}$$

where $\varepsilon_1 = \langle T, T\rangle$, $\varepsilon_2 = \langle N, N\rangle$ and $\varepsilon_3 = \langle B, B\rangle$. Let

$$\mathcal{L}(\gamma) = \int_\gamma f(k, \tau) ds \tag{8}$$

be the action for any real function f defined on an open set of $\mathbb{R}^2$. Let $\Gamma = \Gamma(t, r) : [0, L] \times (-\delta, \delta) \to M$ be a variation of a curve $\gamma : [0, L] \to M^3_\nu(C)$ with $\Gamma(t, 0) = \gamma(t)$. Associated with Γ we consider the variation vector field $W = W(t) = \frac{\partial \Gamma}{\partial r}(t, 0)$ along $\gamma(t)$. We also write $V = V(t, r) = \frac{\partial \Gamma}{\partial t}(t, r)$, $W = W(t, r)$, $v = v(t, r)$, $T = T(t, r)$, $N = N(t, r)$, $B = B(t, r)$, etc., with the obvious meanings. Let s denote the arclength parameter, and let $V(s, r)$, $W(s, r)$, etc., be the corresponding reparametrizations.

Then, by using standard arguments involving the above formulas and integration by parts, the first variation of $\mathcal{L}(\gamma)$ along γ in the direction of W is given by

$$\mathcal{L}'(0) = [\mathcal{B}(\gamma, W)]_0^L - \int_0^L \left\langle \nabla_T P - \varepsilon_1 C f_k N + \varepsilon_1 \varepsilon_2 C \frac{f_\tau'}{k} B, W \right\rangle ds, \tag{9}$$

where the vector P is given by

$$P = \varepsilon_1 (f - (2k f_k + \tau f_\tau))T + \varepsilon_1 k f_\tau B - \nabla_T (f_k N) + \varepsilon_2 \nabla_T \left(\frac{f_\tau'}{k} B \right)$$

and the boundary term is

$$\mathcal{B}(\gamma, W) = \left\langle \nabla_T^2 W, \varepsilon_2 \frac{f_\tau}{k} B \right\rangle + \left\langle \nabla_T W, f_k N - \frac{\varepsilon_2}{k} f_\tau' B \right\rangle + \left\langle W, P + \frac{\varepsilon_1 \varepsilon_2 C f_\tau}{k} B \right\rangle.$$

the critical curves are characterized by the vanishing of the Euler-Lagrange operator $\mathcal{E}$

$$\mathcal{E} := -\left(\nabla_T P - \varepsilon_1 C f_k N + \varepsilon_1 \varepsilon_2 C \frac{f_\tau'}{k} B \right) = 0. \tag{10}$$

It is a straightforward computation to show that equation (10) is equivalent to the Euler-Lagrange equations

$$-\varepsilon_1 \varepsilon_2 k f - \varepsilon_2 (\varepsilon_3 \tau^2 - \varepsilon_1 k^2) f_k + 2\varepsilon_1 \varepsilon_2 k \tau f_\tau + f_k'' + \left(\tau \frac{f_\tau'}{k} \right)' + \tau \left(\frac{f_\tau'}{k} \right)' + \varepsilon_1 C f_k = 0, \tag{11}$$

$$\varepsilon_3 \tau f_k' + \varepsilon_3 \frac{\tau^2}{k} f_\tau' + \varepsilon_3 (\tau f_k)' - \varepsilon_1 (k f_\tau)' - \varepsilon_2 \left(\frac{f_\tau'}{k} \right)'' - \varepsilon_1 \varepsilon_2 C \frac{f_\tau'}{k} = 0. \tag{12}$$

The critical curves of the Lagrangian (8) admit two Killing vector fields P and J given by

$$P = \varepsilon_1 \left(f - (kf_k + \tau f_\tau)\right) T - \left(f_k' + \frac{\tau}{k} f_\tau'\right) N + \left(-\varepsilon_3 \tau f_k + \varepsilon_1 k f_\tau + \varepsilon_2 \left(\frac{f_\tau'}{k}\right)'\right) B, \quad (13)$$

$$J = -\varepsilon_1 f_\tau T - \frac{f_\tau'}{k} N - \varepsilon_3 f_k B, \quad (14)$$

satisfying that

i) $\mathcal{E} = -(\nabla_T P + \varepsilon C J \wedge T)$

ii) $\nabla_T J = -P \wedge T$

where $\varepsilon = \varepsilon_1 \varepsilon_2 \varepsilon_3$.

The critical curves of the Lagrangian (8) satisfy the integral equations

$$\begin{cases} \langle P, P \rangle + \varepsilon C \langle J, J \rangle = d, \\ \langle P, J \rangle = e, \end{cases} \quad (15)$$

for suitable constants d and e.

In [7] we have studied actions in $D = 3$ spacetimes whose Lagrangian is a linear function $m + nk + p\tau$ on the curvature and torsion of the particle path, finding out that trajectories are Lancret curves, or generalized helices. Indeed, the critical curves are

always Lancret curves, which are obtained by geometrical integration involving the Hopf fibrations (see also [4]). Here we go further by assuming that the Lagrangian density is an arbitrary function on the curvature and torsion of the particle path which is lying in a 3-dimensional pseudo-Riemannian space form. We have got two Killing vector fields along curves P and J and exploited the machinery supplied by them, which became a fruitful tool in our earlier and recent paper. Actually, the integral equations are reduced to a system involving P and J, which is equivalent to the Euler-Lagrange equations if, and only if, $\langle J, J\rangle$ is not constant. We note that when the Lagrangian density is $m + nk + p\tau$, then $\langle J, J\rangle$ is constant. Then we have solved the motion equations and found out solutions which, as a pretty interesting fact, are not generalized helices.

To obtain explicitly the critical curves of the Lagrangian, we have chosen suitable coordinate frames where the Frenet equations have been integrated. With the help of the corresponding Lie algebras, a complete system of solutions is given in the de Sitter $\mathbb{S}^3_1$ and anti de Sitter $\mathbb{H}^3_1$ worlds as well as in the non-flat Riemannian space forms $\mathbb{S}^3$ and $\mathbb{H}^3$.

8. Variational problems with torsion in greater dimensions ([6])

We consider the action $\mathcal{L} : \Lambda \to \mathbb{R}$ given by

$$\mathcal{L}(\gamma) = \int_\gamma (pk_2 + q) ds,$$

where p and q are constants. The simplest action describing the motion of a particle is achieved when $p = 0$, so that it is proportional to the proper time. The worldlines of the particles are geodesic curves in the background space.

To compute the first-order variation of this action along the elementary fields space Λ, and so the field equations describing the dynamics of the particles, we use a standard argument involving some integrations by parts. Then by using the Frenet equations we have

$$\mathcal{L}'(0) = [\mathcal{B}(\gamma, W)]_0^L - \int_0^L \langle \nabla_T P, W \rangle \, v dt, \tag{16}$$

where the vector field P is given by

$$P = \varepsilon_1 \varepsilon_3 p \nabla_T \left(\frac{k_3}{k_1} N_3 \right) + \varepsilon_0 p k_1 N_2 + \varepsilon_0 q T$$

and the boundary term reads

$$\mathcal{B}(\gamma, W) \;=\; \left\langle \nabla_T^2 W, \varepsilon_1 \frac{p}{k_1} N_2 \right\rangle + \left\langle \nabla_T W, -\varepsilon_1 \varepsilon_3 p \frac{k_3}{k_1} N_3 \right\rangle + \langle W, P \rangle \,,$$

W standing for a generic variational vector field along γ. We take curves with the same endpoints and having there the same Frenet frame, so that $[\mathcal{B}(\gamma, W)]_0^L$ vanishes. From here we obtain the following result.

The trajectory $\boldsymbol{\gamma \in \Lambda}$ is the worldline of a relativistic particle in the $\boldsymbol{d}$-dimensional background $\boldsymbol{\mathbb{R}^d_\nu}$ if and only if

(i) The Frenet apparatus is well defined ion the whole world trajectory.

(ii) The vector field $\boldsymbol{P}$ is constant along $\boldsymbol{\gamma}$.

In some sense, the vector field P can be interpreted as the linear momentum of the particle and then the above is a consequence of the conserved linear momentum law. Note that the vector field P possesses a non-vanishing space-like component orthogonal to the particle trajectory, which seems to be a manifestation of a generic feature of higher-derivative theories.

A straightforward computation shows that P is constant if and only if the following equations of motion hold:

$$
\begin{aligned}
pk_2(1-\varepsilon\varphi^2)-q &= 0, && (17)\\
k_1'(1-\varepsilon\varphi^2)-3\varepsilon k_1\varphi\varphi' &= 0, && (18)\\
-\varepsilon_2\varepsilon_3\varphi''+\varepsilon_2\varepsilon_4\varphi k_4^2-\varepsilon k_1^2\varphi(1-\varepsilon\varphi^2) &= 0, && (19)\\
2k_4\varphi'+\varphi k_4' &= 0, && (20)\\
k_3k_4k_5 &= 0, && (21)
\end{aligned}
$$

where $\varepsilon=\varepsilon_0\varepsilon_1\varepsilon_2\varepsilon_3$ and $\varphi=k_3/k_1$. The last equation of motion yields $k_5=0$, so that the motion will be restricted to (at most) a 5-dimensional subspace. On the other hand, from Eq. (20) we easily find that φ^2k_4 is a constant B, which determines k_4 in terms of the lower curvatures.

The solutions of the equations of motion

The main goal of this section is to integrate the motion equations of Lagrangians giving models for relativistic particles that linearly involve the torsion of the worldline. We have already integrate Eqs. (20) and (21) in the last section, so that we are going to integrate here Eqs. (17) to (19).

8.1. The 4-dimensional case

In the first place we will integrate the motion equations in $\mathbb{R}^4_\nu$. Let Z_1 and Z_2 be constant vector fields and consider the vector field $W = \gamma \wedge Z_1 \wedge Z_2$, then the boundary term reads

$$\mathcal{B}(\gamma, W) = \langle (p\, N_1 \wedge N_2 - \varepsilon_1\varepsilon_3 p\varphi\, T \wedge N_3 + \gamma \wedge P) \wedge Z_1, Z_2 \rangle\,.$$

From this we define a map $\Phi : \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma)$ by

$$\Phi(Z) = (p\, N_1 \wedge N_2 - \varepsilon_1\varepsilon_3 p\varphi\, T \wedge N_3 + \gamma \wedge P) \wedge Z,$$

where $\mathfrak{X}(\gamma)$ denotes the algebra of differentiable vector fields along the trajectory path of the particle. Observe that $\langle \Phi(Z), Z \rangle = 0$ for every vector field Z. It is not difficult to see that Φ is covariantly constant, i.e. $\nabla_T \Phi = 0$, so that the vector fields $Q = \Phi(P)$ and $V = \Phi(Q)$ are also constant vectors and

$$\begin{aligned} Q &= p\, N_1 \wedge N_2 \wedge P - \varepsilon_1\varepsilon_3 p\varphi\, T \wedge N_3 \wedge P, \\ V &= p\, N_1 \wedge N_2 \wedge Q - \varepsilon_1\varepsilon_3 p\varphi\, T \wedge N_3 \wedge Q + \gamma \wedge P \wedge Q. \end{aligned}$$

Then $J = -\gamma \wedge P \wedge Q + V$ is a Killing vector field along γ. The vector fields P, Q and J read

$$P = \varepsilon_0 q\, T + \varepsilon_0 p k_1 (1 - \varepsilon\varphi^2)\, N_2 + \varepsilon_1\varepsilon_3 p\varphi'\, N_3, \tag{22}$$

$$Q = -\varepsilon_0\varepsilon_1\varepsilon_3 p^2\varphi'\, T + \varepsilon_0\varepsilon_3 p^2\varphi k_1 (1 - \varepsilon\varphi^2)\, N_1 + \varepsilon_0\varepsilon_3 pq\, N_3, \tag{23}$$

$$J = p^2(-\varepsilon_3 q\, T - \varepsilon_0\varepsilon_1\varepsilon_2 p\varphi^2 k_1 (1 - \varepsilon\varphi^2)\, N_2 - \varepsilon_0\varepsilon_1 p\varphi' N_3), \tag{24}$$

and they can be interpreted as generators of the particle mass M and spin S, with the mass-shell condition and the Majorana-like relation between M and S given by $\langle P, P\rangle = M^2$ and $\langle P, J\rangle = MS$. Note that there will be the possibility of tachyonic energy flow, since the mass could be positive, negative or zero, according to the causal character of the vector field P. Time-like and light-like trajectories are the natural ones in space-time geometries, but some recent experiments point out the existence of superluminal particles (space-like trayectories) without any breakdown of the principle of relativity; theoretical developments exist suggesting that neutrinos might be instances of "tachyons" as their square mass appears to be negative.

By using that P and Q are constant vector fields along γ, so that $\langle P, P\rangle = \varepsilon_P u^2$ and $\langle Q, Q\rangle = \varepsilon_Q v^2$ also are, we obtain the following two first integrals for φ:

$$\varepsilon_3 p^2(\varphi')^2 + \varepsilon_0 q^2 + \varepsilon_2 p^2 k_1^2(1-\varepsilon\varphi^2)^2 = \varepsilon_P u^2, \tag{25}$$

$$p^2[\varepsilon_0 p^2(\varphi')^2 + \varepsilon_1\varphi^2 p^2 k_1^2(1-\varepsilon\varphi^2)^2 + \varepsilon_3 q^2] = \varepsilon_Q v^2. \tag{26}$$

From here one easily finds that there exists a constant A such that

$$k_1^2(1-\varepsilon\varphi^2)^3 = A.$$

If A is nonzero, this equation jointly with Eq. (25) yield

$$(\varphi')^2 = \frac{(\varepsilon_P u^2 - \varepsilon_0 q^2)(1-\varepsilon\varphi^2) - \varepsilon_2 p^2 A}{\varepsilon_3 p^2(1-\varepsilon\varphi^2)}.$$

This ODE can be integrate and its solution reads

$$a\mathrm{E}\left(\arcsin\left(b\varphi\right),\frac{\varepsilon}{b^2}\right)=t+C_1,$$

where C_1 is an arbitrary constant, $a=\sqrt{\frac{\varepsilon_3 p^2}{\varepsilon(\varepsilon_P u^2-\varepsilon_0 q^2)}}$, $b=\sqrt{\frac{\varepsilon(\varepsilon_P u^2-\varepsilon_0 q^2)}{-\varepsilon_2 p^2 A+\varepsilon_P u^2-\varepsilon_0 q^2}}$ and E stands for the elliptic function of second kind. From here and Eq. (17) we obtain the curvatures

$$k_1^2=\frac{A}{\left(1-\varepsilon\varphi^2\right)^3},\qquad k_2=\frac{q}{p\left(1-\varepsilon\varphi^2\right)},$$

$$k_3^2=\frac{A\varphi^2}{\left(1-\varepsilon\varphi^2\right)^3}.$$

If $A=0$ then we easily obtain $\varphi^2=\varepsilon$, so that $k_3=\pm k_1$, and from equation (17) we deduce $q=0$. Note that this case can not appear in the Lorentzian background.

8.2. The 5-dimensional case

In this section we study the motion equations when the forth curvature k_4 is non-zero, so the curve γ is fully in $\mathbb{R}^5_\nu$. Note that most of the computations in the 4-dimensional case are useful here.

The linear momentum vector field P reads now as

$$P = \varepsilon_0 q\, T + \varepsilon_0 p k_1 (1 - \varepsilon\varphi^2)\, N_2 + \varepsilon_1\varepsilon_3 p\varphi'\, N_3 + \varepsilon_1\varepsilon_3\varepsilon_4\varphi k_4\, N_4,$$

from which we get the following ODE

$$\varepsilon_3 p^2 (\varphi')^2 = \frac{(\varepsilon_P u^2 - \varepsilon_0 q^2)(1 - \varepsilon\varphi^2)\varphi^2 - \varepsilon_2 p^2 A\varphi^2 - \varepsilon_4 B^2 (1 - \varepsilon\varphi^2)}{(1 - \varepsilon\varphi^2)\varphi^2}.$$

This differential equation can be integrated to obtain the function φ. Now and using a similar reasoning as in the 4-dimensional case we obtain the following curvature functions

$$k_1^2 = \frac{A}{(1 - \varepsilon\varphi^2)^3}, \qquad k_2 = \frac{q}{p\,(1 - \varepsilon\varphi^2)},$$

$$k_3^2 = \frac{A\varphi^2}{(1 - \varepsilon\varphi^2)^3}, \qquad k_4 = \frac{B}{\varphi^2}.$$

9. Variational problems concerning light-like curves

(9.1) Lagrangian density is linear in the curvature in $M_1^3(C)$ ([16])

Let M_1^3 denote a 3-dimensional space-time with background gravitational field $\langle , \rangle$, constant curvature G and Levi-Civita connection ∇.

We consider mechanical systems with Lagrangians which linearly depend on the curvature of a light-like curve. This curvature function is sometimes called torsion since it is obtained from the third derivative of the relativistic null path. The space of elementary fields in this theory is the set Λ of null Cartan curves, [15], satisfying given first order boundary data to drop out the boundary terms which appear when computing the first order variation of the action.

Let $\gamma : I = [a, b] \to M_1^3$ be a null Cartan curve such that $\{\gamma'(s), \gamma''(s), \gamma'''(s)\}$ is positively oriented for all $s \in I$ with Cartan frame $\{L, W, N\}$, where $\langle L, L\rangle = \langle N, N\rangle = 0$ and $\langle L, N\rangle = -1$. The Cartan equations are given by (see [15] for details):

$$\begin{aligned} L' &= W, \\ W' &= -kL + N, \\ N' &= -kW, \end{aligned} \tag{27}$$

where the prime $()'$ denotes covariant derivative.

We consider the action $S : \Lambda \to \mathbb{R}$ given by

$$S(\gamma) = 2c \int_\gamma (\lambda + \mu k(s)) ds.$$

When $\lambda = 1$ and $\mu = 0$ it leads to the action studied by Nersessian and Ramos in [22, 23]. The case $\mu = 1$ has been considered by Nersessian in [24].

To compute the first-order variation of this action, along the elementary fields space Λ, and so the field equations describing the dynamics of this particle, we use a standard argument involving some integrations by parts. Then by using the Cartan equations we have

$$S'(0) = [\Omega]_a^b - c \int_a^b \langle V, (\mu k''' + 3\mu k k' - \lambda k')L \rangle \, ds \tag{28}$$

where

$$\begin{aligned} \Omega = & -c\mu L(h) + 2c\mu \left\langle \nabla_L^2 V, N \right\rangle + c(\mu k + \lambda) \left\langle \nabla_L V, W \right\rangle \\ & - c \left\langle V, \nabla_L((\mu k + \lambda)W) \right\rangle + 2c(\frac{1}{2}\mu k'' - \lambda k + 2\mu G) \left\langle V, L \right\rangle + 2c\mu k \left\langle V, N \right\rangle, \end{aligned} \tag{29}$$

V standing for a generic variational vector field along γ and $h = -\left\langle \nabla_L^2 V, W \right\rangle$.

We take curves with the same endpoints and having the same Cartan frame in them, so that $[\Omega]_a^b$ vanishes. Under these conditions, the first-order variation is

$$S'(0) = -c \int_a^b \langle V, (\mu k''' + 3\mu k k' - \lambda k')L \rangle \, ds,$$

from which we obtain the following statement.

The trajectory $\gamma \in \Lambda$ is the null worldline of a relativistic particle in the (2+1)-dimensional spacetime if and only if:

(i) W, N and k are well defined on the whole world trajectory.

(ii) The following differential equation is satisfied

$$\mu k''' + 3\mu k k' - \lambda k' = 0. \tag{30}$$

A first integration of the equation gives us

$$k'' + \frac{3}{2}k^2 - \lambda k + C = 0,$$

where C is a constant. By standard techniques of integration, this equation leads to

$$(k')^2 + k^3 - \lambda k^2 + 2Ck + D = 0, \tag{31}$$

where D is another constant. Note that constants C and D are not arbitrary, since they are related with the mass m and the spin s of the particle.

Now we are going to analyze all possible cases and present pictures of the corresponding curvature functions.

To get the explicit solution of the motion equation, put $(k')^2 = P(k)$, where P is a polynomial of degree 3. By using standard techniques involving the elliptic Jacobi functions, the solution can be found according to the roots α_1, α_2 and α_3 ($\alpha_1 \leq \alpha_2 \leq \alpha_3$) of the equation $P(t) = 0$.

Before obtaining all the solutions, note that since $P(k) = (k')^2$ then k takes values only where P is non negative. Trivial solutions are $k(s) = \alpha_i$, where α_i is a real root of P. Now we are going to analyze all possible cases and present pictures of the corresponding curvature functions.

I. P has a real root of multiplicity 3: $\alpha = \alpha_1 = \alpha_2 = \alpha_3$

We have that $\alpha = \lambda/3$ and the curvature function is given by

$$k(s) = \frac{\lambda}{3} - \frac{4}{(s+E)^2}, \qquad s \in (-\infty, \lambda/3)$$

where E is a constant of integration depending on the initial condition satisfying that $s + E$ is always different from zero (see Figure **??** (i)).

II. P has two real roots, the lowest with multiplicity 2: $\alpha = \alpha_1 = \alpha_2 < \alpha_3$

The root α_3 is given by $\lambda - 2\alpha$. There are two possibilities:

$$k(s) = \lambda - 2\alpha + (3\alpha - \lambda)\coth^2\left(\frac{1}{2}\sqrt{\lambda - 3\alpha}(s + E)\right), \quad s \in (-\infty, \alpha)$$

$$k(s) = \lambda - 2\alpha + (3\alpha - \lambda)\coth^2\left(\frac{1}{2}\sqrt{\lambda - 3\alpha}(s + E)\right), \quad s \in (\alpha, \lambda - 2\alpha]$$

where E is a constant (see Figure **??** (ii)-(iii)).

III. P has two real roots, the greatest with multiplicity 2: $\alpha = \alpha_1 < \alpha_2 = \alpha_3$

We obtain that $\alpha_2 = \alpha_3 = (\lambda - \alpha)/2$, and the solution is given by

$$k(s) = \alpha + \frac{3\alpha - \lambda}{2}\tan^2\left(\frac{1}{2}\sqrt{\frac{\lambda - 3\alpha}{2}}(s + E)\right), \qquad s \in (-\infty, \alpha]$$

where E is a constant (see Figure **??** (iv)).

IV. P has three distinct real roots: $\alpha_1 < \alpha_2 < \alpha_3$

Let us denote $\alpha = \alpha_1$ and $\beta = \alpha_2$, then $\alpha_3 = \lambda - \alpha - \beta$. There are two possibilities for the curvature:

$$k(s) = \alpha - (\beta - \alpha)\,\mathrm{tn}^2\left(\frac{1}{2}\sqrt{\lambda - 2\alpha - \beta}(s + E), \sqrt{\frac{\lambda - \alpha - 2\beta}{\lambda - 2\alpha - \beta}}\right),$$

$$k(s) = \lambda - \alpha - \beta + (\alpha + 2\beta - \lambda)\,\mathrm{sn}^2\left(\frac{1}{2}\sqrt{\lambda - 2\alpha - \beta}(s + E), \sqrt{\frac{\lambda - \alpha - 2\beta}{\lambda - 2\alpha - \beta}}\right),$$

defined on the intervals $(-\infty, \alpha]$ or $[\beta, \lambda - \alpha - \beta]$, respectively (see Figure **??** (v)-(vi)).

V. P has complex roots

Let us suppose that α_1 and α_2 are complex (so α_3 is real). Then the curvature is given by

$$k(s) = \alpha_3 - (\alpha_3 - \alpha_2)\,\mathrm{sn}^2\left(\frac{1}{2}\sqrt{\alpha_3 - \alpha_1}(s + E), \sqrt{\frac{\alpha_2 - \alpha_3}{\alpha_1 - \alpha_3}}\right), \qquad s \in (-\infty, \alpha_3].$$

(See Figure **??** (vii)).

Sketching worldlines

Once we know the curvature functions, the worldlines of the relativistic particles can be obtained by integrating the Cartan equations. The explicit integration of these equations is a difficult task, sometimes impossible (even when the curvature is a nice function). In our case, the goal of finding the exact worldlines can be reached by numeric integration. Now we we will sketch (with the help of MATHEMATICA) the particle worldlines in all discussed cases in the preceding section.

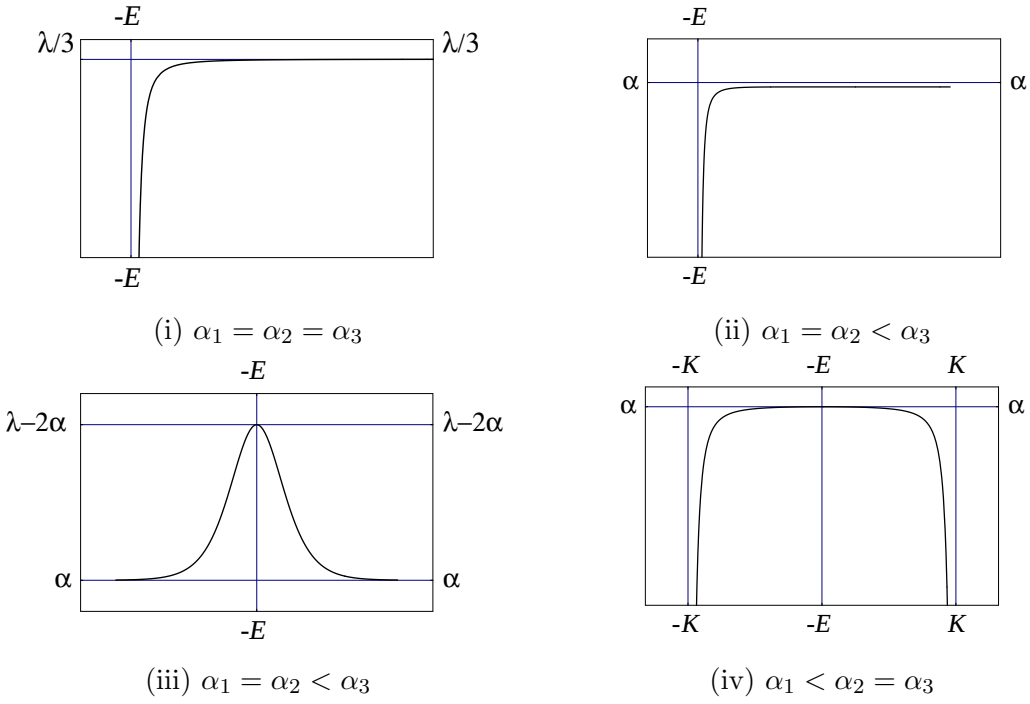
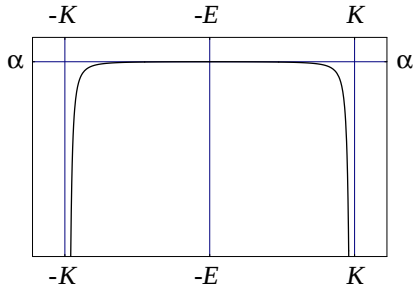
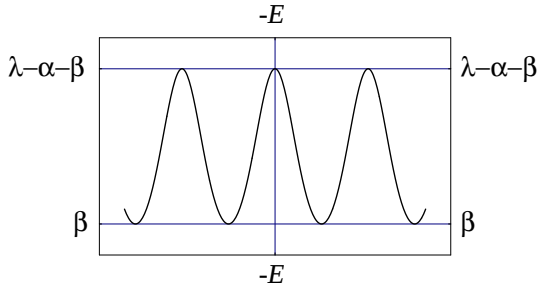


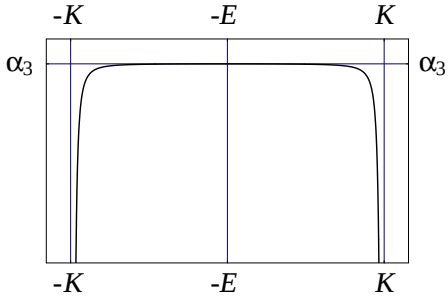
Fig.1: *Curvature function for the different possibilities of the roots of the polynomial P. It is quite interesting to remark that in cases (i), (ii) and (iii), as s ncreases, $k(s)$ approaches to a constant, said otherwise, the trajectory looks like a helix.*



(v) $\alpha_1 < \alpha_2 < \alpha_3$

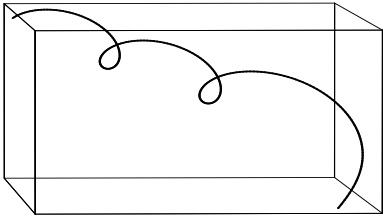


(vi) $\alpha_1 < \alpha_2 < \alpha_3$

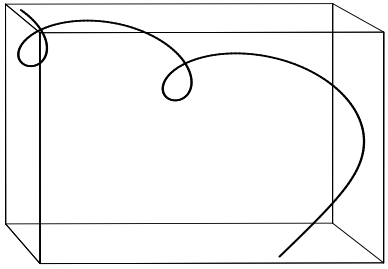


(vii) Complex roots

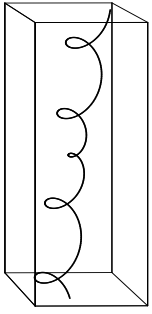
Fig.2: *Curvature function for the different possibilities of the roots of the polynomial P.*



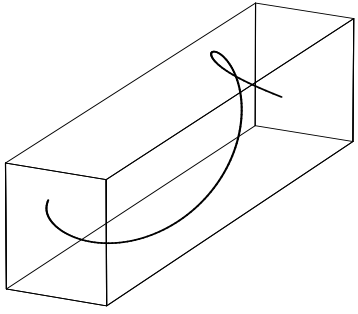
$\alpha_1 = \alpha_2 = \alpha_3,$
$k(s) \in (-\infty, \alpha_1]$



$\alpha_1 = \alpha_2 < \alpha_3,$
$k(s) \in (-\infty, \alpha_1]$

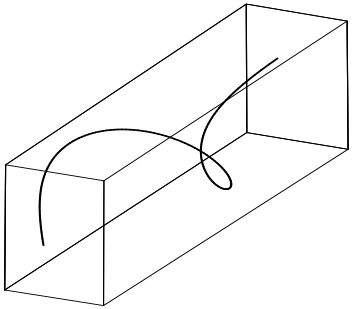


$\alpha_1 = \alpha_2 < \alpha_3,$
$k(s) \in (\alpha_2, \alpha_3]$

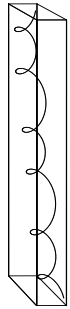


$\alpha_1 < \alpha_2 = \alpha_3,$
$k(s) \in (-\infty, \alpha_1]$

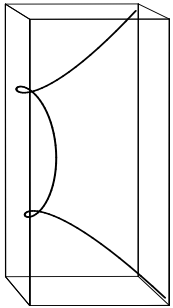
Fig.3: *Worldlines for corresponding curvature functions.*



$\alpha_1 < \alpha_2 < \alpha_3,$
$k(s) \in (-\infty, \alpha_1]$



$\alpha_1 < \alpha_2 < \alpha_3,$
$k(s) \in [\alpha_2, \alpha_3]$



$\alpha_1 = \alpha + \beta i,$
$\alpha_2 = \alpha - \beta i,\ \alpha_3$ real,
$k(s) \in (-\infty, \alpha_3]$

Fig.4: *Worldlines for corresponding curvature functions.*

(9.2) Lagrangian density linear in the curvature in $\mathbb{L}^n$ ([18])

Let $\mathbb{L}^n$ be an n-dimensional Lorentz-Minkowski space with background gravitational field $\langle , \rangle$ and Levi-Civita connection ∇. First of all, we will describe the geometry of null curves in $\mathbb{L}^n$ in terms of the Cartan frame of the curve (see [15] for details).

Let $\gamma : [a, b] \to \mathbb{L}^n$ be a null Cartan curve such that the frame $\left\{\gamma'(\sigma), \gamma''(\sigma), \ldots, \gamma^{(n)}(\sigma)\right\}$ is positively oriented, for all $\sigma \in [a, b]$, σ being the pseudo-arc parameter. Let us consider its corresponding Cartan frame $\{L = \gamma', W_1, N, W_2, \ldots, W_{n-2}\}$, where

$$\begin{aligned} \langle L, L\rangle = \langle N, N\rangle = 0, \qquad & \langle L, N\rangle = -1, \\ \langle W_i, L\rangle = \langle W_i, N\rangle = 0, \qquad & \langle W_i, W_j\rangle = \pm 1. \end{aligned}$$

The Cartan equations read

$$\begin{aligned} L' &= W_1, \\ W_1' &= -k_1 L + N, \\ N' &= -k_1 W_1 + k_2 W_2, \\ W_2' &= k_2 L + k_3 W_3, \\ W_i' &= -k_i W_{i-1} + k_{i+1} W_{i+1} \quad i \in \{3, \ldots, n-3\}, \\ W_{n-2}' &= -k_{n-2} W_{n-3}, \end{aligned} \tag{32}$$

where $()'$ means covariant derivative and k_i are the *Cartan curvatures* of the curve.

The actions $\mathcal{L}$ for the curve depend locally on its geometry and they possess various symmetries, both local and global. The local symmetry is reparametrization invariance and it restricts severely the form of $\mathcal{L}$. We consider the action $\mathcal{L} : \Lambda \to \mathbb{R}$ given by

$$\mathcal{L}(\gamma) = \int_\gamma (\mu k_1 + \lambda) d\sigma, \tag{33}$$

μ and λ both being constant. The simplest action describing the motion of a particle is achieved when it is proportional to the pseudo-arc length parameter (i.e. $\mu = 0$), which has been studied by Nersessian and Ramos in [22, 23] when $n = 2, 3$. When the action is linear on the curvature of the particle path, some advances have been achieved in [16, 24].

A null curve γ is said to be a critical point of the action $\mathcal{L}$ when

$$\left.\frac{d}{d\omega}\right|_{\omega=0} \mathcal{L}(\gamma_\omega) = \left.\frac{d}{d\omega}\right|_{\omega=0} \int_{\gamma_\omega} (\mu k_1 + \lambda) d\sigma = 0,$$

for all variation throughout null curves γ_ω of γ.

To compute the first-order variation of this action along the elementary fields space Λ, and so the field equations describing the dynamics of the particle, we use a standard argument involving some integrations by parts. Then the Cartan equations yield

$$\mathcal{L}'(0) = \frac{1}{2}[\Omega(\gamma, V)]_a^b - \frac{1}{2}\int_a^b \langle V, \mathcal{E}_1(\gamma)L + \mathcal{E}_2(\gamma)W_2 + \mathcal{E}_3(\gamma)W_3 + \mathcal{E}_4(\gamma)W_4\rangle \, d\sigma, \tag{34}$$

where

$$
\begin{aligned}
\mathcal{E}_1(\gamma) &= \mu k_1''' + 2\mu k_2 k_2' + 3\mu k_1 k_1' - \lambda k_1', \\
\mathcal{E}_2(\gamma) &= 2\mu k_2'' - k_2(2\mu k_3 - \mu k_1 + \lambda), \\
\mathcal{E}_3(\gamma) &= 2\mu(k_2 k_3' - 2k_2' k_3), \\
\mathcal{E}_4(\gamma) &= k_2 k_3 k_4,
\end{aligned}
\tag{35}
$$

and the boundary term reads

$$
\begin{aligned}
\Omega(\gamma, V) = &\left\langle \nabla_L^3 V, \mu W_1 \right\rangle + \left\langle \nabla_L^2 V, -\mu k_1 L + 3\mu N \right\rangle \\
&+ \left\langle \nabla_L V, (\mu k_1 + \lambda) W_1 - \mu k_2 W_2 \right\rangle + \left\langle V, P_1 \right\rangle,
\end{aligned}
\tag{36}
$$

where P_1 is the vector field given by

$$
P_1 = \left(\mu k_1'' + \mu k_1^2 - \lambda k_1\right) L - \mu k_1' W_1 + (\mu k_1 - \lambda) N + 2\mu k_2' W_2 + 2\mu k_2 k_3 W_3, \tag{37}
$$

and V stands for a generic variational vector field along γ.

To drop $[\Omega(\gamma, V)]_a^b$ we have to consider curves with the same endpoints and having the same Cartan frame there. Under these conditions, the first-order variation reads

$$
\mathcal{L}'(0) = -\frac{1}{2} \int_a^b \left\langle V, \mathcal{E}_1(\gamma) L + \mathcal{E}_2(\gamma) W_2 + \mathcal{E}_3(\gamma) W_3 + \mathcal{E}_4(\gamma) W_4 \right\rangle d\sigma.
$$

As a consequence we have

A null curve $\gamma \in \Lambda$ is critical for the linear action $\mathcal{L}(\gamma)$ in $\mathbb{L}^n$ if and only if the following statements hold:

(i) W_i, N and k_j are well defined along the whole trajectory; and

(ii) The following differential equations are fulfilled:

$$\mathcal{E}_1(\gamma) = 0, \quad \mathcal{E}_2(\gamma) = 0, \quad \mathcal{E}_3(\gamma) = 0, \quad \mathcal{E}_4(\gamma) = 0.$$

These equations are called the *Euler-Lagrange equations*. The following is an easy consequence from the last equation of (35).

The critical points for the linear action $\mathcal{L}(\gamma)$ in $\mathbb{L}^n$ lie in a Lorentzian subspace of dimension not greater than five.

By considering the special case where the action is constant ($\mu = 0$), the Euler-Lagrange equations are reduced to

$$-\lambda k_1' = 0, \qquad -\lambda k_2 = 0, \qquad k_2 k_3 k_4 = 0.$$

As a consequence we have

The critical points for the constant action in $\mathbb{L}^n$ are just null helices in 3-dimensional Lorentzian linear subspaces.

We have made a more general treatment of Lagrangian in the 3-dimensional case (see [17]). There we have explicitly obtained all solutions for a linear action as well as

got remarkable progress regarding other more difficult Lagrangians. Therefore, it seems reasonable to investigate the critical points of the linear action in the 4-dimensional case.

(9.3) Lagrangian density is a general function on the curvature in $\mathbb{L}^3$ ([17])

We consider the action $\mathcal{L} : \Lambda \to \mathbb{R}$ given by

$$\mathcal{L}(\gamma) = \int_\gamma f(k)dt,$$

where f is a differentiable function. The simplest action describing the motion of a particle is achieved when $f(k)$ is proportional to the pseudo-arc length parameter, and it is studied by Nersessian and Ramos in [22, 23]. When the action is linear on the curvature of the particle path, some advances have been produced in [24, 16]. No other cases appear to have been considered.

A null curve γ will be a critical point of the action $\mathcal{L}$ if

$$\left.\frac{d}{d\omega}\right|_{\omega=0} \mathcal{L}(\gamma_\omega) = \left.\frac{d}{d\omega}\right|_{\omega=0} \int_{\gamma_\omega} f(k_\omega)dt = 0,$$

for all variation of null curves γ_ω of γ.

Then the Cartan equations yield

$$\mathcal{L}'(0) = \frac{1}{2}\left[\Omega(\gamma, V)\right]_a^b - \frac{1}{2}\int_a^b \langle V, \mathcal{E}(\gamma)L\rangle\, dt, \tag{38}$$

where

$$\mathcal{E}(\gamma) = \varphi''' + (k\varphi)' + k\varphi', \qquad \varphi = -f(k) + 2kf'(k) + k''f''(k) + (k')^2 f^{(3)}(k) \tag{39}$$

and the boundary term reads

$$\begin{aligned}\Omega(\gamma, V) &= \left\langle \nabla_L^3 V, f'(k)W \right\rangle + \left\langle \nabla_L^2 V, f'(k)(3N - kL) - f''(k)k'W \right\rangle \\ &\quad + \left\langle \nabla_L V, (f(k) + f''(k)k'' + f^{(3)}(k)(k')^2)W - 2f''(k)k'N \right\rangle + \langle V, P \rangle .\end{aligned}$$

Here the vector field P is given by

$$P = (\varphi'' + k\varphi)L - \varphi' W + \varphi N, \tag{40}$$

V standing for a generic variational vector field along γ.

The trajectory $\boldsymbol{\gamma} \in \boldsymbol{\Lambda}$ is the null worldline of a relativistic particle in the (2+1)-dimensional spacetime if and only if

(i) $\boldsymbol{W}$, $\boldsymbol{N}$ and $\boldsymbol{k}$ are well defined in the whole world trajectory.

(ii) The following differential equation is fulfilled: $\boldsymbol{\mathcal{E}(\gamma) = 0}$.

The vector field X given by

$$X = \left((k')^2 f^{(3)}(k) + k'' f''(k) + f(k)\right) L - 2k' f''(k) W + 2f'(k) N + P \times \gamma$$

is constant along γ. Then

$$J = -P \times \gamma + X = \left((k')^2 f^{(3)}(k) + k'' f''(k) + f(k)\right) L - 2k' f''(k) W + 2f'(k) N \tag{41}$$

is a Killing vector field along γ that jointly with the constant vector field P allow us to find non-trivial first integrals of the Euler-Lagrange equations.

Bearing in mind Eqs. (39) and (41) we obtain that f and k have to satisfy the following ordinary differential equations

$$(\varphi')^2 - 2\varphi(\varphi'' + k\varphi) = \varepsilon p^2,$$
$$-2f'(k)\varphi'' + 2k'f''(k)\varphi' - 2f(k)\varphi - \varphi^2 = \omega.$$

9.1. P is either non-null or null

$\boldsymbol{f}$ being a quadratic function

$f(k) = \rho k^2 + \mu k + \lambda$

Case 1: $\rho = \mu = 0$, $\lambda \neq 0$ (the constant case)

This case represents the simplest action describing the motion of a particle, since it is proportional to the proper time along the light-like trajectory of the particle in space-time. We have $P = -\lambda(kL + N)$ and $J = \lambda L$, so that $\langle P, P\rangle = -2\lambda^2 k = \varepsilon p^2$ and $\langle P, J\rangle = \lambda^2 = \omega$, and therefore

$$k = -\frac{\varepsilon p^2}{2\omega}.$$

This shows that γ is a Cartan helix, [8, 15], with axis given by the vector P. Note that $\omega \neq 0$, otherwise $\lambda = 0$ which can not hold. Moreover, massive (tachyonic) solutions correspond to the null helices with negative (positive) curvature. This was shown by Nersessian and Ramos using a Hamiltonian formulation for this geometrical model, [23]. Here we offer an alternative proof which exploits the geometry of the particle trajectories.

Case 2: $\rho = 0$, $\mu \neq 0$ (the linear case)

Without loss of generality we normalize the constant μ to be one, then we find $P = (k'' + k^2 - \lambda k)L - k'W + (k - \lambda)N$ and $J = (k + \lambda)L + 2N$. In this case $\varphi = k - \lambda$ and we obtain a first solution when $\varphi = 0$, or equivalent $k = \lambda$, that is, γ is a Cartan helix. So, the constant vector field $J = 2\lambda L + 2N$ provided us a constant of the motion given by $\langle J, J\rangle = -8\lambda$.

If $\varphi \neq 0$, the first integrals provided by the vector fields P and J read

$$\begin{aligned} (k')^2 - 2(k-\lambda)(k'' + k^2 - \lambda k) - \varepsilon p^2 = 0, \\ -2k'' - 3k^2 + 2\lambda k + \lambda^2 - \omega = 0. \end{aligned} \tag{42}$$

From that we obtain

$$(k')^2 + k^3 - \lambda k^2 + (\omega - \lambda^2)k + \lambda^3 - \omega\lambda - \varepsilon p^2 = 0, \tag{43}$$

which can be written as $(k')^2 + Q(k) = 0$, Q being the polynomial $Q(X) = X^3 - \lambda X^2 + (\omega - \lambda^2)X + \lambda^3 - \omega\lambda - \varepsilon p^2$. Putting $q = k + \lambda$ we recover Eq. (39) in [24], showing that

the system under consideration contains massive and tachyonic branches. Later we will come back to this, when we determine the curvature functions of the particle trajectories in both sectors.

By using standard techniques involving the elliptic Jacobi functions, the solution can be found in terms of the roots α_1, α_2 and α_3 of the equation $Q(X) = 0$. First, assume that all roots of Q are real, $\alpha_1 \leq \alpha_2 \leq \alpha_3$. Then it is well-known that

$$\begin{aligned} \lambda &= \alpha_1 + \alpha_2 + \alpha_3, \\ \omega - \lambda^2 &= \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3, \\ \varepsilon p^2 + \omega\lambda - \lambda^3 &= \alpha_1\alpha_2\alpha_3, \end{aligned} \tag{44}$$

from which we easily deduce

$$\alpha_1 \leq \frac{\lambda}{3}, \quad \alpha_2 \leq \frac{\lambda - \alpha_1}{2}. \tag{45}$$

Before obtaining all solutions, note that since $Q(k) = -(k')^2$ then k takes values only where Q is negative. Trivial solutions are $k(s) = \alpha_i$, where α_i is a real root of Q, so that we find again the null Cartan helices. In this case $\langle P, P\rangle = -2k(k-\lambda)^2$ and $\langle P, J\rangle = -3k^2 + 2\lambda k + \lambda^2$. As before, the massive and tachyonic sectors correspond with negative or positive curvature, respectively. Now we are going to analyze all possible cases.

I. Q has a real root of multiplicity 3: $\alpha = \alpha_1 = \alpha_2 = \alpha_3$

We have that $\alpha = \lambda/3$ and the curvature function is given by

$$k(t) = \frac{\lambda}{3} - \frac{4}{(t+E)^2}, \quad s \in (-\infty, \lambda/3)$$

where E is a constant of integration, depending on the initial conditions, satisfying that $t+E$ is always different from zero. From Eq. (42) or (44) we find the relations $8\lambda^3+27\varepsilon p^2 = 0$ and $4\lambda^2 - 3\omega = 0$. Note that the constant of the motion εp^2 and ω are completely determined by the constant λ.

II. Q has two real roots, the lowest with multiplicity 2: $\alpha = \alpha_1 = \alpha_2 < \alpha_3$

The root α_3 is given by $\lambda - 2\alpha$. There are two possibilities:

$$k(t) = \lambda - 2\alpha + (3\alpha - \lambda)\coth^2\left(\frac{1}{2}\sqrt{\lambda - 3\alpha}(t+E)\right), \quad s \in (-\infty, \alpha)$$

$$k(t) = \lambda - 2\alpha + (3\alpha - \lambda)\coth^2\left(\frac{1}{2}\sqrt{\lambda - 3\alpha}(t+E)\right), \quad s \in (\alpha, \lambda - 2\alpha]$$

where E is a constant. In this case the relations among α, λ, ω and p are $-2\alpha(\alpha-\lambda)^2 = \varepsilon p^2$ and $-(\alpha - \lambda)(3\alpha + \lambda) = \omega$.

III. Q has two real roots, the greatest with multiplicity 2: $\alpha = \alpha_1 < \alpha_2 = \alpha_3$

We obtain that $\alpha_2 = \alpha_3 = (\lambda - \alpha)/2$, and the solution is given by

$$k(t) = \alpha + \frac{3\alpha - \lambda}{2} \tan^2 \left(\frac{1}{2} \sqrt{\frac{\lambda - 3\alpha}{2}} (t + E) \right), \quad s \in (-\infty, \alpha]$$

where E is a constant. Now the mass-shell condition and the Majorana-type relation read $(1/4)(\alpha - \lambda)(\alpha + \lambda)^2 = \varepsilon p^2$ and $-(1/4)(\alpha + \lambda)(3\alpha - 5\lambda) = \omega$.

IV. Q has three distinct real roots: $\alpha_1 < \alpha_2 < \alpha_3$

Let us denote $\alpha = \alpha_1$ and $\beta = \alpha_2$, then $\alpha_3 = \lambda - \alpha - \beta$. There are two possibilities for the curvature:

$$k(t) = \alpha - (\beta - \alpha) \operatorname{tn}^2 \left(\frac{1}{2} \sqrt{\lambda - 2\alpha - \beta} (t + E), \sqrt{\frac{\lambda - \alpha - 2\beta}{\lambda - 2\alpha - \beta}} \right),$$

$$k(t) = \lambda - \alpha - \beta + (\alpha + 2\beta - \lambda) \operatorname{sn}^2 \left(\frac{1}{2} \sqrt{\lambda - 2\alpha - \beta} (t + E), \sqrt{\frac{\lambda - \alpha - 2\beta}{\lambda - 2\alpha - \beta}} \right),$$

defined in the intervals $(-\infty, \alpha]$ or $[\beta, \lambda - \alpha - \beta]$, respectively. In this case we have the following relations among constants: $-(\alpha + \beta)(\alpha - \lambda)(\beta - \lambda) = \varepsilon p^2$ and $(\alpha + \lambda)(\beta + \lambda) - (\alpha + \beta)^2 = \omega$.

V. Q has complex roots

Let us suppose that α_1 and α_2 are complex (so α_3 is real). Then the curvature is given by

$$k(t) = \alpha_3 - (\alpha_3 - \alpha_2)\,\mathrm{sn}^2\left(\frac{1}{2}\sqrt{\alpha_3 - \alpha_1}(t+E), \sqrt{\frac{\alpha_2 - \alpha_3}{\alpha_1 - \alpha_3}}\right), \quad s \in (-\infty, \alpha_3].$$

Write $\alpha_1 = \alpha + \beta i$ and $\alpha_2 = \alpha - \beta i$, then the mass-shell condition and the Majorana-type relation read $-2\alpha\left((\alpha-\lambda)^2 + \beta^2)\right) = \varepsilon p^2$ and $\lambda^2 + 2\alpha\lambda - 3\alpha^2 + \beta^2 = \omega$.

We use cylindrical coordinates to integrate the Cartan equations of the curves.

Case 3: $\rho \neq 0$ (the quadratic case)

As before, without loss of generality we can assume that $\rho = 1$. The Euler-Lagrange equation is given by

$$2k^{(5)} + (10k + \mu)\,k^{(3)} + 20k''k' + k'\left(15k^2 + 3\mu k - \lambda\right) = 0, \tag{46}$$

In this case $\varphi = 2k'' + 3k^2 + \mu k - \lambda$ and we have two families of solutions. If $\varphi = 0$, then $P = 0$ and $J = -2(k^2 - \lambda)L - 4k'W + 2(2k+\mu)N$ is a constant vector field verifying $\langle J, J\rangle = \varepsilon j^2$. Then, the first family of solutions satisfies the equation

$$(k')^2 + k^3 + \frac{\mu}{2}k^2 - \lambda k - \left(\frac{\mu}{2} + \frac{\varepsilon}{16}j^2\right) = 0.$$

This equation has the same nature that the equation (43) and the solutions are seemed.

We now suppose that $\varphi \neq 0$, then the vector fields P and J read

$$\begin{aligned} P = &\left(2k^{(4)} + k''(8k+\mu) + 6(k')^2 + 3k^3 + \mu k^2 - \lambda k\right) L \\ &- \left(2k^{(3)} + k'(6k+\mu)\right) W + \left(2k'' + 3k^2 + \mu k - \lambda\right) N, \end{aligned}$$

$$J = \left(2k'' + k^2 + \mu k + \lambda\right) L - 4k'W + 2(2k+\mu)N.$$

If $\varphi \neq 0$, using the above equations we obtain the following first integrals:

$$\begin{aligned} &-2\left(2k^{(4)} + k''(8k+\mu) + 6(k')^2 + 3k^3 + \mu k^2 - \lambda k\right)\left(2k'' + 3k^2 + \mu k - \lambda\right) \\ &+ \left(2k^{(3)} + k'(6k+\mu)\right)^2 - \varepsilon p^2 = 0, \end{aligned}$$

$$\begin{aligned} &-(8k+4\mu)k^{(4)} + 8k'k^{(3)} - 4(k'')^2 - (40k^2 + 24\mu k + 2\mu^2)k'' \\ &- 8\mu(k')^2 - 15k^4 - 14\mu k^3 + (2\lambda - 3\mu^2)k^2 + 2\lambda\mu k + \lambda^2 - \omega = 0. \end{aligned}$$

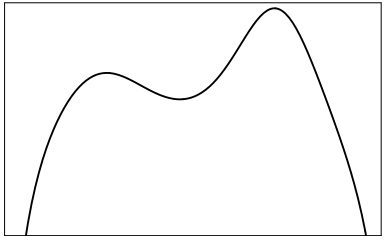
On the other hand, it is easy to see that another first integral is given by

$$2k^{(4)} + 10kk'' + \mu k'' + 5(k')^2 + 5k^3 + \frac{3}{2}\mu k^2 - \lambda k + c = 0,$$

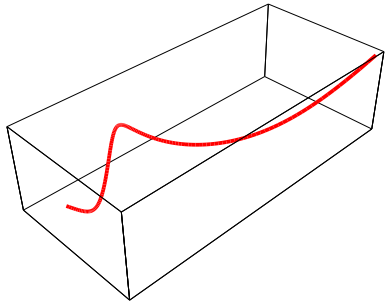
c being a constant. These three first integrals can be combined to obtain the following ordinary differential equation of degree two:

$$\frac{1}{16}(k')^2\left(-4(k'')^2-2(2k+\mu)(k')^2+5k^4+2\mu k^3-2\lambda k^2+4ck+2c\mu+\lambda^2+\omega\right)^2$$
$$+(2k''+3k^2+\mu k-\lambda)(4kk''-2(k')^2+4k^3+\mu k^2+2c)-\varepsilon p^2 \quad = \quad 0.$$

The integration of this equation is very complicated, but we can use computing methods to make us an idea of their solutions (see the following pictures).

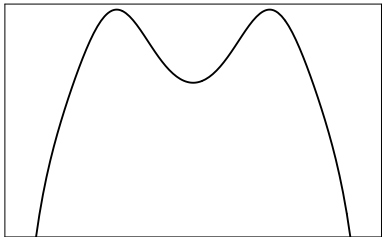


$k(t)$

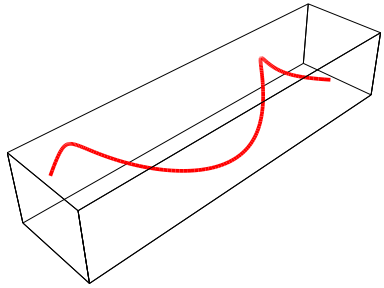


$\gamma(t)$

Fig. 5



$k(t)$



$\gamma(t)$

Fig. 6

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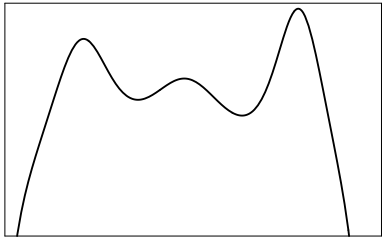
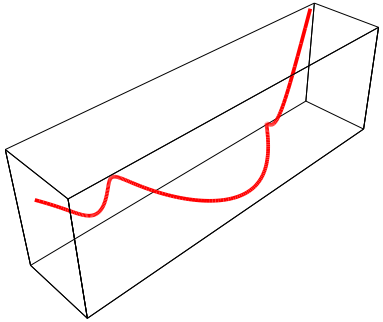


$k(t)$



$\gamma(t)$

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